TOPOLOGICAL AND LOGICAL ASPECTS OF THE PROBLEM OF HIDDEN VARIABLES IN QUANTUM MECHANICS

PRELIMINARY NOTES PREPARED BY J. IOVINO

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Abstract. We examine two proofs of the Bell–Kochen–Specker theorem, which rules out non-contextual hidden variables in quantum mechanics. Bell's proof combines Gleason's Theorem with elementary topology and is infinitary; Kochen and Specker's proof exhibits a finite configuration of rays and is constructive. The two arguments were found independently in the mid-1960s. We show that the compactness theorem of propositional logic links them: compactness converts Bell's argument into a proof of the existence of a finite configuration of the kind that Kochen and Specker built by hand. We also show that quantum measures have no counterpart of Dirac measures and, unlike classical measures, do not form a lattice. In a timeline we trace the history of the problem of hidden variables.

0. Preliminaries (all the linear algebra and physics you will need to read these notes)

We assume that the reader is familiar with Hilbert spaces, with eigenvalues and eigenvectors of linear operators, and with orthogonal projections. In this short section we recall everything else that we need, from linear algebra and from physics alike.

Unless stated otherwise (as in Remark 1.6 and Section 5), the symbol $\mathcal{H}$ denotes a complex Hilbert space with $3 \leq \dim \mathcal{H} \leq \infty$, where ∞ stands for $\aleph_0$. All Hilbert spaces are separable, except where we say otherwise (as in Remarks 1.7 and 1.8). Separability is not a technicality: in Remark 1.7 below we explain what large cardinals have to do with it. With the obvious changes, everything below holds for real and quaternionic Hilbert spaces as well; for instance, a density operator on a real space must be required to be self-adjoint, since positivity no longer implies self-adjointness.

We use the physicists' notation for vectors and functionals; as this notation is not standard in mathematics, we explain it first. The inner product $\langle \xi | \eta \rangle$ of $\xi, \eta \in \mathcal{H}$ is linear in the second variable and conjugate-linear in the first. A vector η is also written $|\eta\rangle$ (a *ket*), and $\langle \xi |$ (a *bra*) denotes the linear functional $\eta \mapsto \langle \xi | \eta \rangle$. Juxtaposition is composition: for a bounded operator A, the number $\langle \xi | A | \eta \rangle$ means $\langle \xi | A\eta \rangle$, and the expression $|\xi\rangle\langle\xi|$ is the operator $\eta \mapsto \langle \xi | \eta \rangle \, \xi$. We write $S(\mathcal{H})$ for the unit sphere of $\mathcal{H}$.

Let $L(\mathcal{H})$ be the lattice of closed subspaces of $\mathcal{H}$, an *orthomodular lattice* in the usual sense; we identify each closed subspace with the orthogonal projection onto it. Recall that the orthogonal projection P onto a closed subspace V is defined by writing each $\eta \in \mathcal{H}$ uniquely as a sum $\eta = \eta_V + \eta_\perp$ with $\eta_V \in V$ and $\eta_\perp \in V^\perp$, and letting $P\eta = \eta_V$. Two properties follow at once. First, the projection P is idempotent, $P^2 = P$, because $P\eta$ lies in V and P fixes every vector of V. Second, it is self-adjoint, $P^* = P$, because for $\xi, \eta \in \mathcal{H}$ we

have

$$\langle \xi | P\eta \rangle = \langle \xi_V | \eta_V \rangle = \langle P\xi | \eta \rangle,$$

since the cross terms $\langle \xi_\perp | \eta_V \rangle$ and $\langle \xi_V | \eta_\perp \rangle$ vanish. Conversely, a bounded operator P with $P^2 = P = P^*$ is the orthogonal projection onto its range, which is closed, being the kernel of $I - P$. Thus, the elements of $L(\mathcal{H})$ are exactly the bounded operators P with $P^2 = P = P^*$, and these two identities yield

$$\langle \eta | P | \eta \rangle = \langle \eta | P^* P \eta \rangle = \langle P\eta | P\eta \rangle = \|P\eta\|^2 \qquad (\eta \in \mathcal{H}). \tag{1}$$

The lattice $L(\mathcal{H})$ carries a partial order $P \le Q$ (inclusion of ranges), a meet $P \wedge Q$ (intersection of ranges), a join $P \vee Q$ (closed span of the ranges), an orthocomplement $P \mapsto P^\perp = I - P$, a least element 0 (the zero subspace) and a greatest element I (the whole space $\mathcal{H}$). Two projections P and Q are *orthogonal*, written $P \perp Q$, when their ranges are orthogonal or, equivalently, when $P \le Q^\perp$. A *basis* of $L(\mathcal{H})$ is the set of rank-one projections onto the lines spanned by the vectors of an orthonormal basis of $\mathcal{H}$.

For $\psi \in S(\mathcal{H})$, the operator $P_\psi = |\psi\rangle\langle\psi|$ is the orthogonal projection onto the one-dimensional subspace $\mathbb{C}\psi$: the projection P_ψ fixes ψ and kills every vector orthogonal to ψ, and it satisfies $P_\psi^2 = P_\psi = P_\psi^*$. The rank-one projections are exactly the minimal nonzero elements, the *atoms*, of the lattice $L(\mathcal{H})$.

We now turn to the physical meaning of these objects. In quantum mechanics, a projection $P \in L(\mathcal{H})$ represents a yes/no question about a physical system, that is, an experiment with two possible outcomes. The projection $P^\perp = I - P$ represents the same question with the answers exchanged. A state of the system assigns to each projection P the probability $\mu(P)$ of obtaining the answer "yes" to P. The question I is always answered "yes", whence $\mu(I) = 1$. If $P_0, P_1, \ldots$ are pairwise orthogonal, then the corresponding questions are mutually exclusive and can be tested together; the probability of "yes" to their join $\bigvee_i P_i$ is the sum of the probabilities $\mu(P_i)$. These requirements are the axioms of a quantum measure (see Definition 1.1 below). The Born rule decides which assignments μ actually occur in physics, and it leads to the notion of a density operator.

Remark 0.1 (The Born rule). The simplest states of a physical system, the *pure states*, are described by unit vectors $\psi \in S(\mathcal{H})$. The *Born rule* says that in the state ψ the probability of the answer "yes" to the question P is

$$\mu(P) = \|P\psi\|^2 = \langle \psi | P | \psi \rangle$$

(the second equality is (1)). An important special case is the question "is the system in the state ξ?" for another unit vector $\xi \in S(\mathcal{H})$. This question is represented by the rank-one projection $P_\xi = |\xi\rangle\langle\xi|$ onto the line $\mathbb{C}\xi$. Since $P_\xi \psi = \langle \xi | \psi \rangle \, \xi$ and $\|\xi\| = 1$, the Born rule yields

$$\mu(P_\xi) = \|P_\xi \psi\|^2 = |\langle \xi | \psi \rangle|^2,$$

the familiar probability that a system in the state ψ is found in the state ξ.

Within the quantum formalism, a pure state gives the fullest specification of the state of an individual system. Even then, the probabilities $\|P\psi\|^2$ may lie strictly between 0 and 1. They therefore need not arise from uncertainty about which quantum state was prepared. Whether a more detailed description could account for them is a further question. The theorems of these notes exclude the non-contextual, predetermined answers described in the next section; they do not exclude every hidden-variable description.

In practice, however, even this complete knowledge is rarely available. Real physical sources seldom produce a single pure state: the light of an ordinary lamp, for instance, consists of photons whose polarizations vary at random from one photon to the next, so no single vector

describes the beam. Because of the imperfections of the preparing apparatus, we know at best which pure states may have been produced, and with what probabilities. Such incomplete knowledge introduces a second kind of randomness, entirely different in nature from the first. This second randomness is classical, like that of a shuffled deck of cards: it reflects only our ignorance of the preparation, not any property of the system itself. A state that combines the two kinds of randomness is called a *mixed state*, or a *mixture*.[1]

Suppose, then, that we know only that the system has been prepared in the state ψ_k with probability λ_k, for a finite or countable family of states $\psi_k \in S(\mathcal{H})$ and weights $\lambda_k \geq 0$ with $\sum_k \lambda_k = 1$. Here the weights λ_k express our ignorance of the preparation, while each $\langle\psi_k|P|\psi_k\rangle$ is a quantum probability given by the Born rule. When the two are combined, the probability of "yes" to P is the weighted average (or expected value)

$$\sum_k \lambda_k \, \langle\psi_k|P|\psi_k\rangle. \tag{2}$$

We show below that this expression depends on the preparation only through the operator

$$\rho = \sum_k \lambda_k \, |\psi_k\rangle\langle\psi_k| = \sum_k \lambda_k \, P_{\psi_k},$$

and that a single formula computes (2) from ρ and P. The tool is the trace, which we now recall.

When $\mathcal{H}$ is finite-dimensional, the *trace* of a linear operator T on $\mathcal{H}$ is

$$\mathrm{Tr}(T) = \sum_i \langle e_i|Te_i\rangle,$$

where (e_i) is an orthonormal basis of $\mathcal{H}$. In other words, the trace $\mathrm{Tr}(T)$ is the sum of the diagonal entries of the matrix of T in the basis (e_i); this sum does not depend on the choice of basis. When $\mathcal{H}$ is infinite-dimensional, the same formula defines the trace of the *trace-class* operators, for which the sum converges absolutely and is again independent of the basis. We need only a few facts about trace-class operators. Every operator of finite rank, in particular every rank-one projection P_ψ, is trace-class. If T is trace-class and A is bounded, then the operators TA and AT are trace-class, with trace norms at most $\|T\|_1\|A\|$, where $\|T\|_1 = \mathrm{Tr}\big(\sqrt{T^*T}\big)$. Finally, the trace-class operators form a Banach space under the trace norm, on which the trace is a continuous linear functional; hence a series $\sum_k T_k$ of trace-class operators with $\sum_k \|T_k\|_1 < \infty$ converges to a trace-class operator, whose trace is $\sum_k \mathrm{Tr}(T_k)$. For the definition and these facts, see the book of Reed and Simon [RS80, Section VI.6]. Readers who prefer to do so may assume that $\mathcal{H}$ is finite-dimensional throughout this section; the infinite-dimensional case differs only in these points of analysis.

The following observation will play an important role in these notes.

Observation 0.2. *Let A be a bounded operator on $\mathcal{H}$ and let $\psi \in S(\mathcal{H})$ be given. Then*

$$\mathrm{Tr}(AP_\psi) = \mathrm{Tr}(P_\psi A) = \langle\psi|A|\psi\rangle.$$

In particular, we have $\mathrm{Tr}(P_\psi P) = \langle\psi|P|\psi\rangle = \|P\psi\|^2$ for every $P \in L(\mathcal{H})$, and $\mathrm{Tr}(P_\xi P_\psi) = |\langle\xi|\psi\rangle|^2$ for $\xi \in S(\mathcal{H})$.

[1] The same mathematical description arises in a situation where ignorance is not the explanation. A system A that has interacted with another system B is in general entangled with B; it then has no pure state of its own, even when the combined system is in a pure state, and the statistics of A are again those of a mixture.

Proof. We compute both traces in an orthonormal basis (e_i) of $\mathcal{H}$ with $e_0 = \psi$. The projection P_ψ fixes $e_0 = \psi$ and kills every e_i with $i \geq 1$, because e_i is orthogonal to ψ. Hence only the first term survives:

$$\operatorname{Tr}(AP_\psi) = \sum_i \langle e_i | AP_\psi e_i \rangle = \langle \psi | A\,\psi \rangle = \langle \psi | A | \psi \rangle.$$

Similarly, since $P_\psi A e_i = \langle \psi | A e_i \rangle\, \psi$ and $\langle e_i | \psi \rangle = 0$ for $i \geq 1$, we have

$$\operatorname{Tr}(P_\psi A) = \sum_i \langle e_i | P_\psi A e_i \rangle = \sum_i \langle e_i | \psi \rangle \langle \psi | A e_i \rangle = \langle \psi | A\,\psi \rangle.$$

For $P \in L(\mathcal{H})$, the identity (1) turns $\langle \psi | P\psi \rangle$ into $\langle P\psi | P\psi \rangle = \|P\psi\|^2$. Finally, taking $A = P_\xi$ gives $\operatorname{Tr}(P_\xi P_\psi) = \langle \psi | P_\xi | \psi \rangle = \|P_\xi \psi\|^2 = |\langle \xi | \psi \rangle|^2$, since $P_\xi \psi = \langle \xi | \psi \rangle\, \xi$ and $\|\xi\| = 1$. ☺

We now return to the Born rule: by Observation 0.2 and the linearity and continuity of the trace, the probability (2) is

$$\sum_k \lambda_k \, \langle \psi_k | P | \psi_k \rangle = \sum_k \lambda_k \operatorname{Tr}(P_{\psi_k} P) = \operatorname{Tr}(\rho P).$$

Thus the Born rule, for pure and mixed preparations alike, takes the single form $P \mapsto \operatorname{Tr}(\rho P)$. The operator ρ has three properties that do not depend on how it was obtained. First, it is positive, since $\langle \xi | \rho | \xi \rangle = \sum_k \lambda_k |\langle \psi_k | \xi \rangle|^2 \geq 0$ for every $\xi \in \mathcal{H}$. Second, it is trace-class, because the series that defines it converges in the trace norm: each P_{ψ_k} has trace norm 1, and $\sum_k \lambda_k = 1$. Third, its trace is 1; this property is nothing but the requirement that the weights λ_k add up to 1. Indeed, each P_{ψ_k} has trace $\operatorname{Tr}(P_{\psi_k}) = \langle \psi_k | \psi_k \rangle = 1$ (by Observation 0.2 with $A = I$), and therefore

$$\operatorname{Tr}(\rho) = \sum_k \lambda_k \operatorname{Tr}(P_{\psi_k}) = \sum_k \lambda_k = 1.$$

In physical terms, the number $\operatorname{Tr}(\rho) = \operatorname{Tr}(\rho I)$ is the probability of the answer "yes" to the question I, which is always answered "yes". These are the properties singled out in the following definition.

Definition 0.3 (Density operator). A *density operator* (or *density matrix*) on $\mathcal{H}$ is a trace-class operator ρ that is positive, i.e. $\langle \xi | \rho | \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$, and has $\operatorname{Tr}(\rho) = 1$.

Conversely, every density operator arises from a preparation as in Remark 0.1. Suppose first that $\mathcal{H}$ is finite-dimensional. A positive operator ρ on a complex Hilbert space is self-adjoint, so the spectral theorem provides an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors ψ_k of ρ; the corresponding eigenvalues $\lambda_k = \langle \psi_k | \rho | \psi_k \rangle$ are nonnegative because ρ is positive, and they add up to $\operatorname{Tr}(\rho) = 1$. Hence the operator ρ can be written as

$$\rho = \sum_k \lambda_k \, |\psi_k\rangle\langle\psi_k|, \qquad \lambda_k \geq 0, \quad \sum_k \lambda_k = 1, \tag{3}$$

where ψ_k has eigenvalue λ_k. When $\mathcal{H}$ is infinite-dimensional, the same decomposition holds with a countable orthonormal basis (ψ_k) of eigenvectors, by the spectral theorem for compact self-adjoint operators [RS80, Section VI.5] (density operators, being trace-class, are compact, and the space $\mathcal{H}$ is separable). Thus density operators are exactly the probabilistic mixtures of pure states, the pure state $\psi \in S(\mathcal{H})$ being represented by the projection P_ψ.

