

SESSION 1: PARITY CHECKS THAT DO NOT READ THE MESSAGE

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These pages collect, in full, the computations that the first session carries out at the board. They accompany [Section 1](#) of the notes and the first three subsections of [Section 2](#) of the notes, where the definitions and proofs are given; nothing here is needed to follow the notes, and nothing here is new. Throughout, $[u]$ is an arbitrary state of the repetition code and $u = \alpha|000\rangle + \beta|111\rangle$ a chosen normalized representative, with $|\alpha|^2 + |\beta|^2 = 1$. Kets below denote representative vectors and matrices denote linear lifts of transformations; the signs of check lifts specify their outcome labels.

1. The checks on the eight basis vectors. Since $Z|b\rangle = (-1)^b|b\rangle$, the checks $g_1 = Z_1Z_2$ and $g_2 = Z_2Z_3$ are diagonal in the computational basis:

b	000	001	010	011	100	101	110	111
$(-1)^{b_1+b_2}$	+	+	−	−	−	−	+	+
$(-1)^{b_2+b_3}$	+	−	−	+	+	−	−	+

The four joint eigenspaces are therefore $\text{span}\{|000\rangle, |111\rangle\}$ (outcome bits 00), $\text{span}\{|100\rangle, |011\rangle\}$ (10), $\text{span}\{|010\rangle, |101\rangle\}$ (11), and $\text{span}\{|001\rangle, |110\rangle\}$ (01); the corresponding projections P_σ of [Proposition 1.3](#) of the notes are the diagonal projections onto these pairs of basis vectors.

2. Measuring after a bit flip. After the error X_2 the vector is $X_2u = \alpha|010\rangle + \beta|101\rangle$, which lies in the eigenspace with bits 11. Hence

$$\|P_{11}X_2u\|^2 = |\alpha|^2 + |\beta|^2 = 1, \quad P_\sigma X_2u = 0 \quad (\sigma \neq 11),$$

and the post-measurement state is $[X_2u]$. Both conclusions hold for every (α, β) : the outcome reveals the error and nothing about the encoded state. By contrast, measuring Z_1 alone on $[u]$ uses the projections $(I \pm Z_1)/2$, which give representatives $\alpha|000\rangle$ and $\beta|111\rangle$, with probabilities $|\alpha|^2$ and $|\beta|^2$; the output states, when their probabilities are positive, are $[|000\rangle]$ and $[|111\rangle]$. This measurement destroys a superposition with both coefficients nonzero.

3. A rotation that the measurement discretizes. Let $U = \cos \theta I + i \sin \theta X_1$ ([Remark 1.7](#) of the notes). Then $Uu = \cos \theta u + i \sin \theta X_1u$, with the two terms in the eigenspaces 00 and 10. The joint measurement gives

$$P_{00}Uu = \cos \theta u, \quad P_{10}Uu = i \sin \theta X_1u,$$

i.e., outcome 00 with probability $\cos^2 \theta$ and state $[u]$, or outcome 10 with probability $\sin^2 \theta$ and state $[X_1u]$. Applying X_1 after the outcome 10 restores $[u]$ in both branches.

4. The phase flip. $Z_1u = \alpha|000\rangle - \beta|111\rangle$ lies in the eigenspace 00, so the measurement does not notice it. It moves a general projective code state: for $\alpha = \beta = 1/\sqrt{2}$, the states $[u]$ and $[Z_1u]$ are orthogonal. It fixes the computational state $[|000\rangle]$, which is why a classical analysis of the repetition code cannot see this failure.

5. The dimension formula for the repetition code. The stabilizer is $S = \{I, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$, and

$$P = \frac{I + Z_1Z_2}{2} \cdot \frac{I + Z_2Z_3}{2} = \frac{I + Z_1Z_2 + Z_2Z_3 + Z_1Z_3}{4}.$$

Its diagonal entry at b is $(1 + (-1)^{b_1+b_2} + (-1)^{b_2+b_3} + (-1)^{b_1+b_3})/4$, which equals 1 for $b \in \{000, 111\}$ and 0 otherwise. Thus $\text{Tr } P = 2 = 2^{3-2}$, as [Theorem 2.2](#) of the notes predicts: only the identity contributes to the trace. The underlying code subspace has dimension two; its projectivization is a line, encoding one qubit.

6. Syndromes add. The syndrome map is a homomorphism ([Proposition 2.3](#) of the notes). For example, $s(X_1X_2) = s(X_1) + s(X_2) = (1, 0) + (1, 1) = (0, 1) = s(X_3)$, so X_1X_2 and X_3 cannot be distinguished. Their quotient $X_1X_2X_3$ commutes with both checks and exchanges $|000\rangle$ and $|111\rangle$; it is a logical operator of weight three. The repetition code has distance one only because of the weight-one logical operator Z_1 .

7. Syndromes on the nine-qubit patch. For the patch of [Figure 1](#) of the notes, the Z checks (B_1, B_2, B_3, B_4) detect X errors and the X checks (A_1, A_2, A_3, A_4) detect Z errors. A check flips exactly when it contains the qubit:

qubit	1	2	3	4	5	6	7	8	9
syndrome of X_j in B	1000	1000	0001	1010	1100	0101	0010	0100	0100
syndrome of Z_j in A	1000	1100	0100	0010	0110	0100	0010	0011	0001

Coincidences occur exactly for pairs that differ by a two-qubit check: X_1, X_2 (by A_1), X_8, X_9 (by A_4), Z_3, Z_6 (by B_4), and Z_4, Z_7 (by B_3). Correcting one member of such a pair by the other leaves a stabilizer, which acts trivially on the code; this is the first instance of degeneracy. The checks commute: for instance $A_2 = X_2X_3X_5X_6$ meets B_1, B_2, B_4 in $\{2, 5\}, \{5, 6\}, \{3, 6\}$, two qubits each, and is disjoint from B_3 .