Compactness and the classical excess

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We prove that a general probabilistic system has a countably additive classical simulation at a prescribed error as soon as each of its finitely generated subsystems has a finite classical simulation at that error, so that the countably additive classical excess is attained and is determined by the finitely generated subsystems. The key is compactness: an ultraproduct of the finite models, followed by Loeb's construction, gives one model that meets the bound on the whole system. For bounded finite-dimensional systems we show that the classical excess of Catani, Galley and Gonda is attained on a number of sample points bounded by the dimensions, which makes their stabilization criterion unconditional. Two systems on Cantor space show that for the countable target both conclusions fail, and fail independently.

1 Introduction

Suppose every finite collection of preparations and effects, together with the mixtures it generates, admits a classical simulation at a prescribed error. Must one classical model meet that error bound for the entire system? The error bounds the discrepancy between original and simulated outcome probabilities uniformly over preparations and effects. We prove that one model meets the bound when it may assign countably additive probability measures to the preparations.

The question arises from the classical excess of Catani, Galley and Gonda [1], the infimum of the errors of univalent simulations by the countable classical system $\Delta_{\mathbb{N}}$. Such simulations preserve mixtures of preparations and effects. For bounded finite-dimensional systems, we show that the excess is attained in a finite simplex whose size is bounded by the affine dimension of the state space and the dimension of the effect span. This makes the stabilization criterion in their Remark 18 unconditional for their finite-dimensional systems (Corollary 3.3). For a general system, attainment depends on the choice of classical target.

A finitely generated subsystem consists of the convex hulls of finitely many preparations and effects, with the zero effect included and the pairing inherited from the full system. The following theorem gives the passage from finite simulations to one global model.

Theorem A (Uniformization)**.** *Let A be a system and let $r \geq 0$. The following are equivalent:*

(i) *every finitely generated subsystem of A has a univalent simulation with error at most r by a classical system with finitely many outcomes;*

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(ii) *A has a univalent simulation with error at most r by a countably additive classical system.*

To rule out a countably additive classical simulation at a prescribed error, it suffices to find finitely many preparations and effects whose subsystem admits no finite classical simulation at that error. If every finitely generated subsystem admits such a simulation, one countably additive model meets the bound on the whole system. The local models may use different sample spaces and may simulate a shared preparation or effect differently.

It follows that the countably additive excess $\varepsilon_\sigma(A)$ is attained and equals the supremum of the classical excesses of the finitely generated subsystems (Corollary 4.4). For an increasing union, it is the limit of its values at the stages (Corollary 4.5).

For the passage from a global model to a finite one, Carathéodory's theorem compresses a simulation of a finite-dimensional subsystem without changing its simulated probabilities (Proposition 3.1). In the other direction, an ultraproduct enforces the finite requirements simultaneously, and Loeb's construction [2] realizes the limiting probabilities on one measurable space (Lemma 4.1).

The choice of a countably additive target is essential. The following examples separate two ways in which the countable target can fail.

Theorem B. *There are systems A and A', with the same states, each the increasing union of finite-dimensional systems simulated exactly by finite classical systems, such that:*

(1) *every finitely generated subsystem of A has classical excess 0, and the classical excess of A is 1;*

(2) *every finitely generated subsystem of A' has classical excess 0, the classical excess of A' is 0, and no univalent simulation of A' by $\Delta_{\mathbb{N}}$ has error 0.*

Both systems are simulated exactly by the classical system of Borel probability measures on Cantor space.

The two systems use the same conditional laws on Cantor space. The system A admits all cylinder-constant effects, while A' keeps only those that are 1-Lipschitz for the ultrametric (Section 5). The failures are independent. In A the finite subsystems do not determine the countable excess, which is attained at one. In A' they determine it at zero, but no countable model attains that value.

Section 6 gives a model-theoretic formulation of Theorem A. In the dual pairs of Eduardo Dueñez [3], structures with a sort of effects and a sort of states over which one may quantify, a univalent simulation with error at most ϵ is a realization of a set of conditions in the standard dual pair of $\Delta_{\mathbb{N}}$. The number $\varepsilon_\sigma(A)$ is the least ϵ for which these conditions are finitely approximately satisfiable in the standard pair, and the classical excess is the infimum of the ϵ for which they are realized in it (Theorem 6.2). The systems of Theorem B give conditions that are finitely approximately satisfiable in the standard pair and omitted by it.