Finally, if we take $A = \rho$ in Observation 0.2, we see that the probability of finding a system in the state ρ in the pure state ψ is

$$\operatorname{Tr}(\rho P_\psi) = \langle \psi | \rho | \psi \rangle.$$

If ψ is an eigenvector of ρ with eigenvalue λ, then this probability equals $\lambda\|\psi\|^2 = \lambda$; in general, the number $\langle\psi|\rho|\psi\rangle$ is a weighted average of the eigenvalues of ρ, namely $\sum_k \lambda_k |\langle\psi_k|\psi\rangle|^2$, with weights $|\langle\psi_k|\psi\rangle|^2$ that add up to $\|\psi\|^2 = 1$.

This is all the quantum mechanics and linear algebra needed for the rest of these notes.

1. Quantum measures and Gleason's Theorem

In this section we define quantum measures, state the main theorem of these notes, and recall Gleason's Theorem, which describes all quantum measures when $\dim \mathcal{H} \geq 3$.

Definition 1.1 (Quantum measure). A *quantum measure* (or *state*[2]) on $L(\mathcal{H})$ is a function $\mu : L(\mathcal{H}) \to [0,1]$ with $\mu(I) = 1$ that is countably additive on orthogonal families: for every finite or countable family $\{P_i\}$ of pairwise orthogonal projections, we have

$$\mu\Big(\bigvee_i P_i\Big) = \sum_i \mu(P_i).$$

A *two-valued quantum measure* is a quantum measure $v : L(\mathcal{H}) \to \{0,1\}$.

The meaning of a two-valued quantum measure. Definition 1.1 is purely mathematical: it concerns functions on a lattice and mentions nothing physical. Read physically, however, a two-valued quantum measure expresses a precise hypothesis about nature. Recall from Section 0 that a projection P represents a yes/no question about a physical system. A two-valued quantum measure v answers every such question in advance: the answer to P is "yes" if $v(P) = 1$ and "no" if $v(P) = 0$, and this answer does not depend on which other compatible questions are asked together with P. Such a system of predetermined answers is what physicists call a *non-contextual hidden-variable assignment*. In these notes, a two-valued quantum measure is also called a *valuation*.

The "only one winner per basis" lemma. The first mathematical consequence of Definition 1.1 is elementary, but it plays a pivotal role in these notes: of the questions P_{e_i} attached to an orthonormal basis $\{e_i\}$, a valuation answers exactly one with "yes".

Lemma 1.2. *If v is a two-valued quantum measure on $L(\mathcal{H})$ and $\{e_i\}$ is an orthonormal basis of $\mathcal{H}$, then $v(P_{e_i}) = 1$ for exactly one index i, and $v(P_{e_i}) = 0$ for all the others.*

Proof. The projections P_{e_i} are pairwise orthogonal, and their join is I, because the closed span of a basis is $\mathcal{H}$. The basis is countable, since $\mathcal{H}$ is separable, and countable additivity therefore yields $\sum_i v(P_{e_i}) = v(I) = 1$. A sum of terms in $\{0,1\}$ equals 1 exactly when one term is 1 and all the others are 0, and the lemma follows. ☺

Remarks 1.3.

(1) The proof of Lemma 1.2 does not use the hypothesis $\dim \mathcal{H} \geq 3$: it applies to every countable orthonormal basis, in particular in dimension 2. For this reason the Main Theorem (Theorem 1.4 below), which fails in dimension 2 (see Remark 1.6), cannot follow from the lemma alone; what is missing is described in item 2.

(2) The lemma pins down only one privileged ray in *each* orthonormal basis, separately, and says nothing about how the privileged ray of one basis relates to that of a different, overlapping basis. To prove Theorem 1.4, we must make the choices in different bases interact, and we give two ways of doing so. Bell's argument (see Section 2 below) invokes Gleason's Theorem to force the choices to fit together into one continuous

[2]Gleason called this object a *measure* [Gle57]; the quantum-logic tradition (Mackey [Mac63], Piron [Pir64]) renamed it a *state*, the name used in much of the older literature; later authors (e.g. Hamhalter [Ham03]) returned to *quantum measure*, the term used throughout these notes.

function on the sphere $S(\mathcal{H})$, and then turns the connectedness of $S(\mathcal{H})$ against that function.[3] Kochen and Specker's argument (see Section 4 below) uses instead a finite family of bases, overlapping in a pattern that no choice of privileged rays can respect.

The role of valuations. A deterministic, non-contextual hidden-variable explanation would represent a quantum state as an average of valuations, just as a classical probability measure is an average of Dirac measures. Quantum probabilities would then record our uncertainty about which valuation describes the system. The main theorem of these notes rules out even the individual valuations that such an explanation would require.

Theorem 1.4 (Main Theorem, Bell–Kochen–Specker). *If $3 \leq \dim \mathcal{H} \leq \infty$, then no two-valued quantum measure exists on $L(\mathcal{H})$.*

We give two independent proofs of Theorem 1.4: a topological one, due to Bell, which is the simpler of the two (see Section 2 below), and a combinatorial one, due to Kochen and Specker (see Section 4 below). Bell's proof relies on Gleason's Theorem, which describes all quantum measures in terms of density operators (see Definition 0.3). For every density operator ρ, the map $P \mapsto \operatorname{Tr}(\rho P)$ is a quantum measure. Its values lie in $[0,1]$, because (3) and Observation 0.2 express $\operatorname{Tr}(\rho P)$ as the weighted average $\sum_k \lambda_k \|P\psi_k\|^2$ of numbers in $[0,1]$, and its value at I is $\operatorname{Tr}(\rho) = 1$. It is additive on finite orthogonal families, because the join of finitely many pairwise orthogonal projections is their sum and the trace is linear. In finite dimension every orthogonal family of nonzero projections is finite, so finite additivity is all that is needed. In infinite dimension, the partial sums $P_1 + \cdots + P_n$ of an orthogonal sequence converge strongly to the join $\bigvee_i P_i$, and countable additivity follows from the continuity of $A \mapsto \operatorname{Tr}(\rho A)$ along bounded, strongly convergent sequences of operators [RS80, Section VI.6]. By Remark 0.1, these are exactly the quantum measures given by the Born rule. Gleason's Theorem, which we now state, asserts the converse: in dimension at least 3, every quantum measure arises in this way, and the Born rule is therefore forced by the lattice structure of $L(\mathcal{H})$ alone.

Theorem 1.5 (Gleason's Theorem [Gle57]). *If $3 \leq \dim \mathcal{H} \leq \infty$, then every quantum measure μ on $L(\mathcal{H})$ has the form $\mu(P) = \operatorname{Tr}(\rho P)$ for a unique density operator ρ (see Definition 0.3).*

We take Gleason's Theorem as given, since every known proof (Gleason's original argument and its later simplifications alike) is a substantial piece of real analysis. The lower bound $\dim \mathcal{H} \geq 3$ cannot be weakened to $\dim \mathcal{H} \geq 2$; the counterexample in the following remark goes back to Gleason's paper [Gle57, p. 886]. Remark 1.6 is not needed for our proofs and may be skipped without loss of continuity.

Remark 1.6. On $\mathcal{H} = \mathbb{R}^2$, we parametrize the rays by their angle $\theta \in [0, \pi)$ with the first coordinate axis; the ray of angle θ is spanned by the unit vector $u_\theta = (\cos\theta, \sin\theta)$. Every orthonormal basis then corresponds to a pair of angles $\{\theta, \theta + \pi/2 \pmod{\pi}\}$. We define a

[3] The sphere $S(\mathcal{H})$ is path-connected, and hence connected. If $u, w \in S(\mathcal{H})$ and $w \neq -u$, then the vector $(1-t)u + tw$ never vanishes, and the path $t \mapsto \frac{(1-t)u+tw}{\|(1-t)u+tw\|}$ with $t \in [0,1]$ joins u to w inside $S(\mathcal{H})$; the path $\theta \mapsto e^{i\theta}u$ with $\theta \in [0, \pi]$ joins u to $-u$. (In a real space, one joins u to $-u$ through a unit vector orthogonal to u.)

function $v : [0, \pi) \to \{0, 1\}$ by

$$v(\theta) = \begin{cases} 1, & \text{if } \theta \in [0, \pi/2), \\ 0, & \text{if } \theta \in [\pi/2, \pi). \end{cases}$$

In every such pair, exactly one of the two angles lies in $[0, \pi/2)$, so additivity holds on every basis. Extended by $v(0) = 0$ and $v(I) = 1$, the function v is therefore a two-valued quantum measure on $L(\mathbb{R}^2)$ in the sense of Definition 1.1. It jumps at $\theta = \pi/2$, whereas the function $\theta \mapsto \langle u_\theta|\rho|u_\theta\rangle$ is continuous for every density operator ρ. Hence the measure v is not of the form $\mathrm{Tr}(\rho\,\cdot\,)$: the conclusion of Gleason's Theorem fails when $\dim \mathcal{H} = 2$, and so does that of Theorem 1.4. The same construction works for $\mathcal{H} = \mathbb{C}^2$: in each pair of orthogonal rays, we assign the value 1 to one ray and 0 to the other, and we set $v(0) = 0$ and $v(I) = 1$.

The upper bound on the dimension is also there for a reason, but one that has nothing to do with geometry. In Remark 1.7 below we show that, in nonseparable Hilbert spaces, the validity of Gleason's Theorem depends on large cardinals.

Remark 1.7 (Gleason's Theorem and real-valued measurable cardinals). An uncountable cardinal κ is called *real-valued measurable* if it carries a countably additive probability measure on $\mathcal{P}(\kappa)$ that vanishes on singletons and whose null sets are closed under unions of fewer than κ members. We use this κ-additive convention throughout the remark. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_i\}_{i\in N}$, and suppose that $|N|$ is at least a real-valued measurable cardinal. There is then a countably additive probability measure ν on the power set of N that vanishes on singletons. We define

$$\mu(P) = \int_N \langle e_i|P|e_i\rangle \, d\nu(i) \qquad (P \in L(\mathcal{H})),$$

which makes sense because ν measures every subset of N. The function μ satisfies $\mu(I) = 1$ and is countably additive on orthogonal families: if $P = \bigvee_k P_k$ with the P_k pairwise orthogonal, then the identity $\langle e_i|P|e_i\rangle = \sum_k \langle e_i|P_k|e_i\rangle$ holds for every i, and the monotone convergence theorem lets us integrate term by term. For every index i, however, we have $\mu(P_{e_i}) = \nu(\{i\}) = 0$, whereas a measure $\mathrm{Tr}(\rho\,\cdot\,)$ satisfies $\sum_{i\in N} \mathrm{Tr}(\rho P_{e_i}) = \mathrm{Tr}(\rho) = 1$. Thus the function μ is a quantum measure that is not of the form $\mathrm{Tr}(\rho\,\cdot\,)$. Dvurečenskij and Solovay proved that this is the only obstruction [Dvu93, Sol09]: a Hilbert space of dimension at least 3 carries a quantum measure that is not of the form $\mathrm{Tr}(\rho\,\cdot\,)$ if and only if its dimension is at least some real-valued measurable cardinal. Gleason's Theorem therefore holds verbatim for every Hilbert space whose dimension is at least 3 and lies below the least real-valued measurable cardinal, in particular for every separable one. Theorem 1.4, on the other hand, holds in every dimension at least 3, with no set-theoretic hypothesis (see Remark 1.8 below).

If $|N|$ is at least a measurable cardinal, then the measure ν can be taken two-valued, that is, the indicator of a nonprincipal σ-complete ultrafilter $\mathcal{U}$ on N; the measure μ is then given by $\mu(P) = \lim_{i,\mathcal{U}} \langle e_i|P|e_i\rangle$. Even so, it is not two-valued, because the numbers $\langle e_i|P|e_i\rangle$ need not tend to 0 or 1 along $\mathcal{U}$. For instance, when the indices are grouped in pairs $\{a, b\}$ and P is the projection onto the closed span of the vectors $(e_a + e_b)/\sqrt{2}$, every index i satisfies $\langle e_i|P|e_i\rangle = 1/2$, whence $\mu(P) = 1/2$. By Ulam's theorem [Ula30], a real-valued measurable cardinal is either at most $2^{\aleph_0}$ or measurable. Solovay showed that adding κ random reals to a model with a measurable cardinal κ makes κ equal to $2^{\aleph_0}$ and real-valued measurable [Sol71]. The constructible universe, on the other hand, has no real-valued measurable cardinals at all. Hence, relative to the consistency of a measurable cardinal, Gleason's Theorem for Hilbert spaces of dimension $2^{\aleph_0}$ is independent of ZFC.

Remark 1.8 (Ultrafilters on bases, and the Main Theorem in every dimension). In this remark we drop separability, and we weaken the notion of a two-valued measure as far as possible: let $v\colon L(\mathcal{H}) \to \{0,1\}$ satisfy $v(I) = 1$ and $v(P \vee Q) = v(P) + v(Q)$ for orthogonal P, Q, with no countable additivity at all. Let $\{e_i\}_{i\in N}$ be an orthonormal basis of $\mathcal{H}$. The projections onto the closed spans of the sets $\{e_j : j \in J\}$, for $J \subseteq N$, form a Boolean algebra isomorphic to the power set of N; on this algebra, the function v is the indicator of an ultrafilter $\mathcal{U}_v$ on N. If v is countably additive and N is countable, then the ultrafilter $\mathcal{U}_v$ is σ-complete on a countable set, hence principal, and its generator is the unique winner of Lemma 1.2: the lemma *is* the principality of σ-complete ultrafilters on countable sets. A single basis, taken alone, allows more: finite additivity on its Boolean algebra permits any ultrafilter on N, and countable additivity permits any σ-complete one, which by Ulam's theorem is principal unless $|N|$ is at least a measurable cardinal.

Nevertheless, Theorem 1.4 survives in every dimension at least 3, even for such finitely additive v; the proof amplifies a finite witness. In finite dimension, finite additivity is all of additivity, and the theorem applies as stated. If $\dim \mathcal{H}$ is infinite, we write $\mathcal{H} = \mathbb{C}^3 \otimes K$ for a Hilbert space K of the same dimension. By Corollary 3.2 below, there are finitely many rays $u_1, \dots, u_m$ of $\mathbb{C}^3$ and orthonormal bases $B_1, \dots, B_n$ drawn from them such that no function from $\{u_1, \dots, u_m\}$ to $\{0,1\}$ takes the value 1 at exactly one ray of each basis. For each ray u, let W_u be the projection onto $u \otimes K$. Orthogonal rays $u \perp w$ yield orthogonal projections $W_u \perp W_w$, and for each basis $\{a, b, c\}$ among the B_k the three projections satisfy $W_a + W_b + W_c = I$. Finite additivity therefore forces $v(W_a) + v(W_b) + v(W_c) = 1$, and the map $u \mapsto v(W_u)$ is a function of exactly the kind that the configuration forbids. Thus, in dimension at least 3, the lattice $L(\mathcal{H})$ has no points in the strongest sense: it carries no two-valued measure, not even a finitely additive one, although the Boolean algebra attached to each orthonormal basis, a copy of the power set of the basis, carries ultrafilters in abundance.

2. Bell's 1964 proof of Theorem 1.4, based on Gleason's Theorem and elementary topology

In this section we present Bell's proof of Theorem 1.4, which combines Gleason's Theorem with the connectedness of the unit sphere. The Kochen–Specker proof, found independently at about the same time, is given in Section 4; we trace the history of both proofs in the Timeline (see Section 6 below).

Proof of Theorem 1.4, via Gleason's Theorem and connectedness. The idea is that Gleason's Theorem makes a valuation continuous on the unit sphere, whereas Lemma 1.2 forces it to take both of its values on every orthonormal basis. Suppose that v is a two-valued quantum measure on $L(\mathcal{H})$. By Gleason's Theorem, there is a density operator ρ with $v(P) = \mathrm{Tr}(\rho P)$ for every $P \in L(\mathcal{H})$. The function $g\colon S(\mathcal{H}) \to \{0,1\}$ defined by $g(\psi) = v(P_\psi)$ is then given by $g(\psi) = \langle\psi|\rho|\psi\rangle$ (see Observation 0.2); as its values are matrix elements of the fixed bounded operator ρ, it is continuous. Since $S(\mathcal{H})$ is connected, a continuous function from $S(\mathcal{H})$ to $\{0,1\}$ is constant, whence the function g takes only one value. On the other hand, every orthonormal basis of $\mathcal{H}$ has at least three vectors, and, by Lemma 1.2, the function g equals 1 at exactly one of them and 0 at the others. The function g thus takes both values, and this contradiction completes the proof. ☺

Remark 2.1. In the preceding proof, the hypothesis $\dim \mathcal{H} \geq 3$ is used only to invoke Gleason's Theorem; the rest of the argument needs only an orthonormal basis with at least two vectors.

Remark 2.2 (*Sfumato*). Bell's proof has a pictorial reading. Let us color each unit vector ψ by the value $\mu(P_\psi)$ of a quantum measure μ, from black (0) to white (1). A two-valued quantum measure would paint the sphere $S(\mathcal{H})$ in pure black and white, and, by Lemma 1.2, both colors would appear on every orthonormal basis; the picture would have hard contours. When $\dim \mathcal{H} \geq 3$, Gleason's Theorem allows only Leonardo's *sfumato*: for every quantum measure μ, the function $\psi \mapsto \mu(P_\psi)$ has the form $\psi \mapsto \langle\psi|\rho|\psi\rangle$, a continuous function on the path-connected space $S(\mathcal{H})$, and by the intermediate value theorem every shade of gray occurs along any path from a white vector to a black one. For the pure state $\rho = |u\rangle\langle u|$, the gradation away from u is the Born rule $\psi \mapsto |\langle u|\psi\rangle|^2$ itself. In dimension 2, where Gleason's Theorem fails, hard contours are possible (see Remark 1.6).

Historical note. Gleason proved his theorem in 1957 to answer a question of Mackey's about measures on closed subspaces [Mac57, Gle57], without discussing hidden variables in the paper. In 1963 Jauch pointed out to Bell that the theorem bears on the hidden-variables question. Bell turned this remark into the argument above, and he thanks Jauch in a footnote of his paper [Bel66]. Bell wrote the paper in 1964, but it appeared only in 1966. In it he also criticizes two earlier impossibility arguments, von Neumann's of 1932 and Jauch and Piron's of 1963. Each of them imposes a rule on the answers to questions that cannot be tested together. Nothing in physics justifies such a rule. We tell the story in more detail in the Timeline (see Section 6 below).

Provenance of the argument. The proof of Theorem 1.4 given above is, in substance, due to Bell; what we have presented is the now-standard topological restatement of his argument. Bell's own argument [Bel66] did not use the language of continuity or connectedness. Using an auxiliary third orthogonal direction, available only because $\dim \mathcal{H} \geq 3$, he showed directly that two rays with v-values 1 and 0 cannot be arbitrarily close. On the other hand, since both values occur on every basis (see Lemma 1.2) and the sphere is connected, there are oppositely valued rays arbitrarily close to each other, and the contradiction follows.

3. Obtaining a finite obstruction via logical compactness

Bell's proof of Theorem 1.4 in Section 2 shows that no two-valued quantum measure exists, but it argues over the whole uncountable sphere and never exhibits a finite culprit. In Corollary 3.2 below we show that a finite culprit was nevertheless *guaranteed* to exist the moment Bell's proof was known: finitely many orthonormal bases already admit no consistent choice of one winner per basis. The tool is compactness, which appears here in its logical and its topological forms at once.

Throughout this section, we assume that $\dim \mathcal{H} = d$ with $3 \leq d < \infty$.

Let $\mathcal{R} = S(\mathcal{H})/\!\sim$ be the set of rays, that is, of unit vectors modulo phase factors, the relation $\sim$ identifying u with $e^{i\theta}u$ for every $\theta \in \mathbb{R}$. Let $\mathcal{E}$ be the set of all orthonormal bases of $\mathcal{H}$, each basis being regarded as a set of exactly d rays. We attach to each ray $u \in \mathcal{R}$ a propositional symbol σ_u, and to each basis $B = \{u_1, \ldots, u_d\} \in \mathcal{E}$ the propositional formula

$$\varphi_B :- \big(\sigma_{u_1} \vee \cdots \vee \sigma_{u_d}\big) \wedge \bigwedge_{1 \leq i < j \leq d} \neg\big(\sigma_{u_i} \wedge \sigma_{u_j}\big) \qquad (\text{“exactly one } \sigma_{u_i} \text{ is true”}),$$

a formula in the d propositional symbols of B alone. The two-valued quantum measures on $L(\mathcal{H})$ correspond one-to-one to the truth assignments to $\{\sigma_u : u \in \mathcal{R}\}$ that satisfy every φ_B. In one direction, by Lemma 1.2, a two-valued quantum measure v yields such an assignment, namely $\sigma_u \mapsto v(P_u)$, where P_u denotes the projection onto the ray u. Conversely, let such an assignment be given. For $P \in L(\mathcal{H})$, let $v(P)$ be the number of true symbols σ_u with u in an orthonormal basis of the range of P. This number does not depend on the choice of basis.

Indeed, if we extend a basis of the range of P by a fixed basis of the range of $P^\perp$, the result is a basis of $\mathcal{H}$, which contains exactly one true ray; the count for the range of P is therefore 1 minus the count for the fixed basis of the range of $P^\perp$, whichever basis of the range of P we started from. The same extension shows that $v(P) \in \{0,1\}$, and the case $P = I$ gives $v(I) = 1$. Finally, if $P_1, \dots, P_n$ are pairwise orthogonal, the union of bases of their ranges is a basis of the range of $P_1 \vee \cdots \vee P_n$, whence $v(P_1 \vee \cdots \vee P_n) = v(P_1) + \cdots + v(P_n)$; in finite dimension this is all of additivity, since only finitely many members of an orthogonal family are nonzero and $v(0) = 0$. Thus the function v is a two-valued quantum measure, and by additivity the two constructions are inverse to each other.

The topological picture. A truth assignment to the symbols σ_u is a point of the Cantor cube $2^{\mathcal{R}} = \{0,1\}^{\mathcal{R}}$, which is compact by Tychonov's theorem. For each basis $B \in \mathcal{E}$, the set

$$C_B = \{x \in 2^{\mathcal{R}} : \ x \text{ satisfies } \varphi_B\}$$

depends only on the d coordinates indexed by the rays of B, so the set C_B is clopen. (In the language of Stone duality, the space $2^{\mathcal{R}}$ is the Stone space of the free Boolean algebra on the symbols σ_u, and the clopen set of the formula φ_B is C_B.) By the preceding paragraph, the two-valued quantum measures on $L(\mathcal{H})$ correspond to the points of $\bigcap_{B\in\mathcal{E}} C_B$, and Theorem 1.4 says that this intersection is empty. Since $2^{\mathcal{R}}$ is compact, the finite intersection property then produces a finite subfamily of $\{C_B : B \in \mathcal{E}\}$ with empty intersection. The compactness theorem of propositional logic, for the uncountable set of symbols $\{\sigma_u : u \in \mathcal{R}\}$, is exactly this statement about $2^{\mathcal{R}}$ (see, for example, [End01]). We record the argument in the next proposition and its corollary.

Proposition 3.1. *If every finite set $\mathcal{E}_0 \subseteq \mathcal{E}$ of bases admits* some *truth assignment to the finitely many rays occurring in $\mathcal{E}_0$ that satisfies φ_B for every $B \in \mathcal{E}_0$, then a two-valued quantum measure exists on all of $L(\mathcal{H})$.*

Proof. We apply the finite intersection property to the closed sets C_B. Let $\mathcal{E}_0 \subseteq \mathcal{E}$ be finite. By hypothesis, some assignment to the rays occurring in $\mathcal{E}_0$ satisfies every φ_B with $B \in \mathcal{E}_0$; extended arbitrarily to the remaining rays, it becomes a point of $\bigcap_{B\in\mathcal{E}_0} C_B$, and therefore this intersection is nonempty. Since the sets C_B are closed in the compact space $2^{\mathcal{R}}$, the finite intersection property provides a point $x \in \bigcap_{B\in\mathcal{E}} C_B$. The point x is a truth assignment that satisfies every φ_B, and by the correspondence above it defines a two-valued quantum measure on $L(\mathcal{H})$, as required. ☺

Corollary 3.2. *If $3 \le \dim \mathcal{H} < \infty$, then there are finitely many orthonormal bases of $\mathcal{H}$, made of finitely many rays, that admit no assignment of the values 0 and 1 to these rays taking the value 1 at exactly one ray of each basis.*

Proof. Bell's proof of Theorem 1.4 in Section 2, via Gleason's Theorem, shows that no two-valued quantum measure exists on $L(\mathcal{H})$. The conclusion of Proposition 3.1 therefore fails, and so does its hypothesis: some finite set $\mathcal{E}_0 \subseteq \mathcal{E}$ of bases admits no assignment to the rays occurring in $\mathcal{E}_0$ that satisfies φ_B for every $B \in \mathcal{E}_0$. Since each basis consists of d rays, only finitely many rays occur in $\mathcal{E}_0$, and the corollary is proved. ☺

Remark 3.3. By Corollary 3.2, a finite Kochen–Specker configuration exists; we derived its existence from Bell's argument in Section 2 without constructing one. The derivation is non-constructive in two ways. First, it uses the compactness of $2^{\mathcal{R}}$, which is not provable in ZF: the set $\mathcal{R}$ has the cardinality of the continuum, and the compactness of $2^{\mathcal{R}}$ produces a nonprincipal ultrafilter on ω, whose existence ZF does not prove. Over ZF, the compactness of every Cantor cube 2^X is equivalent to the Boolean prime ideal theorem, a weak form of

the Axiom of Choice. Second, the argument gives no bound on the size of the finite witness, nor any hint of where to find it.

The proof in the next section, in contrast, exhibits an explicit configuration of 18 vectors in dimension 4 and uses no choice at all. The search for small explicit configurations ran from 1967, when Kochen and Specker used 117 vectors in dimension 3, to 1996, when 18 vectors were found to suffice in dimension 4; in dimension 3 the smallest size is still unknown (see the **2024** entry of the Timeline in Section 6).

The appeal to choice is nevertheless harmless for the conclusion, though not for the proof: for each d, the statement of Corollary 3.2 asserts the existence of finitely many complex numbers that satisfy finitely many polynomial equations together with a finite combinatorial condition, and it is therefore a Σ^1_1 statement. Since ZFC proves the statement and ZF proves that the constructible universe L satisfies ZFC, the statement holds in L. By Mostowski's absoluteness theorem, every Σ^1_1 statement true in L is true in the universe, whence ZF alone proves Corollary 3.2. The compactness argument uses choice; the theorem it proves does not need it.

The topology also hides a trap. One might try to run the compactness argument over a countable dense set of rays, say the rays with rational coordinates, where the compactness of 2^ω, which is provable in ZF, would suffice. In dimension 4 this happens to work, because the 18 vectors of Section 4 have coordinates in $\{0, \pm 1\}$. A warning in dimension 3 comes from a different dense set: Meyer [Mey99] showed that the rays through the rational points of the sphere S^2 admit an assignment that takes the value 1 at exactly one ray of every orthonormal basis made of such rays. Consequently, in dimension 3 every finite witness made of real rays contains a ray with no rational unit representative. This does not require an irrational direction: a vector with rational coordinates may become irrational when normalized. A set of rays that is dense among the real rays need not carry any obstruction at all. Density is no substitute for continuity: Bell's argument uses continuity on the whole sphere, and an assignment on a dense set of rays need not extend continuously.

The same picture, Boolean algebra by Boolean algebra. In this paragraph only, we drop the assumption that $\mathcal{H}$ is finite-dimensional. The lattice $L(\mathcal{H})$ is glued together from Boolean algebras: each family of commuting projections generates a Boolean subalgebra, and for an orthonormal basis indexed by a set I, the projections onto the closed spans of its subsets form a Boolean subalgebra isomorphic to the power set of I, whose Stone space is βI. A finitely additive two-valued measure on $L(\mathcal{H})$ (in finite dimension, simply a two-valued quantum measure) amounts to a family of ultrafilters, one on each Boolean subalgebra, that agree on common elements. The product of the Stone spaces of all Boolean subalgebras is compact by Tychonov's theorem. In this product, each agreement condition involves only two coordinates and defines a clopen set, and the two-valued measures correspond to the points of the intersection of these clopen sets. For a single Boolean algebra, or for a chain of Boolean algebras, such an intersection is never empty. By Theorem 1.4, the intersection is empty for $L(\mathcal{H})$ in finite dimension, and the finite intersection property then shows that finitely many agreement conditions already contradict one another: this is Corollary 3.2 once more. In infinite dimension, the Stone space βI of each basis has $2^{2^{|I|}}$ points, and still no choice of one point in each Stone space is consistent (see Remark 1.8).

4. A finite proof of Theorem 1.4, after Kochen and Specker

In this section we give a proof of Theorem 1.4 that uses no measure theory and no continuity. A finite configuration, which can be checked by hand, replaces Bell's use of Gleason's

Theorem and connectedness, and supplies what both arguments need: a way to force the separate "one winner per basis" choices of Lemma 1.2 to interact. We use the 18-vector configuration of Cabello, Estebaranz, and García-Alcaine [CEGA96], the end point of a search for small configurations that began with Kochen and Specker's original 117 vectors (see the historical note at the end of this section). One feature should be said at once: the configuration lives in $\mathbb{R}^4$. Kochen and Specker worked in $\mathbb{R}^3$, the smallest dimension in which the theorem holds (see Remark 1.6), and paid for it with 117 vectors; Peres later needed only 33 there (see the **1991** entry). Below we show that in $\mathbb{R}^4$ eighteen vectors and a parity count suffice, and that an embedding carries the argument to every dimension at least 4, finite or infinite, real or complex. For dimension 3, these notes rely on Bell's proof.