Organization of the paper

Section 2 fixes the definitions. Section 3 contains the compression of finite-dimensional models and the attainment of the classical excess in finite dimension. Section 4 contains the proof of Theorem A and its consequences. Section 5 contains Theorem B and the other examples for the countable target. Section 6 gives the formulation in dual pairs and concludes with the omitted-type examples.

2 Systems, simulations and the classical excess

In a classical simulation a preparation gives a probability measure on a sample space, and an effect gives a response function on that space. The integral of the response against the measure is the simulated outcome probability. The definitions below specify the mixture identities and the error bound imposed on these probabilities.

A *system* A consists of a nonempty convex subset Ω_A of a real vector space, whose points are the *states* of A, a convex subset E_A of a real vector space with $0 \in E_A$, whose points are the *effects* of A, and a pairing $(e, \omega) \mapsto e \cdot \omega \in [0,1]$ that is affine in each argument, with $0 \cdot \omega = 0$ for every state ω. The objects of the resource theory of [1] are finite-dimensional systems in this sense and carry a unit effect, to which we return in Remark 3.4. The extension to infinite dimension is needed to state Theorems A and B, and it is the setting in which continuous ontological models live.

For a nonempty sample space Ω with a σ-algebra $\mathcal{F}$, the *classical system* $\Delta_{(\Omega,\mathcal{F})}$ has as states the countably additive probability measures on $\mathcal{F}$, as effects the $\mathcal{F}$-measurable functions $\Omega \to [0,1]$, and as pairing $f \cdot \gamma = \int f \, d\gamma$. For a set Ω that is finite or countable we take $\mathcal{F}$ to be the power set and write Δ_Ω; in particular $\Delta_k = \Delta_{\{1,\dots,k\}}$ for a positive integer k, and $\Delta_{\mathbb{N}}$ is the countable classical system, with $f \cdot \gamma = \sum_n f(n)\gamma(n)$.

Following [1, Definition 6], for systems A and B and a real number $\epsilon \geq 0$, a *univalent ϵ-simulation* of A by B is a pair of maps $\Gamma\colon \Omega_A \to \Omega_B$ and $\Theta\colon E_A \to E_B$ that preserve convex mixtures, satisfy $\Theta(0) = 0$, and satisfy

$$|e \cdot \omega - \Theta(e) \cdot \Gamma(\omega)| \leq \epsilon \qquad (\omega \in \Omega_A,\ e \in E_A).$$

The least such ϵ is the *error* of (Γ, Θ), and the function $(\omega, e) \mapsto \Theta(e) \cdot \Gamma(\omega)$ is its *simulated pairing*. For a classical system B we write

$$\varepsilon(A, B) = \inf\{\epsilon \geq 0 : \text{ there is a univalent } \epsilon\text{-simulation of } A \text{ by } B\},$$

and we write $\varepsilon_\sigma(A)$ for the infimum of $\varepsilon(A, \Delta_{(\Omega,\mathcal{F})})$ over all measurable spaces $(\Omega, \mathcal{F})$. The *classical excess* of A is $\varepsilon(A, \Delta_{\mathbb{N}})$ [1, Definition 11, Section 4.3]. Every system has a univalent simulation by every classical system, with error at most 1: send every state to a fixed point mass and every effect to 0. Since Δ_k is contained in $\Delta_{\mathbb{N}}$, which is a countably additive classical system,

$$\varepsilon_\sigma(A) \leq \varepsilon(A, \Delta_{\mathbb{N}}) \leq \varepsilon(A, \Delta_k) \leq 1 \qquad (k \geq 1),$$

and the sequence $k \mapsto \varepsilon(A, \Delta_k)$ is nonincreasing.