Lemma 4.1 (Cabello–Estebaranz–García-Alcaine [CEGA96]). *After normalization, each of the following* 9 *sets of* 4 *vectors is an orthonormal basis of* $\mathbb{R}^4$*. Exactly* 18 *vectors occur in these bases, no two of them proportional, and each occurs in exactly* 2 *of the* 9 *bases.*

$$
\begin{aligned}
B_1 &= \{(0,0,0,1),\ (0,0,1,0),\ (1,1,0,0),\ (1,-1,0,0)\}\\
B_2 &= \{(0,0,0,1),\ (0,1,0,0),\ (1,0,1,0),\ (1,0,-1,0)\}\\
B_3 &= \{(1,-1,1,-1),\ (1,-1,-1,1),\ (1,1,0,0),\ (0,0,1,1)\}\\
B_4 &= \{(1,-1,1,-1),\ (1,1,1,1),\ (1,0,-1,0),\ (0,1,0,-1)\}\\
B_5 &= \{(0,0,1,0),\ (0,1,0,0),\ (1,0,0,1),\ (1,0,0,-1)\}\\
B_6 &= \{(1,-1,-1,1),\ (1,1,1,1),\ (1,0,0,-1),\ (0,1,-1,0)\}\\
B_7 &= \{(1,1,-1,1),\ (1,1,1,-1),\ (1,-1,0,0),\ (0,0,1,1)\}\\
B_8 &= \{(1,1,-1,1),\ (-1,1,1,1),\ (1,0,1,0),\ (0,1,0,-1)\}\\
B_9 &= \{(1,1,1,-1),\ (-1,1,1,1),\ (1,0,0,1),\ (0,1,-1,0)\}.
\end{aligned}
$$

Verification. Orthogonality within each B_k is a check of $\binom{4}{2} = 6$ dot products; for instance, for B_3 we have $(1,-1,1,-1)\cdot(1,-1,-1,1) = 1+1-1-1 = 0$, and the other five products in B_3 vanish in the same way. Table 1 lists, for each of the 18 vectors, two bases that contain it, and an inspection of the nine bases confirms every entry; a glance at the table also shows that no two of the 18 vectors are proportional. The table thus records $18 \times 2 = 36$ incidences, exactly as many as the $9 \times 4 = 36$ places in the bases. Hence no vector occurs in a third basis and no other vector occurs at all, which proves the lemma. ☺

vector	bases	vector	bases	vector	bases
$(0,0,0,1)$	B_1, B_2	$(1,0,-1,0)$	B_2, B_4	$(1,0,0,1)$	B_5, B_9
$(0,0,1,0)$	B_1, B_5	$(1,-1,1,-1)$	B_3, B_4	$(1,0,0,-1)$	B_5, B_6
$(1,1,0,0)$	B_1, B_3	$(1,-1,-1,1)$	B_3, B_6	$(0,1,-1,0)$	B_6, B_9
$(1,-1,0,0)$	B_1, B_7	$(0,0,1,1)$	B_3, B_7	$(1,1,-1,1)$	B_7, B_8
$(0,1,0,0)$	B_2, B_5	$(1,1,1,1)$	B_4, B_6	$(1,1,1,-1)$	B_7, B_9
$(1,0,1,0)$	B_2, B_8	$(0,1,0,-1)$	B_4, B_8	$(-1,1,1,1)$	B_8, B_9

TABLE 1. The 18 vectors of Lemma 4.1, each with the two bases that contain the vector.

Table 1 can be read as a graph: the vertices are the 9 bases, and each vector is an edge joining the two bases that contain the vector (see Figure 1). Each vertex has degree 4, because each basis has 4 vectors. A two-valued quantum measure would select, at every

vertex, exactly one of the 4 edges, and an edge selected at one endpoint is selected at the other endpoint too, since the edge is a single ray. The selected edges would therefore form a perfect matching of a graph with 9 vertices, which is impossible: a perfect matching pairs off the vertices, and the number 9 is odd. In the proof below, we write this parity argument as a double count.

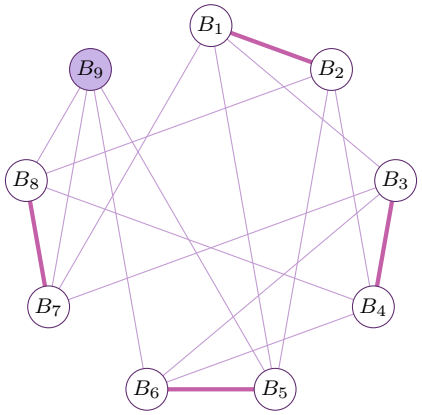


FIGURE 1. The 18 vectors as the edges of a 4-regular graph on the 9 bases, with a largest matching (pink) and the vertex that it leaves uncovered (lavender).

Proof of Theorem 1.4 for $\mathcal{H} = \mathbb{R}^4$. The idea is to count the value 1 twice, once basis by basis and once vector by vector, and to compare the parities. Suppose that v is a two-valued quantum measure on $L(\mathbb{R}^4)$. We restrict attention to the 18 vectors of Lemma 4.1, and for each such vector u we write $v(u)$ for the value of v at the projection onto the line $\mathbb{R}u$. Since each B_k is, after normalization, an orthonormal basis, additivity gives

$$\sum_{u \in B_k} v(u) = v(I) = 1 \qquad \text{for each } k = 1, \ldots, 9.$$

Summing over all 9 bases, we obtain

$$\sum_{k=1}^{9} \sum_{u \in B_k} v(u) = 9.$$

We now reorganize the left side by vector instead of by basis. Each of the 18 vectors occurs in exactly 2 of the bases B_k, and therefore contributes twice its value to the double sum:

$$\sum_{k=1}^{9} \sum_{u \in B_k} v(u) = \sum_{u} 2v(u) = 2 \sum_{u} v(u),$$

where u runs over the 18 vectors. The double sum is thus an even integer, which contradicts its value, the odd integer 9. Hence no two-valued quantum measure exists on $L(\mathbb{R}^4)$. ☺

Other dimensions. The same argument proves Theorem 1.4 for every real or complex Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} \geq 4$. Indeed, suppose that v is a two-valued quantum measure on $L(\mathcal{H})$. By Lemma 1.2, some unit vector ψ has $v(P_\psi) = 1$. We choose a 4-dimensional subspace K of $\mathcal{H}$ containing ψ, and we let P_K be the projection onto K. Since P_K is the sum of the orthogonal projections P_ψ and $P_K - P_\psi$, additivity yields $v(P_K) \geq v(P_\psi) = 1$, whence

$v(P_K) = 1$. Next we fix an orthonormal basis $f_1, \ldots, f_4$ of K. The map $(x_1, \ldots, x_4) \mapsto \sum_i x_i f_i$ preserves inner products of real vectors and sends nonproportional vectors to nonproportional vectors; it therefore carries the 9 bases of Lemma 4.1, after normalization, to 9 orthonormal bases of K with the same incidence pattern. For each of these bases, the 4 projections add up to P_K, and their values under v add up to $v(P_K) = 1$; the double count above now produces the same contradiction. Dimension 3, as announced at the beginning of the section, requires a configuration in $\mathbb{R}^3$ or Bell's argument.

Historical note. Kochen and Specker's 1967 proof [KS67] uses 117 vectors in $\mathbb{R}^3$ (see Figure 2). Unlike the proof above, it therefore covers dimension 3.

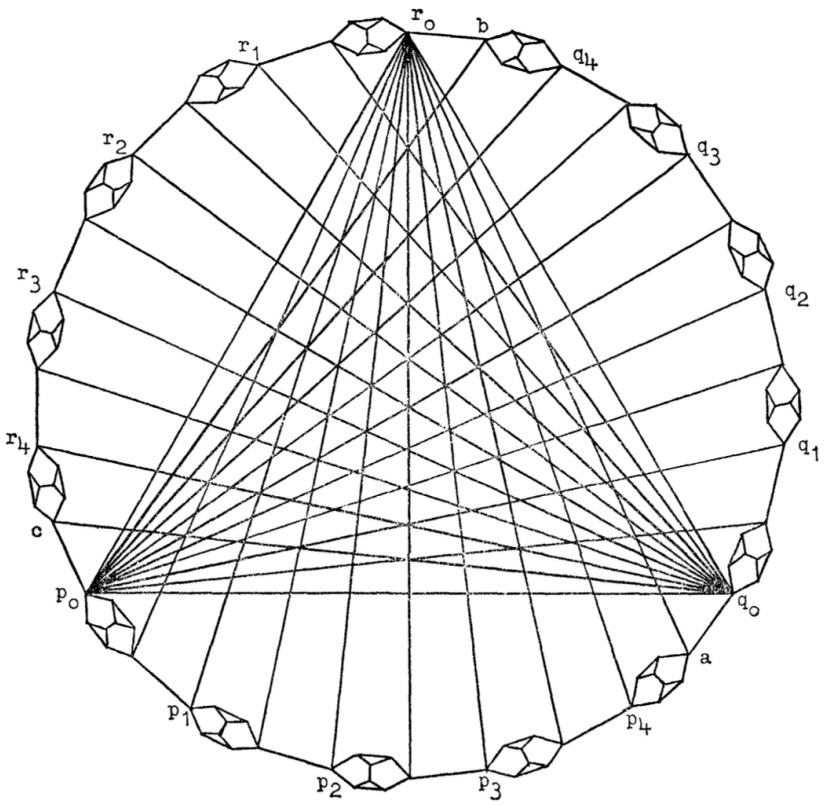


FIGURE 2. The graph Γ_2 of Kochen and Specker's 1967 paper [KS67, p. 69], whose 117 vertices are the vectors of their original proof, joined by an edge when orthogonal; the vertices are the points on the rim once p_0, q_0 and r_0 are identified with a, b and c, and the fifteen small gadgets on the rim are copies of their ten-vertex graph Γ_1.

Both authors were logicians. Ernst Specker (1920–2011), of ETH Zürich, had shown in 1953 that the Axiom of Choice fails in Quine's New Foundations. In 1960 he had announced the theorem for three-dimensional space, without proof [Spe60]. Simon Kochen (b. 1934) had written his Princeton thesis on ultraproducts under Alonzo Church. At the time, he was working with James Ax on the papers that became the Ax–Kochen theorem. Their formulation of the problem is accordingly logical. The questions that can be tested together form a Boolean algebra. These Boolean algebras, glued along their common parts, form a *partial Boolean algebra*. A valuation is a homomorphism from this partial Boolean algebra

to the two-element Boolean algebra, and the theorem asserts that none exists when the dimension is at least 3. The 18-vector configuration used above was found in 1996 [CEGA96], after reductions by Peres and by Kernaghan (see the **1991**, **1994**, and **1996** entries of the Timeline).

5. Quantum measures have no counterpart of Dirac measures and do not form a lattice

We have now proved the Main Theorem by continuity and, in dimension at least 4, also by counting. In this section we show that, in the language of ultrafilters, the theorem says that the quantum lattice has no points. This reading was already in view at the end of Section 3, where a two-valued measure appeared as a family of ultrafilters, one on each Boolean subalgebra of $L(\mathcal{H})$, that agree on common elements; now it becomes the starting point.

Classical probability rests on two facts so familiar that they are easy to overlook. The first is that a single outcome decides every event: the events that contain a point ω of the sample space form an ultrafilter on the event algebra, and the Dirac measure δ_ω is nothing but the indicator function of that ultrafilter. The second is that measures can always be combined, even when they cannot be compared: any two finite measures on the same σ-algebra have a greatest lower bound and a least upper bound, so the measures form a lattice (see Proposition 5.10 below). We show that, once $\dim \mathcal{H} \geq 3$, quantum measures have neither property. The first failure is the Bell–Kochen–Specker theorem in another guise. The second is of a different nature altogether: it occurs already in dimension 2, where two-valued measures exist. Throughout the section, the symbol $\mathcal{H}$ denotes a Hilbert space with $\dim \mathcal{H} \geq 2$; apart from Remark 5.8, which rests on Corollary 3.2, none of the arguments uses Theorem 1.4.

The classical properties (C1) and (C2). Let Σ be a σ-algebra of subsets of a sample space Ω. The two properties read as follows.

(C1) Each outcome decides every event, and its verdicts respect the Boolean operations. In terms of the Dirac measures, for every $\omega \in \Omega$ and all $A, B \in \Sigma$, we have

$$\delta_\omega(A \cap B) = \min\big(\delta_\omega(A), \delta_\omega(B)\big), \qquad \delta_\omega(\Omega \setminus A) = 1 - \delta_\omega(A). \tag{4}$$

(Each side of each equation equals 1 if ω belongs to the event on the left-hand side, and 0 otherwise.)

(C2) Let $\mathcal{M}(\Sigma)$ be the set of finite positive measures on Σ, ordered by declaring $\mu \leq \nu$ if $\mu(A) \leq \nu(A)$ for every $A \in \Sigma$. Then the ordered set $(\mathcal{M}(\Sigma), \leq)$ is a lattice (see Proposition 5.10 below).

The quantum questions (Q1) and (Q2). To test (C1) and (C2) in quantum mechanics, we replace Σ by the lattice $L(\mathcal{H})$ and the classical measures by the quantum measures of Definition 1.1; for a density operator ρ we write $\mu_\rho(P) = \operatorname{Tr}(\rho P)$. Each classical property then becomes a question.

(Q1) For a quantum measure μ on $L(\mathcal{H})$, the natural counterpart of the first equation in (4) is

$$\mu(P \wedge Q) = \min\big(\mu(P), \mu(Q)\big). \tag{5}$$

The question is whether there is a quantum measure μ on $L(\mathcal{H})$ that satisfies (5) whenever $P, Q \in L(\mathcal{H})$ commute.

The second equation in (4) needs no counterpart, since additivity already gives $\mu(P^\perp) = 1 - \mu(P)$. The restriction to commuting pairs is the fair one: commuting projections lie in a common Boolean subalgebra of $L(\mathcal{H})$, and Boolean algebras are

where the classical equations (4) live. (If P and Q commute, the projections PQ, $P(I-Q)$, $(I-P)Q$ and $(I-P)(I-Q)$ are pairwise orthogonal with sum I; the joins of subsets of these four projections form a Boolean subalgebra of $L(\mathcal{H})$ containing $P = PQ + P(I-Q)$ and $Q = PQ + (I-P)Q$.) Without the restriction the question would be too easy, for no quantum measure satisfies (5) for all pairs $P, Q \in L(\mathcal{H})$ (see Proposition 5.2 below).

(Q2) For the counterpart of (C2), let $\mathcal{C}(\mathcal{H})$ be the set of functions $P \mapsto \mathrm{Tr}(TP)$ on $L(\mathcal{H})$, where T ranges over the positive trace-class operators on $\mathcal{H}$. When $\dim \mathcal{H} \geq 3$, Gleason's Theorem identifies $\mathcal{C}(\mathcal{H})$ with the set of nonnegative multiples of quantum measures. As in (C2), we admit every positive trace-class operator T, not only the density operators, because a greatest lower bound of two probability measures is in general not a probability measure: the greatest lower bound of two distinct Dirac measures is 0. We order $\mathcal{C}(\mathcal{H})$ pointwise:

$$\mu \leq \nu \quad \text{if } \mu(P) \leq \nu(P) \text{ for every } P \in L(\mathcal{H}).$$

The question is whether $(\mathcal{C}(\mathcal{H}), \leq)$ is a lattice.

The main result of this section.

- When $\dim \mathcal{H} \geq 3$, the answer to (Q1) is negative (see Theorem 5.6 below); when $\dim \mathcal{H} = 2$, the answer is positive (see Remark 1.6).
- When $\dim \mathcal{H} \geq 2$, the answer to (Q2) is negative (see Proposition 5.11 below).

Thus, when $\dim \mathcal{H} \geq 3$, neither (C1) nor (C2) has a quantum counterpart.

Interpretation of the answers. As announced, the negative answer to (Q1) is the Bell–Kochen–Specker theorem in disguise: in Lemma 5.1 below, we show that a quantum measure satisfies (5) for all commuting P, Q exactly when it is two-valued, that is, exactly when it behaves like a Dirac measure. For $\dim \mathcal{H} \geq 3$, the negative answer to (Q1) therefore says that there is no two-valued quantum measure on $L(\mathcal{H})$, which is Theorem 1.4, or, in the language of Section 1, that there is no non-contextual hidden-variable assignment. The measures that physics actually uses fare even worse: in every dimension, no measure μ_ρ given by a density operator satisfies (5) for all commuting pairs (see Corollary 5.7 below).

The negative answer to (Q2) says that quantum measures are not ordered like classical ones. Recall that an *order isomorphism* between partially ordered sets $(X, \leq)$ and $(Y, \leq)$ is a bijection $\Phi : X \to Y$ such that, for all $x, x' \in X$, we have $x \leq x'$ if and only if $\Phi(x) \leq \Phi(x')$. Whatever the σ-algebra Σ, there is no order isomorphism between $(\mathcal{C}(\mathcal{H}), \leq)$ and $(\mathcal{M}(\Sigma), \leq)$ (see Corollary 5.12 below). The corollary holds already in dimension 2, where two-valued quantum measures exist (see Remark 1.6); the failure of (C2) therefore owes nothing to the Bell–Kochen–Specker theorem.

Ultrafilters, principal and otherwise. The answer to (Q1) is best read in the language of ultrafilters. On a Boolean algebra $\mathcal{B}$, a finitely additive two-valued measure v is the indicator function of the ultrafilter $v^{-1}(1)$, and every ultrafilter arises in this way, so the two-valued measures on $\mathcal{B}$ are the points of the Stone space of $\mathcal{B}$. On the power set of a set Ω, the Dirac measures are exactly the indicators of the principal ultrafilters. (Countable additivity is a strong demand: a countably additive two-valued measure on the power set of Ω that vanishes on singletons is the indicator function of a nonprincipal σ-complete ultrafilter, and such ultrafilters exist only if $|\Omega|$ is at least a measurable cardinal [Ula30]; compare Remark 1.7.)

In $L(\mathcal{H})$, a two-valued quantum measure v defines the set $\mathcal{F}_v = \{P \in L(\mathcal{H}) : v(P) = 1\}$. This set is upward closed, contains exactly one of P and $P^\perp$ for each P, and, by Lemma 5.1 below, is closed under meets of commuting pairs. For every Boolean subalgebra $\mathcal{B}$, the intersection $\mathcal{F}_v \cap \mathcal{B}$ is an ultrafilter on $\mathcal{B}$, and these ultrafilters agree on common projections. Thus (Q1) asks for a compatible family of Boolean ultrafilters whose induced two-valued assignment is countably additive on orthogonal families. When $\dim \mathcal{H} \geq 3$, the Main Theorem rules out such families: in this Boolean-context sense, the quantum lattice has no points. Its atoms do generate principal filters,

$$\mathcal{F}_\psi = \{P \in L(\mathcal{H}) : P \geq P_\psi\} = \{P \in L(\mathcal{H}) : \psi \in \operatorname{ran} P\} \qquad (\psi \in S(\mathcal{H})),$$

which are even closed under arbitrary meets. The filter $\mathcal{F}_\psi$ is a maximal proper lattice filter: adjoining any projection Q with $\psi \notin \operatorname{ran} Q$ would make $0 = Q \wedge P_\psi$ a member of $\mathcal{F}_\psi$. It nevertheless contains P or $P^\perp$ only when ψ is an eigenvector of P, so it does not supply an ultrafilter in every Boolean context. This failure to decide every projection is the superposition principle in lattice-theoretic dress. What survives of $\mathcal{F}_\psi$ is the pure state μ_ψ, given by $\mu_\psi(P) = \langle\psi|P|\psi\rangle = \|P\psi\|^2$. The pure state takes the value 1 at P exactly when $P\psi = \psi$, that is, exactly when $P \in \mathcal{F}_\psi$; on the questions that $\mathcal{F}_\psi$ leaves undecided, its values lie strictly between 0 and 1.

The comparison also has a topological side. The Stone space of a Boolean algebra is compact and zero-dimensional, and its clopen sets are the events. The natural candidate for a space of quantum points, the space of rays, is connected (see Proposition 5.3 below), and so it has no nontrivial clopen sets at all. In Section 5.2 we turn this contrast into a proof.

The plan of the section. We treat (Q1) first and (Q2) second, and in each case the classical property comes right before its quantum failure. Property (C1) needs no proof beyond the parenthetical remark after (4). In Section 5.1 we prove Lemma 5.1, which turns (Q1) into a question about two-valued measures, and we show that the restriction to commuting pairs cannot be dropped. Section 5.2 contains the answer to (Q1), obtained by playing sliding against jumping: a unit vector can be moved continuously around the unit sphere, Gleason's Theorem forces every quantum measure to follow it continuously, and a two-valued measure can only jump between 0 and 1. There, in Remark 5.8, we also use the compactness argument of Section 3 to show that finitely many projections in a single 3-dimensional subspace already force the negative answer to (Q1). For (Q2), we first prove (C2) by the Hahn decomposition (see Section 5.3); then, in Section 5.4, an explicit pair of commuting 2×2 matrices shows where the Hahn decomposition argument breaks down, and (Q2) receives its negative answer.

5.1. **Quantum measures that preserve commuting meets.** We begin with (Q1) and show first that its condition is far more restrictive than it looks: a quantum measure that satisfies it takes only the values 0 and 1, just as a Dirac measure does. The answer to (Q1) is therefore positive exactly when two-valued quantum measures exist. We say that a quantum measure μ on $L(\mathcal{H})$ *preserves commuting meets* if μ satisfies (5) whenever $P, Q \in L(\mathcal{H})$ commute.

Lemma 5.1. *Let μ be a quantum measure on $L(\mathcal{H})$. Then the measure μ preserves commuting meets if and only if it is two-valued.*

Proof. The commuting pair $P, P^\perp$ rules out every value strictly between 0 and 1, and, for a two-valued measure, only one case of (5) needs an argument. Additivity yields $\mu(0) = 0$ and $\mu(P^\perp) = 1 - \mu(P)$ for every P. Suppose first that μ preserves commuting meets. For every P, the projections P and $P^\perp$ commute and have meet $P \wedge P^\perp = 0$. Applying (5) to the pair

$P, P^\perp$ gives

$$\min\big(\mu(P), 1-\mu(P)\big) = \min\big(\mu(P), \mu(P^\perp)\big) = \mu(P \wedge P^\perp) = \mu(0) = 0.$$

Since $\min(r, 1-r) > 0$ for every $r \in (0,1)$, this forces $\mu(P) \in \{0,1\}$.

Conversely, suppose that μ is two-valued, and let $P, Q \in L(\mathcal{H})$ be commuting projections. We first recall that $PQ = P \wedge Q$. Since P and Q commute, the operator PQ is self-adjoint and idempotent, hence a projection. Its range lies in the ranges of both P and Q (because $PQ = QP$), and it fixes every vector lying in both ranges; its range is therefore the intersection of the two ranges, and $PQ = P \wedge Q$.

Next, we show that $\mu(P \wedge Q) \leq \min\big(\mu(P), \mu(Q)\big)$. The projection $P = PQ + P(I-Q)$ is a sum of two orthogonal projections, so additivity gives $\mu(P) = \mu(P \wedge Q) + \mu\big(P(I-Q)\big) \geq \mu(P \wedge Q)$. By symmetry, we also have $\mu(Q) \geq \mu(P \wedge Q)$.

Since μ takes only the values 0 and 1, it remains to be shown that $\mu(P \wedge Q) = 1$ whenever $\mu(P) = \mu(Q) = 1$. The projections P and $(I-P)Q$ are orthogonal, so their sum is a projection, and, by additivity, $\mu(P) + \mu\big((I-P)Q\big) \leq 1$. The decomposition $Q = PQ + (I-P)Q$ into two orthogonal projections shows, in the same way, that $\mu\big((I-P)Q\big) = \mu(Q) - \mu(P \wedge Q)$. The two facts combine into the inequality

$$\mu(P) + \mu(Q) - \mu(P \wedge Q) \leq 1,$$

which, when $\mu(P) = \mu(Q) = 1$, forces $\mu(P \wedge Q) \geq 1$, whence $\mu(P \wedge Q) = 1$. This proves that μ preserves commuting meets. ☺

Since the first equation in (4) holds for every pair of events, one might hope to drop the commutativity assumption in (Q1). Without this assumption, however, (Q1) has an immediate negative answer.

Proposition 5.2. *No quantum measure μ on $L(\mathcal{H})$ satisfies $\mu(P \wedge Q) = \min\big(\mu(P), \mu(Q)\big)$ for all $P, Q \in L(\mathcal{H})$.*

Proof. The idea is to find two distinct rays of measure 1: their meet is 0, whereas the minimum of their measures is 1. Suppose that μ is such a quantum measure. In particular, the measure μ preserves commuting meets, so it is two-valued by Lemma 5.1, and, by Lemma 1.2, some unit vector ψ satisfies $\mu(P_\psi) = 1$. Let K be a 2-dimensional subspace containing ψ, and let P_K be the projection onto K. Since $P_K \geq P_\psi$, monotonicity gives $\mu(P_K) = 1$. For every orthonormal basis $\{a, b\}$ of K, the projections P_a and P_b add up to P_K; hence exactly one of the numbers $\mu(P_a)$ and $\mu(P_b)$ equals 1. We fix an orthonormal basis $\{a, b\}$ of K and let $c = (a+b)/\sqrt{2}$ and $d = (a-b)/\sqrt{2}$; the vectors c and d form an orthonormal basis of K with no ray in common with $\{a, b\}$. Interchanging a and b, and c and d, if necessary, we may assume that $\mu(P_a) = \mu(P_c) = 1$. The rays a and c are therefore distinct, and two distinct rays meet in 0. Consequently, the meet $P_a \wedge P_c$ has measure $\mu(0) = 0$, whereas $\min\big(\mu(P_a), \mu(P_c)\big) = 1$, contrary to the hypothesis on μ. ☺

5.2. **Topology and the negative answer to (Q1).** We can now answer (Q1) when $\dim \mathcal{H} \geq 3$. By Lemma 5.1, it suffices to show that no two-valued quantum measure exists, and for this we play continuity against jumps: Gleason's Theorem makes every quantum measure continuous on rays, whereas a two-valued quantum measure, which is not constant on rays, cannot be continuous on the connected unit sphere (see Proposition 5.5 below).

We equip $L(\mathcal{H})$ with the operator-norm topology, in which the distance between P and Q is $\|P - Q\|$. As in Section 0, for $\psi \in S(\mathcal{H})$ we write $P_\psi = |\psi\rangle\langle\psi|$ for the projection onto $\mathbb{C}\psi$; the corresponding pure state $\mu_\psi = \mu_{P_\psi}$ satisfies $\mu_\psi(P) = \langle\psi|P|\psi\rangle$ (see Observation 0.2). We

also need the *trace norm* $\|T\|_1 = \mathrm{Tr}\big((T^*T)^{1/2}\big)$ of a trace-class operator T. For self-adjoint T, the number $\|T\|_1$ is the sum of the absolute values of the eigenvalues of T, counted with multiplicity, and every bounded operator A satisfies $|\mathrm{Tr}(TA)| \le \|T\|_1\|A\|$ (see Reed and Simon [RS80, Section VI.6]). We need three facts, which we collect in the next proposition: rank-one projections are close in norm exactly when they come from unit vectors that are nearly parallel up to a phase factor, the rank-one projections form a connected space, and the number $\mu_\rho(P)$ depends continuously on both the density operator ρ and the projection P.

Proposition 5.3.

(1) *For $\psi, \xi \in S(\mathcal{H})$ we have $\|P_\psi - P_\xi\| = (1 - |\langle\psi|\xi\rangle|^2)^{1/2}$ and $\|P_\psi - P_\xi\|_1 = 2(1 - |\langle\psi|\xi\rangle|^2)^{1/2}$.*

(2) *The set $\{P_\psi : \psi \in S(\mathcal{H})\}$ of rank-one projections is a connected subset of $L(\mathcal{H})$.*

(3) *For density operators ρ, ρ' and $P, Q \in L(\mathcal{H})$ we have $|\mu_\rho(P) - \mu_{\rho'}(Q)| \le \|\rho - \rho'\|_1 + \|P - Q\|$.*

Proof. (1) The operator $X = P_\psi - P_\xi$ is self-adjoint, has trace 0, and has range in $\mathrm{span}(\psi, \xi)$. Its nonzero eigenvalues, at most two in number and with sum 0, are therefore $\pm t$ for some $t \ge 0$, and Observation 0.2 yields $2t^2 = \mathrm{Tr}\, X^2 = \mathrm{Tr}\, P_\psi + \mathrm{Tr}\, P_\xi - 2\,\mathrm{Tr}(P_\psi P_\xi) = 2 - 2|\langle\psi|\xi\rangle|^2$. This gives $t = (1 - |\langle\psi|\xi\rangle|^2)^{1/2}$, whence $\|X\| = t$ and $\|X\|_1 = 2t$.

(2) Every complex number z with $|z| \le 1$ satisfies $1 - |z|^2 \le 2(1 - |z|) \le 2(1 - \mathrm{Re}\, z)$. Taking $z = \langle\psi|\xi\rangle$ and applying the Cauchy–Schwarz inequality, we obtain $1 - |\langle\psi|\xi\rangle|^2 \le 2(1 - \mathrm{Re}\langle\psi|\xi\rangle) = 2\,\mathrm{Re}\langle\psi|\psi - \xi\rangle \le 2\|\psi - \xi\|$, and therefore, by (1), $\|P_\psi - P_\xi\| \le (2\|\psi - \xi\|)^{1/2}$. The map $\psi \mapsto P_\psi$ is thus continuous, and the set of rank-one projections is the continuous image of the connected space $S(\mathcal{H})$.

(3) By the triangle inequality and the trace-norm inequality recalled above, we have $|\mathrm{Tr}(\rho P) - \mathrm{Tr}(\rho' Q)| \le |\mathrm{Tr}((\rho - \rho')P)| + |\mathrm{Tr}(\rho'(P - Q))| \le \|\rho - \rho'\|_1\|P\| + \|\rho'\|_1\|P - Q\|$, which is at most $\|\rho - \rho'\|_1 + \|P - Q\|$, since $\|P\| \le 1$ and $\|\rho'\|_1 = \mathrm{Tr}\,\rho' = 1$. ☺

A first contrast with the classical case. Before proving the main theorem of this subsection, we compare the pure states μ_ψ with their classical counterparts, the Dirac measures. Let Ω be a finite set with the discrete topology, and let Σ be the set of all subsets of Ω. For every $A \in \Sigma$, the function $\omega \mapsto \delta_\omega(A)$ takes only the values 0 and 1, and the set $\{\omega : \delta_\omega(A) = 1\} = A$ is both open and closed in Ω. In the next proposition we show that both properties fail for the measures μ_ψ on $L(\mathcal{H})$.

Proposition 5.4. *Let $P \in L(\mathcal{H})$ be a projection with $P \ne 0$ and $P \ne I$. Then the function $\psi \mapsto \mu_\psi(P)$ on $S(\mathcal{H})$ is continuous and takes every value in $[0, 1]$. The set $\{\psi \in S(\mathcal{H}) : \mu_\psi(P) = 1\}$ is closed, nonempty, and not open.*

Proof. Let $f(\psi) = \mu_\psi(P) = \langle\psi|P|\psi\rangle$ for $\psi \in S(\mathcal{H})$. For $\xi \in S(\mathcal{H})$, part (3) of Proposition 5.3, applied to the density operators P_ψ and P_ξ, and part (1) give $|f(\psi) - f(\xi)| \le \|P_\psi - P_\xi\|_1 = 2\|P_\psi - P_\xi\|$; since the map $\psi \mapsto P_\psi$ is continuous (see the proof of part (2)), the function f is continuous. It equals 1 at the unit vectors in the range of P and 0 at those in the range of $P^\perp$. By the intermediate value theorem along a path in $S(\mathcal{H})$ joining two such vectors, exactly as in Remark 2.2, the function f takes every value in $[0, 1]$. The set $f^{-1}(1)$ is closed by continuity, nonempty because $P \ne 0$, and proper because $P \ne I$. If it were also open, it would be a nontrivial clopen subset of the connected sphere $S(\mathcal{H})$, which is impossible. ☺

In Proposition 5.4 the projection P is fixed and the measure μ_ψ varies. We now reverse the roles: we fix a measure and let the projection vary. The quantum measures that, like Dirac measures, take only the values 0 and 1 are the two-valued ones, and none of them varies continuously with the ray:

Proposition 5.5. *If v is a two-valued quantum measure on $L(\mathcal{H})$, then the function $\psi \mapsto v(P_\psi)$ is not continuous on $S(\mathcal{H})$.*

Proof. The argument is the second half of Bell's proof of Theorem 1.4 in Section 2, which uses Gleason's Theorem only to obtain continuity. Suppose that the function $g(\psi) = v(P_\psi)$ is continuous on $S(\mathcal{H})$. Since g takes only the values 0 and 1 and the sphere $S(\mathcal{H})$ is connected, the function g is constant. By Lemma 1.2, however, both values occur on every orthonormal basis of $\mathcal{H}$, since such a basis has at least two elements. This contradiction uses neither Gleason's Theorem nor the hypothesis $\dim \mathcal{H} \geq 3$. ☺

Gleason's Theorem supplies the continuity that Proposition 5.5 forbids, and (Q1) is answered.