Lemma 2.1. *Let (Γ, Θ) be a univalent simulation of a system A by a classical system $\Delta_{(\Omega,\mathcal{F})}$. Then Θ extends to a linear map from the linear span of E_A into the space of bounded measurable functions on Ω, and Γ extends to an affine map from the affine hull of Ω_A into the space of finite signed measures on $\mathcal{F}$.*

Proof. Preservation of binary mixtures implies preservation of finite convex combinations, by induction on their length. Suppose that $\sum_i t_i e_i = \sum_j s_j f_j$ for finitely many effects e_i, f_j of A and nonnegative reals t_i, s_j, not all zero, and let $T = \max(\sum_i t_i, \sum_j s_j)$. Dividing by T and adding the effect 0 with the missing weight on each side, we obtain two convex combinations of effects of A with the same value; since Θ preserves mixtures and $\Theta(0) = 0$, the functions $\sum_i t_i \Theta(e_i)$ and $\sum_j s_j \Theta(f_j)$ are equal. Every vector of the span of E_A is a difference of two nonnegative combinations of effects, so the formula $\Theta(\sum_i t_i e_i - \sum_j s_j f_j) = \sum_i t_i \Theta(e_i) - \sum_j s_j \Theta(f_j)$ defines a linear extension. For Γ the argument is the same, with affine relations $\sum_i t_i \omega_i = \sum_j s_j \omega'_j$, $\sum_i t_i = \sum_j s_j$, in place of linear ones, and no padding. □

3 In finite dimension the minimum is attained on finitely many points

The simulated probabilities on a basis of preparations and a basis of effects determine the simulated pairing. We record these probabilities as moments of a classical model and preserve them by replacing the original sample space with finitely many points.

Throughout this section A is a system whose states and effects form bounded subsets of finite-dimensional real vector spaces. Let d be the dimension of the affine hull of Ω_A, let m be the dimension of the linear span of E_A, and let $D = (d+1)(m+1)$. Fix affinely independent states $v_0, \dots, v_d$ of A whose affine hull contains Ω_A, and effects $e_1, \dots, e_m$ of A that form a basis of the span of E_A. For a state ω, let $a_0(\omega), \dots, a_d(\omega)$ be its barycentric coordinates with respect to $v_0, \dots, v_d$, so that $\sum_i a_i(\omega) = 1$, and for an effect e let $c_1(e), \dots, c_m(e)$ be its coordinates with respect to $e_1, \dots, e_m$. These coordinates are bounded functions on Ω_A and on E_A, because Ω_A and E_A are bounded. Boundedness holds for the systems of [1], in which the effects separate the states and the states separate the effects.

By Lemma 2.1, every univalent simulation (Γ, Θ) of A by a classical system satisfies $\Gamma(\omega) = \sum_i a_i(\omega)\Gamma(v_i)$ and $\Theta(e) = \sum_j c_j(e)\Theta(e_j)$, so its simulated pairing is

$$\Theta(e) \cdot \Gamma(\omega) = \sum_{i,j} a_i(\omega) c_j(e) M_{ij}, \qquad M_{ij} = \Theta(e_j) \cdot \Gamma(v_i).$$

We call M the *matrix* of (Γ, Θ); it determines the simulated pairing, and hence the error. For a real $(d+1) \times m$ matrix M put

$$\operatorname{err}(M) = \sup_{\omega \in \Omega_A,\, e \in E_A} \Big| e \cdot \omega - \sum_{i,j} a_i(\omega) c_j(e) M_{ij} \Big|.$$

Since the coordinates are bounded, err is a supremum of uniformly Lipschitz functions of M, and is therefore continuous.

The classical models of A will be described by probability measures on the set

$$X = \Big\{ (\lambda, f) \in [0, d{+}1]^{d+1} \times [0,1]^m : \begin{array}{ll} \sum_i \lambda_i = d{+}1, & \sum_i a_i(\omega)\lambda_i \geq 0 \quad (\omega \in \Omega_A), \\ 0 \leq \sum_j c_j(e) f_j \leq 1 & (e \in E_A) \end{array} \Big\},$$

which is compact, being an intersection of closed half-spaces with a box. A point (λ, f) of X is an outcome of a classical model, recorded by the relative likelihoods λ_i of the states v_i and by the responses f_j of the effects e_j. Let $\Psi \colon X \to \mathbb{R}^D$ be the map $\Psi(\lambda, f) = \big((\lambda_i f_j)_{i,j}, (\lambda_i)_i\big)$. Since $\sum_i \lambda_i = d+1$ on X, the set $\Psi(X)$ lies in an affine hyperplane of $\mathbb{R}^D$, and its convex hull $\operatorname{conv} \Psi(X)$ is compact.