Theorem 5.6. *A quantum measure on $L(\mathcal{H})$ that preserves commuting meets exists if* $\dim \mathcal{H} = 2$*, and does not exist if* $\dim \mathcal{H} \geq 3$.

Proof. Suppose that $\dim \mathcal{H} \geq 3$, and let μ be a quantum measure that preserves commuting meets. By Lemma 5.1, the measure μ is two-valued. Gleason's Theorem, on the other hand, gives $\mu = \mu_\rho$ for a density operator ρ, and the function $\psi \mapsto \mu(P_\psi) = \langle\psi|\rho|\psi\rangle$ is then continuous on $S(\mathcal{H})$ (see Proposition 5.3). This contradicts Proposition 5.5.

If $\dim \mathcal{H} = 2$, then, as in Remark 1.6, we assign the value 1 to one ray in each pair of orthogonal rays and the value 0 to the other, and we set $v(0) = 0$ and $v(I) = 1$. The resulting function v is a two-valued quantum measure, because the only orthogonal families with more than one nonzero member are the pairs of orthogonal rays, whose values add up to $1 = v(I)$; by Lemma 5.1, it preserves commuting meets. ☺

The second half of the proof of Theorem 5.6 shows what changes in dimension 2. There, Gleason's Theorem fails (see Remark 1.6), and the answer to (Q1) is positive; by Lemma 5.1 and Proposition 5.5, however, every quantum measure μ that gives this positive answer is two-valued, and the function $\psi \mapsto \mu(P_\psi)$ is discontinuous on $S(\mathcal{H})$. The measures μ_ρ used in physics (see Remark 0.1) vary continuously with the ray, and none of them answers (Q1) positively, in any dimension:

Corollary 5.7. *For every density operator ρ, the quantum measure μ_ρ does not preserve commuting meets.*

Proof. If μ_ρ preserved commuting meets, Lemma 5.1 would make μ_ρ two-valued. Since the function $\psi \mapsto \mu_\rho(P_\psi) = \langle\psi|\rho|\psi\rangle$ is continuous, this would contradict Proposition 5.5. ☺

Remark 5.8 (A finite witness). The proof of Theorem 5.6 applies Gleason's Theorem to the whole of $L(\mathcal{H})$, an infinite object. A finite configuration in three dimensions supplies the obstruction; in a larger space, we place it in a subspace to which the hypothetical two-valued measure assigns value 1. The existence of the configuration still follows through Gleason's Theorem: by Corollary 3.2 with $d = 3$, there is a finite family $\mathcal{E}_0$ of orthonormal bases of $\mathbb{C}^3$ such that no function from the rays occurring in $\mathcal{E}_0$ to $\{0, 1\}$ takes the value 1 at exactly one ray of each basis, and a unitary map places $\mathcal{E}_0$ in any 3-dimensional space. Now suppose that $3 \leq \dim \mathcal{H} \leq \infty$, and let v be a quantum measure on $L(\mathcal{H})$ that preserves commuting meets. By Lemma 5.1, the measure v is two-valued, and by Lemma 1.2 some unit vector ψ has $v(P_\psi) = 1$. Let K be a 3-dimensional subspace containing ψ; since $P_K \geq P_\psi$,

monotonicity gives $v(P_K) = 1$, and we place $\mathcal{E}_0$ in K. For each basis $\{u_1, u_2, u_3\}$ in $\mathcal{E}_0$ we have $P_{u_1} + P_{u_2} + P_{u_3} = P_K$, so exactly one of the three rays has value 1, contrary to the choice of $\mathcal{E}_0$. The contradiction uses finitely many commuting triples of rank-one projections in the chosen 3-dimensional subspace, together with $v(P_K) = 1$. The witnesses of Proposition 5.2 (the rays a and c) and of Proposition 5.11 below (the matrices M and τ) likewise lie in a single 2-dimensional subspace.

Until now, the obstruction to (Q1) has come from the continuity of the measures. For pairs that need not commute, the meet operation itself supplies a second obstruction. For a Dirac measure δ_ω, the first equation in (4) says that the number $\delta_\omega(A \cap B)$ is computed from $\delta_\omega(A)$ and $\delta_\omega(B)$ by a fixed continuous function, namely min. By Proposition 5.2, no quantum measure computes all its meets by min; in Proposition 5.9 below, we show that no measure μ_ρ given by a density operator computes them by any continuous function at all. The reason can be seen in a picture: the meet is not continuous for the norm topology. Two distinct lines through the origin meet only at 0, however small the angle between them, whereas a line meets itself in the whole line (see Figure 3).

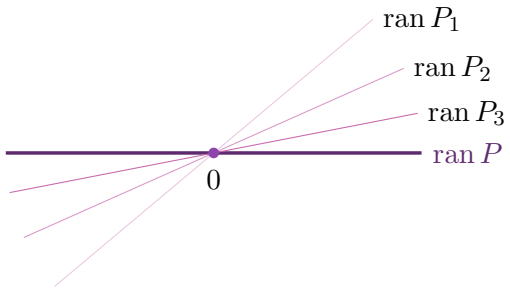


FIGURE 3. Rank-one projections P_n converging to a rank-one projection P, represented by their ranges.

Proposition 5.9. *Let P_n and P be rank-one projections with $P_n \neq P$ and $P_n \to P$ in $L(\mathcal{H})$. Then the meets $P_n \wedge P$ are 0 and the distances $\|P_n \vee P - P\|$ are 1 for every n, while $P \wedge P = P \vee P = P$. Moreover, for every density operator ρ there is no continuous function $m : [0,1]^2 \to [0,1]$ such that $\mu_\rho(Q \wedge R) = m\big(\mu_\rho(Q), \mu_\rho(R)\big)$ for all $Q, R \in L(\mathcal{H})$.*

Proof. Since $P_n \neq P$, the ranges of P_n and P are distinct lines, so their meet $P_n \wedge P$ is 0. The join $P_n \vee P$ is the projection onto a 2-dimensional space containing the range of P, and therefore the operator $P_n \vee P - P$ is a rank-one projection, of norm 1.

For the second assertion, the idea is that a continuous m would carry the identity $\mu_\rho(Q_n \wedge P_\psi) = 0$ to the limit, where the meet jumps from 0 to P_ψ. Let ρ be a density operator, and suppose that $m : [0,1]^2 \to [0,1]$ is continuous and satisfies $\mu_\rho(Q \wedge R) = m\big(\mu_\rho(Q), \mu_\rho(R)\big)$ for all $Q, R \in L(\mathcal{H})$. Let $\psi \in S(\mathcal{H})$ be given; we choose a unit vector $\xi \perp \psi$ and let $Q_n = P_{\psi_n}$ with $\psi_n = (\cos \frac{1}{n})\psi + (\sin \frac{1}{n})\xi$. Since $|\langle \psi | \psi_n \rangle| = \cos \frac{1}{n} < 1$, each Q_n differs from P_ψ, and the distance $\|Q_n - P_\psi\| = \sin \frac{1}{n}$ tends to 0 by Proposition 5.3(1); hence, by the first part of this proof, the meets $Q_n \wedge P_\psi$ are 0, and, by Proposition 5.3(3), the values $\mu_\rho(Q_n)$ converge to $\mu_\rho(P_\psi)$. Letting $n \to \infty$ and using the continuity of m, we obtain

$$0 = \mu_\rho(Q_n \wedge P_\psi) = m\big(\mu_\rho(Q_n), \mu_\rho(P_\psi)\big) \to m\big(\mu_\rho(P_\psi), \mu_\rho(P_\psi)\big) = \mu_\rho(P_\psi \wedge P_\psi) = \mu_\rho(P_\psi).$$

Since ψ was arbitrary, the measure μ_ρ vanishes on every rank-one projection, and summing over an orthonormal basis gives $\mu_\rho(I) = 0$, which contradicts $\mu_\rho(I) = \operatorname{Tr} \rho = 1$. ☺

5.3. **Classical measures form a lattice.** We now turn to (Q2). Before answering it, we recall the argument for its classical counterpart (C2), because the counterexample of Section 5.4 is built to defeat exactly this argument. The idea is simple: given $\mu, \nu \in \mathcal{M}(\Sigma)$, we split Ω into a part where $\mu \geq \nu$ and a part where $\mu \leq \nu$, and on each part we keep the smaller of the two measures. The Hahn decomposition theorem provides the split, and in Proposition 5.10 below we show that the resulting measure is the greatest lower bound of μ and ν.

A *finite signed measure* on Σ is a countably additive function $\lambda : \Sigma \to \mathbb{R}$. The difference $\mu - \nu$ of two measures $\mu, \nu \in \mathcal{M}(\Sigma)$ is one, and it is the one we need. Given a finite signed measure λ, a *Hahn set* for λ is a set $S \in \Sigma$ such that $\lambda(F) \geq 0$ for every measurable $F \subseteq S$ and $\lambda(F) \leq 0$ for every measurable $F \subseteq \Omega \setminus S$. The *Hahn decomposition theorem* says that every finite signed measure has a Hahn set (see, e.g., Rudin [Rud87, Theorem 6.14]).

Given $\mu, \nu \in \mathcal{M}(\Sigma)$ and a Hahn set S for $\mu - \nu$, define measures $\gamma, \eta \in \mathcal{M}(\Sigma)$ by

$$\gamma(A) = \nu(A \cap S) + \mu(A \setminus S), \qquad \eta(A) = \mu(A \cap S) + \nu(A \setminus S) \qquad (A \in \Sigma). \tag{6}$$

Proposition 5.10. *Let $\mu, \nu \in \mathcal{M}(\Sigma)$ be given, and let S be a Hahn set for $\mu - \nu$. Then the measures γ and η defined in (6) are, respectively, the greatest lower bound and the least upper bound of μ and ν in $(\mathcal{M}(\Sigma), \leq)$.*

Proof. The measure γ takes the smaller of μ and ν on each side of S, and the measure η takes the larger. Since S is a Hahn set for $\mu - \nu$, we have $\mu(F) \geq \nu(F)$ for every measurable $F \subseteq S$ and $\mu(F) \leq \nu(F)$ for every measurable $F \subseteq \Omega \setminus S$. In particular, every $A \in \Sigma$ satisfies $\nu(A \cap S) \leq \mu(A \cap S)$ and $\mu(A \setminus S) \leq \nu(A \setminus S)$, whence $\gamma(A) \leq \mu(A)$ and $\gamma(A) \leq \nu(A)$; in other words, the measure γ is a lower bound of μ and ν. Now let $\kappa \in \mathcal{M}(\Sigma)$ be any lower bound of μ and ν. Applying $\kappa \leq \nu$ on $A \cap S$ and $\kappa \leq \mu$ on $A \setminus S$, we obtain

$$\kappa(A) = \kappa(A \cap S) + \kappa(A \setminus S) \leq \nu(A \cap S) + \mu(A \setminus S) = \gamma(A) \qquad (A \in \Sigma),$$

which means that $\kappa \leq \gamma$. This proves that γ is the greatest lower bound of μ and ν. The same argument, with the inequalities reversed, shows that η is the least upper bound. ☺

5.4. **The negative answer to (Q2).** In this subsection we show that the answer to (Q2) is negative as soon as $\dim \mathcal{H} \geq 2$. We first translate pointwise comparison of measures into positivity of an operator difference; the counterexample can then be constructed with 2×2 matrices. Recall that $\mathcal{C}(\mathcal{H})$ consists of the functions $\mu_T(P) = \operatorname{Tr}(TP)$ on $L(\mathcal{H})$, where T ranges over the positive trace-class operators on $\mathcal{H}$, and that $\mathcal{C}(\mathcal{H})$ is ordered pointwise. For bounded self-adjoint operators R and T, the inequality $R \leq T$ means that $T - R$ is positive. For positive trace-class operators R and T, we have $\mu_R \leq \mu_T$ if and only if $R \leq T$. Indeed, if $\mu_R \leq \mu_T$, then every $\psi \in S(\mathcal{H})$ satisfies $\langle \psi | T - R | \psi \rangle = \mu_T(P_\psi) - \mu_R(P_\psi) \geq 0$, so the operator $T - R$ is positive. Conversely, if $T - R \geq 0$, then every $P \in L(\mathcal{H})$ satisfies $\operatorname{Tr}((T - R)P) = \operatorname{Tr}(P(T - R)P) \geq 0$, and therefore $\mu_R \leq \mu_T$.

To show that $\mathcal{C}(\mathcal{H})$ is not a lattice, it is enough to exhibit two of its elements without a greatest lower bound. We choose two that are given by commuting operators, so that the failure cannot be blamed on a lack of commutativity between them. For orthonormal vectors $e_1, e_2 \in \mathcal{H}$, let

$$A = P_{e_1} + P_{e_2}, \qquad D = 2P_{e_1} + \tfrac{1}{2} P_{e_2}. \tag{7}$$

Proposition 5.11. *Let $e_1, e_2 \in \mathcal{H}$ be orthonormal, and let A and D be the operators defined in (7). Then the measures μ_A and μ_D have no greatest lower bound in $\mathcal{C}(\mathcal{H})$, although A and D commute.*

Proof. The idea is to reduce the problem to 2×2 matrices, where the classical recipe produces a single candidate, the diagonal matrix M below, and then to exhibit a nondiagonal lower bound that this candidate fails to dominate. Let $K = \operatorname{span}(e_1, e_2)$, and let μ_T be a lower bound of μ_A and μ_D in $\mathcal{C}(\mathcal{H})$; by the equivalence proved above, the operator T satisfies $0 \le T \le A$ and $T \le D$. If $\xi \perp K$, then the inequalities $0 \le \langle\xi|T|\xi\rangle \le \langle\xi|A|\xi\rangle = 0$ show that the vector $T^{1/2}\xi$ is 0, and hence so is $T\xi$. Since T is self-adjoint, it follows that $\langle\xi|Tx\rangle = \langle T\xi|x\rangle = 0$ for all $x \in K$ and $\xi \perp K$. Thus the operator T maps K into K and vanishes on $K^\perp$. Conversely, a positive operator on K that lies below both A and D on K, extended by 0 on $K^\perp$, is a positive operator of finite rank, and it gives a lower bound of μ_A and μ_D in $\mathcal{C}(\mathcal{H})$, because A and D also vanish on $K^\perp$. We may therefore identify the lower bounds of μ_A and μ_D with the positive 2×2 matrices, written in the basis e_1, e_2, that lie below both A and D. In this basis,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Throughout the argument we use the fact that a self-adjoint 2×2 matrix is positive if and only if both diagonal entries and the determinant are nonnegative. For instance, the diagonal matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

is a lower bound of A and D. Suppose that G is a greatest lower bound of A and D. Since $M \le G \le A$, the $(1,1)$ entries satisfy $1 \le G_{11} \le 1$, and hence $G_{11} = 1$. The positive matrix $G - M$ therefore has a zero $(1,1)$ entry, and its determinant, $-|G_{12}|^2$, can be nonnegative only if its off-diagonal entries vanish. The matrix G is thus diagonal, and the inequalities $M \le G \le D$ now yield $G_{22} = \frac{1}{2}$, whence $G = M$. To reach a contradiction, we exhibit a lower bound of A and D that does not lie below M, namely

$$\tau = \begin{pmatrix} \frac{11}{12} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

The matrix τ is positive (since $\det\tau = \frac{1}{6}$), and so are $A-\tau = \left(\begin{smallmatrix} 1/12 & -1/4 \\ -1/4 & 3/4 \end{smallmatrix}\right)$ (since $\det(A-\tau) = 0$) and $D - \tau = \left(\begin{smallmatrix} 13/12 & -1/4 \\ -1/4 & 1/4 \end{smallmatrix}\right)$ (since $\det(D-\tau) = \frac{5}{24}$); thus the matrix τ is a lower bound of A and D. However, the matrix $M - \tau = \left(\begin{smallmatrix} 1/12 & -1/4 \\ -1/4 & 1/4 \end{smallmatrix}\right)$ has determinant $-\frac{1}{24} < 0$, and consequently the matrix τ does not lie below $M = G$, which contradicts the choice of G. Therefore the measures μ_A and μ_D have no greatest lower bound in $\mathcal{C}(\mathcal{H})$, although the operators A and D commute. ☺

The proof of Proposition 5.11 shows exactly where the classical argument of Proposition 5.10 breaks down. On K, the operator $D - A$ is positive on the line spanned by e_1 and negative on the line spanned by e_2, so these two lines play the part of a Hahn set and its complement. Keeping the smaller of A and D on each line produces the matrix M, just as in the classical case. Among diagonal matrices, that is, among the operators on K that commute with P_{e_1} and P_{e_2}, the matrix M is indeed the greatest lower bound of A and D: a positive matrix $\operatorname{diag}(a, b)$ lies below both A and D if and only if $a \le 1$ and $b \le \frac{1}{2}$. The lower bound τ that defeats M, however, is not diagonal, and therefore does not commute with D.

In Corollary 5.12 below, we show that Proposition 5.11 rules out any identification of the order on quantum measures with the order on classical ones. Recall from the beginning of

this section that such an identification is an order isomorphism: a bijection that preserves the order in both directions.

Corollary 5.12. *If Σ is a σ-algebra on a set Ω, then there is no order isomorphism between $(\mathcal{C}(\mathcal{H}), \leq)$ and $(\mathcal{M}(\Sigma), \leq)$.*

Proof. An order isomorphism carries greatest lower bounds to greatest lower bounds, and it would therefore transport property (C2) from $\mathcal{M}(\Sigma)$ to $\mathcal{C}(\mathcal{H})$. Suppose, to reach a contradiction, that $\Phi : \mathcal{C}(\mathcal{H}) \to \mathcal{M}(\Sigma)$ is an order isomorphism; its inverse Φ^{-1} is then an order isomorphism as well. Let $\mu, \nu \in \mathcal{C}(\mathcal{H})$ be given. By (C2), proved in Proposition 5.10, the measures $\Phi(\mu)$ and $\Phi(\nu)$ have a greatest lower bound γ in $\mathcal{M}(\Sigma)$. We claim that $\Phi^{-1}(\gamma)$ is a greatest lower bound of μ and ν in $\mathcal{C}(\mathcal{H})$. Since γ is a lower bound of $\Phi(\mu)$ and $\Phi(\nu)$, the element $\Phi^{-1}(\gamma)$ is a lower bound of μ and ν, because Φ^{-1} preserves the order. If $\kappa \in \mathcal{C}(\mathcal{H})$ is a lower bound of μ and ν, then the measure $\Phi(\kappa)$ is a lower bound of $\Phi(\mu)$ and $\Phi(\nu)$, and therefore $\Phi(\kappa) \leq \gamma$, whence $\kappa \leq \Phi^{-1}(\gamma)$. This proves the claim. Thus any two elements of $\mathcal{C}(\mathcal{H})$ have a greatest lower bound, which contradicts Proposition 5.11. ☺

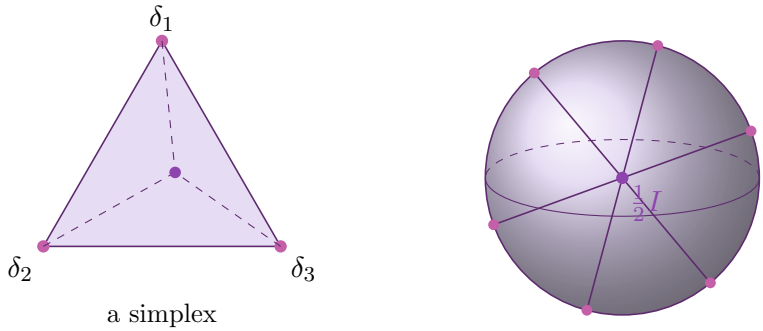
Remark 5.13 (Simplices and the Bloch ball). Two known theorems explain Proposition 5.11 geometrically; they are not used elsewhere in these notes. A point of a convex set is *extreme* if it lies in no open line segment joining two distinct points of the set. A compact convex subset of a finite-dimensional real vector space is a *simplex* if every point has a unique finite convex combination of distinct extreme points with positive coefficients, up to reordering. A *base* of a cone is a convex subset meeting each nonzero ray of the cone exactly once. The first theorem states that a finite-dimensional real vector space ordered by a closed generating cone is a lattice if and only if a compact base of the cone is a simplex [Alf71]. Classically, the probability measures on a finite set Ω form a simplex whose extreme points are the Dirac measures, that is, the indicator functions of the principal ultrafilters once more. For $\mathcal{H} = \mathbb{C}^2$, normalizing by $\mu(I) = 1$ gives a compact base of $\mathcal{C}(\mathcal{H})$: the measures μ_ρ, which we identify with the density operators ρ. Every self-adjoint 2×2 matrix of trace 1 can be written uniquely as $\rho = \frac{1}{2}(I + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)$ with $x \in \mathbb{R}^3$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

the eigenvalues of ρ are $\frac{1}{2}(1 \pm |x|)$, so the matrix ρ is a density operator if and only if $|x| \leq 1$. The density operators thus form a closed ball, the *Bloch ball*, whose extreme points, the pure states, fill the whole boundary sphere. The center $\frac{1}{2}I$ is the midpoint of every diameter, since $\frac{1}{2}I = \frac{1}{2}(P_\psi + P_{\psi'})$ for every orthonormal basis $\{\psi, \psi'\}$ of $\mathbb{C}^2$ (see Figure 4). Hence the Bloch ball is not a simplex, and, by the first theorem, the self-adjoint 2×2 matrices, ordered by the cone of positive ones, do not form a lattice. The second theorem, due to Kadison [Kad51], goes much further: two bounded self-adjoint operators on a Hilbert space have a least upper bound among the bounded self-adjoint operators if and only if they are comparable, that is, if and only if one of them lies below the other.

6. Timeline

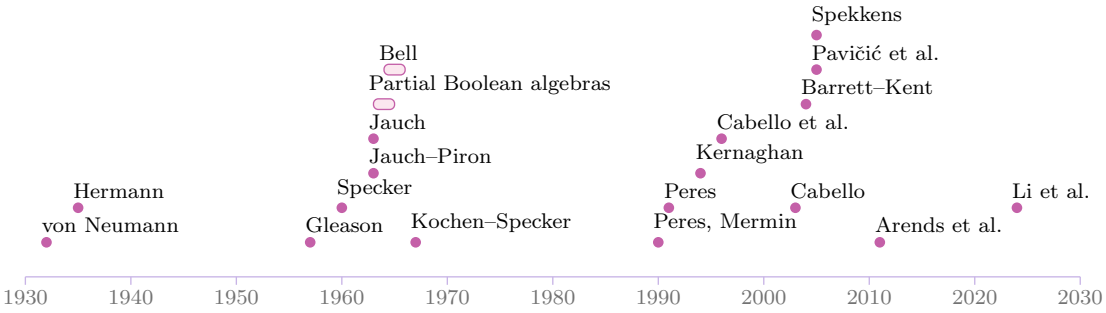
The story has three acts. In the first, von Neumann's argument against hidden variables is accepted by almost everyone for thirty years. Its key assumption is questioned as early as 1935, but the objection goes almost unheard. In the second act, which lasts only from 1957 to 1967, the theorem of these notes is found twice: once as a corollary of Gleason's Theorem, by Bell, and once as a finite configuration of vectors, by Kochen and Specker. In the third,



a simplex — the Bloch ball

FIGURE 4. The probability measures on a three-point space, forming a simplex whose vertices are the Dirac measures $\delta_1, \delta_2, \delta_3$ (left), and the density operators on $\mathbb{C}^2$, forming a ball whose extreme points (pink) are the pure states on the boundary sphere and whose center $\frac{1}{2}I$ is the midpoint of every diameter (right).

which is still going on, the configurations become smaller and smaller, first by hand and then by computer. The theorem itself is pushed beyond projections, beyond exact measurements, and beyond the values 0 and 1.



1932 **von Neumann.** Von Neumann gives the first argument against hidden variables, in his book *Mathematische Grundlagen der Quantenmechanik* [vN32]. The argument assumes that the hidden value of a sum $A+B$ of two observables is the sum of the hidden values of A and B, even when A and B cannot be measured together. The assumption is harmless for averages and unjustified for individual values. Bell makes exactly this objection some thirty years later [Bel66]. In a 1988 interview he states the objection more bluntly: "the proof of von Neumann is not merely false but foolish!" [MC88].

1935 **Hermann.** Grete Hermann, a mathematician and philosopher who had written her doctoral thesis under Emmy Noether, objects to the same assumption in *Die naturphilosophischen Grundlagen der Quantenmechanik* [Her35]. Her objection goes almost unread by physicists until the 1970s [Jam74]. The extent to which it anticipates Bell's objection is still debated [MS18].

1957 **Gleason.** Gleason proves his theorem [Gle57] in answer to a question of Mackey's about measures on the closed subspaces of a Hilbert space [Mac57, Mac63]. Gleason never mentions hidden variables in the paper. Bell later uses the theorem in the hidden-variables debate and credits Jauch with drawing his attention to it [Bel66].

1960 **Specker.** Specker, meanwhile, approaches the same fact from logic. In *Die Logik nicht gleichzeitig entscheidbarer Aussagen* ("The logic of propositions that cannot be decided simultaneously") [Spe60], he announces that the closed subspaces of a three-dimensional Hilbert space cannot be mapped into a Boolean algebra in a way that respects the operations on orthogonal subspaces. For a proof he offers only the phrase "an elementary geometric argument" (*ein elementargeometrisches Argument*). The argument itself appears seven years later, in his joint paper with Kochen.

1963 **Jauch–Piron.** In Geneva, Jauch and Piron publish *Can Hidden Variables Be Excluded in Quantum Mechanics?* [JP63], an argument against hidden variables in the lattice setting of these notes. Their argument assumes that whenever two questions both have probability 1, so does their meet. In his 1966 paper, Bell objects that, for questions that cannot be tested together, the meet is not tested by testing the two questions. Nothing in physics, then, forces this assumption, any more than it forces von Neumann's.

1963 **Jauch.** It is Jauch, nonetheless, who sets Bell on the right track. J. M. Jauch (1914–1974), director of the Institute of Theoretical Physics at the University of Geneva, is then supervising Piron's doctoral thesis [Pir64], which characterizes the lattices of closed subspaces of Hilbert spaces by axioms on propositions. Gleason's Theorem is therefore close to Jauch's own research. He points the theorem out to Bell, and, as Bell later recalled, he stresses exactly the right feature: Gleason's Theorem needs additivity only for questions that can be tested together, and this is the weakening of von Neumann's assumption that Bell is looking for [Bel87]. Bell thanks him in a footnote of his 1966 paper [Bel66]: "I am much indebted to Professor Jauch for drawing my attention to this work."

1963–1965 **Partial Boolean algebras.** Kochen and Specker, for their part, build the framework first. They present *Logical Structures Arising in Quantum Theory* at the 1963 Berkeley symposium on the theory of models, and *The Calculus of Partial Propositional Functions* at the 1964 Jerusalem congress; both appear in 1965 [KS65b, KS65a]. In these papers they introduce *partial Boolean algebras*: families of Boolean algebras, one for each set of questions that can be tested together, glued along their common parts. The application to hidden variables is held back for the 1967 paper.

1964–1966 **Bell.** Bell, a particle physicist at CERN who worked on the foundations of quantum mechanics as a side interest, writes *On the Problem of Hidden Variables in Quantum Mechanics* [Bel66] in 1964, during a leave at the Stanford Linear Accelerator Center [Ber14]. In it, Bell disposes of the

arguments of von Neumann and of Jauch and Piron. He then derives the nonexistence of non-contextual hidden variables from Gleason's Theorem, by an argument close to that of Section 2. The manuscript lies unanswered at the journal for a year, and publication is delayed until 1966 (Rev. Mod. Phys. **38**, 447–452) [Ber14]. The paper thus appears between the two stages of Kochen and Specker's work, and Bell's argument is independent of both.

1967 **Kochen–Specker.** Kochen and Specker publish *The Problem of Hidden Variables in Quantum Mechanics* [KS67]: a self-contained, finite proof with 117 vectors in $\mathbb{R}^3$, written in the language of partial Boolean algebras (see Section 4). Physicists often name the theorem after its two authors alone; we add Bell's name, for the reasons given in the subsection that follows the timeline.

1990 **Peres, Mermin.** The third act takes a new turn in proofs using products of observables. Peres gives a short argument, and Mermin presents a refinement [Per90, Mer90]. The resulting "magic square" is a 3×3 array of observables of a pair of two-dimensional systems, in which the products along the rows and the columns rule out every assignment of predetermined values ± 1 that respects the product identities for commuting observables. The square speaks about observables rather than rays, but at heart it is the same kind of argument as the proof of Section 4: a parity count that cannot come out right.

1991 **Peres.** The shrinking begins. Peres reduces Kochen and Specker's construction to 33 vectors in $\mathbb{R}^3$, and gives a 24-vector construction in $\mathbb{R}^4$ [Per91]. Conway and Kochen soon go down to 31 vectors in $\mathbb{R}^3$, a record that still stands; their configuration is unpublished, and Peres reports it in his book [Per93].

1994 **Kernaghan.** Kernaghan finds 20-vector configurations in $\mathbb{R}^4$ hidden inside Peres's 24 vectors [Ker94].

1996 **Cabello et al.** Cabello, Estebaranz, and García-Alcaine find the 18-vector configuration in $\mathbb{R}^4$ used in Section 4 [CEGA96]. They search inside Peres's 24 vectors, which contain Kernaghan's configurations, and observe that none of their 18-vector configurations lies inside any of the 96 minimal 20-vector configurations found there, "which probably explains why they were not obtained previously."

2003 **Cabello.** With the configurations nearly as small as they would get, attention turns to the hypotheses. Cabello extends the argument to measurements that are not given by projections, on a single two-dimensional system [Cab03] (see Appendix A below).

2004 **Barrett–Kent.** Barrett and Kent test the theorem against measurements of finite precision whose answers may be random [BK04] (see Appendix A below).

2005 **Pavičić et al.** Pavičić, Merlet, McKay, and Megill close the question in dimension 4: an exhaustive computer search shows that no configuration has fewer than 18 vectors [PMMM05]. The configuration of Section 4 is therefore not merely economical but optimal.

2005 **Spekkens.** In the same year, Spekkens proposes a notion of non-contextuality that is phrased in terms of experimental procedures rather than projections, and that therefore applies to imperfect measurements [Spe05] (see Appendix A below).

2011 **Arends et al.** Arends, Ouaknine, and Wampler carry the computer search to dimension 3 and prove that every configuration there needs at least 18 vectors [AOW11].

2024 **Li et al.** The gap in dimension 3 is still open: the best configuration known has the 31 vectors of Conway and Kochen, and a computer search has raised the lower bound to 24 [LBG24]. In the same year, Ramanathan gives finite configurations in $\mathbb{C}^3$ that rule out non-contextual assignments from $\{0, p, 1-p, 1\}$, for $0 \le p \le \frac{1}{2}$ with $p \neq \frac{1}{3}$, under the orthogonality and basis-sum rules [Ram24] (see Appendix A below). With such assignments, the question that von Neumann asked about averages returns, ninety years later, in a sharper form.

6.1. **The fame of the 1967 paper.** As Bell's proof in Section 2 shows, the nonexistence of two-valued quantum measures follows in a few lines from Gleason's Theorem of 1957. Yet many students of the foundations of physics learn the result under the names of Kochen and Specker alone. The reason is not that Kochen and Specker found a deeper fact. It lies partly in accidents of publication and mostly in the form of their proof.

The accidents are quickly told. Gleason never mentions hidden variables in his paper: he answers a question of Mackey's, and nothing in his theorem singles out two-valued measures. Bell credits Jauch with drawing the theorem to his attention (see the **1963** entry on Jauch). Bell then saw the publication of his paper delayed until 1966, a year before Kochen and Specker's (see the **1964–1966** entry).

The form of the proof matters more. Kochen and Specker's argument depends on no theorem of analysis: it is finite and can be checked by hand, so the reader need not take on trust Gleason's Theorem, a substantial piece of analysis. Finiteness has three further consequences. First, it invited thirty years of shrinking the configuration, from 117 to 33 and then 31 vectors in $\mathbb{R}^3$, and from 24 to 20 to 18 in $\mathbb{R}^4$. Second, a finite configuration makes sense in structures that do not come from Hilbert spaces, where Gleason's Theorem says nothing. Third, the argument can be taken to the laboratory: a finite list of measurements can be carried out, whereas "all quantum measures on $L(\mathcal{H})$ at once" cannot.

In short, Gleason's Theorem contained the result from 1957 on, Bell drew it out of that theorem, and Kochen and Specker gave it a form that could be checked by hand, generalized, and tested. For these reasons we speak of the Bell–Kochen–Specker theorem.

Appendix A. Attempts beyond classical propositional logic

Throughout these notes, a hidden-variable assignment answers each yes/no question with 0 or 1, and the constraints on it are propositional: in each orthonormal basis, exactly one

answer is "yes" (see Section 3). In this appendix we consider changes to the logical framework, to the measurements themselves, and to the values assigned to their outcomes. We begin with quantified quantum logics and classical axiomatization, then turn to generalized measurements and probabilistic assignments.

A.1. **Richer logics: quantifiers and first-order quantum logic.** The first sustained attempt to give quantum logic quantifiers came from a set theorist. Gaisi Takeuti, a Japanese logician who spent most of his career at the University of Illinois and was already known for Boolean-valued analysis, published *Quantum Set Theory* in 1981 [Tak81]. His idea was to run the Scott–Solovay construction of Boolean-valued models of ZFC with the complete Boolean algebra replaced by the complete orthomodular lattice $L(\mathcal{H})$ of closed subspaces of a Hilbert space $\mathcal{H}$. The quantum universe $V^{(L(\mathcal{H}))}$ is built by transfinite recursion, as in the Boolean case. Quantifiers receive the literal semantics $[\![(\forall x)\varphi(x)]\!] = \bigwedge_{x\in V^{(L(\mathcal{H}))}} [\![\varphi(x)]\!]$, an infimum taken in $L(\mathcal{H})$, and the real numbers of the new universe turn out to be the self-adjoint operators on $\mathcal{H}$. The result is genuine first-order, indeed set-theoretic, quantification, evaluated in a truth-value algebra that is not Boolean. Takeuti hoped that a "quantum mathematics" based on quantum logic would grow out of the construction. In 1999, in *A Lattice-Valued Set Theory* [Tit99], Satoko Titani placed Takeuti's construction in a wider family: set theories whose truth values form an arbitrary complete lattice, a family that also contains the Heyting-valued (intuitionistic) set theories. She also proved general completeness results.

Takeuti's most persistent heir came to the subject from physics. Masanao Ozawa of Nagoya University developed quantum set theory into a framework for quantified statements about observables. In *Transfer Principle in Quantum Set Theory* [Oza07], he replaced $L(\mathcal{H})$ by the projection lattice of an arbitrary von Neumann algebra and proved the following transfer principle: for every theorem of ZFC expressed by a formula with bounded quantifiers, its truth value at quantum-set parameters is at least their commutator, the projection measuring their degree of commutativity; when this commutator is I, the transferred theorem has truth value I. In *Orthomodular-Valued Models for Quantum Set Theory* [Oza17] he refined the model theory, and in later papers he revised the semantics to recover De Morgan's laws and extended the framework to quantum conditionals. Ozawa's interest in uncertainty was not only logical: his own reformulation of the uncertainty principle, proposed in 2003 [Oza03], was tested with polarized neutrons by Yuji Hasegawa's group in Vienna in 2012 [ESS+12].

The other great tool of modern set theory, Cohen's method of forcing, was brought to the question by William Boos in 1996, in *Mathematical Quantum Theory I: Random Ultrafilters as Hidden Variables* [Boo96]. Where Takeuti had changed the truth-value algebra, Boos changed the hidden variables: his are *random* ultrafilters, generic objects for forcing with measure algebras, through which he interprets quantum probabilities. The paper remains one of very few attempts to attack the hidden-variable question, the question behind Theorem 1.4, with the methods of set theory rather than with those of analysis.

A third line came from physics proper. Between 1998 and 2002 Chris Isham, a quantum-gravity theorist at Imperial College London, and Jeremy Butterfield, a philosopher of physics then at All Souls College, Oxford, with J. Hamilton as a third author of Part III, published the four-part series *A Topos Perspective on the Kochen–Specker Theorem* [IB98, BI99, HIB00, IB02]. They took seriously the objection that the orthomodular lattice, being non-distributive, is too poorly behaved to serve as a logic with an ordinary semantics, and they replaced it by the family of its Boolean subalgebras ("contexts"), each of which is classical. Stripped of the topos language, their reformulation of Theorem 1.4 says that one cannot

choose a point in the Stone space of each context, that is, an ultrafilter on it, so that, whenever one context is contained in another, the ultrafilter on the smaller context is the restriction of the ultrafilter on the larger one. This is the picture at the end of Section 3, and the statement of Section 5 that the quantum lattice has no points. Isham and Andreas Döring later extended the approach into a reformulation of physics, *A Topos Foundation for Theories of Physics* [DI08].

A.1.1. *Classical first-order logic.* Classical first-order logic entered quantum foundations in two quite different ways, and the two are best kept apart. The older tradition, the representation program of Mackey, Piron, Foulis and Randall, Hardy, and Barrett, uses classical logic only as ordinary mathematics is always written: it states physical postulates as axioms about a lattice, a family of tests or a convex set of states, and proves classical representation theorems about the structures that satisfy them. The younger tradition takes first-order logic literally, as an object of study. Goldblatt studied what first-order logic can detect about a space from orthogonality between its vectors; Fritz studied the decidability of laws about its closed subspaces. The two results concern different structures and different signatures. We describe the two traditions in this order.

In the early 1960s George Mackey at Harvard [Mac63] and Constantin Piron in Geneva, Jauch's student [Pir64], stated the physical postulates (a set of "questions", or an atomistic, orthomodular, irreducible lattice satisfying the covering law) as ordinary axioms in ordinary set-theoretic mathematics, with no non-classical logic anywhere. Mackey simply postulated, as his Axiom VII, that the questions form the lattice of closed subspaces of a separable infinite-dimensional complex Hilbert space. Piron derived this structure by a classical representation theorem: every complete lattice of length at least 4 that satisfies his axioms is isomorphic to the lattice of closed subspaces of a generalized Hilbert space, that is, of a vector space over a division ring with an involution, equipped with a suitable Hermitian form. From 1972 on, David Foulis and Charles Randall at the University of Massachusetts continued this tradition with their *manuals* and *test spaces* [FR72], starting from experiments rather than from lattices: outcomes and tests are ordinary sets, and non-Boolean structure appears only later, as a theorem about which propositions can be tested together. Lucien Hardy's *Quantum Theory from Five Reasonable Axioms* [Har01] of 2001 and Jonathan Barrett's *Information Processing in Generalized Probabilistic Theories* [Bar07] of 2007 revived the program in a modern form, in which quantum theory appears as one member of a family of probabilistic theories described by simple axioms about mixing states and combining systems.

Robert Goldblatt, of Victoria University of Wellington, studied a different first-order structure in *Orthomodularity is not elementary* [Gol84] of 1984: its elements are vectors, and its sole relation is orthogonality. An incomplete infinite-dimensional pre-Hilbert space can be elementarily equivalent to a Hilbert space in this language, although its lattice of orthoclosed subspaces is not orthomodular. Here a subspace is *orthoclosed* when it equals its double orthogonal complement. Thus vector orthogonality does not detect metric completeness; this does not establish nonaxiomatizability of Hilbert lattices as structures whose elements are subspaces. (The finite configuration of Section 4, by contrast, lives in a 4-dimensional subspace, which is complete automatically.) In 2016 Tobias Fritz, building on William Slofstra's work on embeddability problems for groups, added a second negative result in *Quantum Logic is Undecidable* [Fri21], published in 2021. In the signature consisting of join, the binary orthogonality relation, and the constants 0 and I, the first-order theory common to all complex Hilbert lattices is undecidable already at the level of quasi-identities: no algorithm decides

whether an implication between equations and orthogonality relations holds in every Hilbert space.

A.1.2. *The approaches compared.* The representation program and the model-theoretic investigations have different aims. A representation theorem derives a Hilbert-space description from physical axioms; elementary axiomatizability asks whether first-order sentences in a specified language can characterize the structures, and decidability asks for an algorithm recognizing their common theory. Goldblatt's result concerns vector orthogonality, while Fritz's concerns the common quasi-identities of complex Hilbert lattices. These results therefore leave the representation theorems intact. Takeuti and the topos approach change the semantics in which quantum propositions are interpreted: Takeuti uses a non-Boolean truth-value algebra, while Isham and Butterfield use the intuitionistic internal logic of a topos. Boos keeps a Boolean truth-value algebra and changes the hidden variables instead, using random ultrafilters.

A.2. **Richer values: $[0,1]$-valued and POVM-based assignments.** The Main Theorem (see Theorem 1.4) rules out two-valued, non-contextual assignments to projections. In this subsection we distinguish two changes: replacing projective measurements by more general measurements, and replacing predetermined answers by probabilities of answers. Retaining predetermined answers while adding measurements imposes further consistency constraints; allowing probabilistic responses instead relaxes that requirement.

Unsharp measurements are modeled as follows. An *effect* on $\mathcal{H}$ is a self-adjoint operator E with $0 \leq E \leq I$, that is, with spectrum in $[0,1]$. A *positive-operator-valued measure* (POVM) with finitely many outcomes is a finite family $(E_1, \ldots, E_n)$ of effects with

$$E_1 + \cdots + E_n = I.$$

By the Born rule, a system in the state given by a unit vector ψ yields outcome i with probability $\langle\psi|E_i|\psi\rangle$; the condition $\sum_i E_i = I$ says exactly that these probabilities sum to 1. A projective measurement, i.e. an orthogonal decomposition of I into projections as in Definition 1.1, is the special case in which every E_i is a projection. Physically, a POVM is what one obtains by coupling the system to an auxiliary system and performing an ordinary projective measurement on the combined system; Naimark's dilation theorem guarantees that every POVM arises in this way. The essential new feature is that the effects of a POVM need not be mutually orthogonal. A POVM may therefore have more nonzero effects than $\dim \mathcal{H}$, and a single effect may belong to many different POVMs, even in dimension 2.

A.2.1. *Fuzzing the measurements, keeping assignments sharp: the no-go strengthens, even in dimension* 2. The analogue of a two-valued quantum measure is now a map v from effects to $\{0,1\}$ such that, for every POVM $(E_1, \ldots, E_n)$ whose effects are pairwise distinct, exactly one E_i has $v(E_i) = 1$: each measurement has exactly one outcome. The restriction to distinct effects is not pedantry: without it, the POVM $(\frac{1}{2}I, \frac{1}{2}I)$ would refute every v at once, in every dimension, since its two outcomes carry the same effect and therefore receive the same value. Non-contextuality is built into the requirement that v is a function of the effect alone, not of the POVM in which the effect appears. When $\dim \mathcal{H} = 2$, the projections give too few constraints for a contradiction: the only projective measurements are the pairs $\{P, I - P\}$, and a two-valued quantum measure exists (see Remark 1.6). Effects supply many more constraints, because the same effect appears in many POVMs, with different companions. Cabello [Cab03] exhibited five POVMs on the qubit $\mathbb{C}^2$, each with eight rank-one effects, built from twenty effects tied to the geometry of the dodecahedron; each effect occurs in exactly

two of the five POVMs, and the parity count of Section 4 then applies verbatim, since five is odd and every effect is counted twice. Thus the Bell–Kochen–Specker obstruction extends past sharp projections, into dimension 2, which Theorem 1.4 does not reach.

A.2.2. *Fuzzing the assignments, keeping measurements sharp: the naive no-go can be evaded, and needs a better physical definition to survive.* Now the deterministic value $v(P) \in \{0,1\}$ gives way to a response function $\xi_P(\lambda) \in [0,1]$: the probability of the answer "yes" to P when the system is in the ontic state λ, a point of a measure space Λ that represents the complete physical state of the system, whether or not anyone knows λ. Non-contextuality again means that ξ_P depends on P alone. In this form the requirement excludes nothing: the quantum state itself is a non-contextual model, with $\Lambda = S(\mathcal{H})$ and $\xi_P(\psi) = \langle\psi|P|\psi\rangle$. A second escape, of a different kind, uses finite precision. Meyer [Mey99], and after him Kent and Clifton, built non-contextual two-valued models on dense sets of rays, and Barrett and Kent [BK04] defended the claim that such models reproduce quantum statistics to any fixed finite precision. Spekkens [Spe05] answered with a stronger notion of non-contextuality: experimental procedures that no experiment can tell apart, preparations as well as measurements, must be represented in the same way in the hidden-variable model. Under this notion, new impossibility theorems hold, some of them already in dimension 2.

A.2.3. *Extending the outcome alphabet directly.* Ramanathan [Ram24] proves a Bell–Kochen–Specker-type theorem for non-contextual assignments with values in the finite alphabet $\{0, p, 1-p, 1\}$, where $0 \le p \le \frac{1}{2}$ and $p \ne \frac{1}{3}$, rather than only in $\{0,1\}$: a finite set of vectors in $\mathbb{C}^3$ admits no assignment whose values sum to 1 on every orthonormal basis and to at most 1 on every set of mutually orthogonal vectors. For $p > 0$, the additional values p and $1-p$ allow nondeterministic responses, while $p = 0$ recovers the original two-valued assignments. The case $p = \frac{1}{2}$ also rules out consistent non-contextual models that are "fundamentally binary", in which at most two outcomes of each measurement can occur.

Remark A.1. Retaining deterministic assignments while extending them to effects yields a further obstruction: Cabello's POVMs give a contradiction already in dimension 2. Fuzzing the *assignments* weakens the theorem under the naive definition of non-contextuality, as the Born-rule model shows, and Spekkens's stronger definition was needed to recover a robust impossibility theorem. Enlarging the set of values to a finite alphabet with intermediate probabilities recovers a close cousin of the original theorem outright (Ramanathan). In each of these impossibility theorems, the substantive work is a fresh rigidity argument, playing the role that Gleason's Theorem and the 18-vector configuration play in these notes, and never a restatement of the bare fact that local assignments respect the connectives.

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