Proposition 3.1. *For every univalent simulation (Γ, Θ) of A by a classical system $\Delta_{(\Omega,\mathcal{F})}$ there is a univalent simulation of A by $\Delta_{\mathbb{N}}$ with the same simulated pairing, whose states are all supported on a common set of at most D points of $\mathbb{N}$.*

Proof. The model as a measure on X. Let $s = \frac{1}{d+1} \sum_i \Gamma(v_i)$, a probability measure on $\mathcal{F}$. Since $0 \leq \Gamma(v_i) \leq (d+1)s$ and $\sum_i \Gamma(v_i) = (d+1)s$, we may choose versions λ_i of the Radon–Nikodym derivatives $d\Gamma(v_i)/ds$ with $0 \leq \lambda_i \leq d+1$ and $\sum_i \lambda_i = d+1$ everywhere on Ω. For each state ω of A, the function $\sum_i a_i(\omega)\lambda_i$ is a version of $d\Gamma(\omega)/ds$, and is therefore nonnegative off an s-null set. Taking the union of these null sets over a countable dense subset of Ω_A, and using that each a_i is continuous, we obtain an s-null set N_0 off which $\sum_i a_i(\omega)\lambda_i \geq 0$ for every state ω. Each function $\Theta(e) = \sum_j c_j(e)\Theta(e_j)$ takes values in $[0,1]$ everywhere. Hence the map $x = (\lambda_0, \dots, \lambda_d, \Theta(e_1), \dots, \Theta(e_m))$, redefined

on N_0 as a fixed point of X, is a measurable map $\Omega \to X$, and its image $\nu = x_* s$ is a Borel probability measure on X with

$$\int_X \lambda_i \, d\nu = 1, \qquad \int_X \lambda_i f_j \, d\nu = \int_\Omega \Theta(e_j) \, d\Gamma(v_i) = M_{ij}.$$

Carathéodory's theorem. The vector $p = \int_X \Psi \, d\nu$ lies in $\operatorname{conv} \Psi(X)$: otherwise a linear functional ℓ on $\mathbb{R}^D$ would satisfy $\ell(p) > \max_{\Psi(X)} \ell$, which is impossible because $\ell(p) = \int_X \ell \circ \Psi \, d\nu$. Since $\Psi(X)$ lies in an affine hyperplane of dimension $D-1$, Carathéodory's theorem writes p as a convex combination $\sum_{k=1}^K r_k \Psi(x_k)$ of $K \leq D$ points $x_k = (\lambda^k, f^k)$ of X. Comparing the last $d+1$ coordinates gives $\sum_k r_k \lambda_i^k = 1$ for every i, and comparing the others gives $\sum_k r_k \lambda_i^k f_j^k = M_{ij}$.

The compressed simulation. On the points $1, \dots, K$ of $\mathbb{N}$ define

$$\Gamma'(\omega)(k) = r_k \sum_i a_i(\omega) \lambda_i^k, \qquad \Theta'(e)(k) = \sum_j c_j(e) f_j^k,$$

and at the other points of $\mathbb{N}$ let $\Gamma'(\omega)$ vanish and let $\Theta'(e)$ take its value at the point 1. The conditions defining X give $\Gamma'(\omega) \geq 0$ and $\Theta'(e) \in [0,1]^{\mathbb{N}}$, and $\sum_k \Gamma'(\omega)(k) = \sum_i a_i(\omega) \sum_k r_k \lambda_i^k = 1$. The map Γ' is affine and Θ' is the restriction of a linear map with $\Theta'(0) = 0$, so both preserve mixtures. Finally $\Theta'(e) \cdot \Gamma'(\omega) = \sum_{i,j} a_i(\omega) c_j(e) \sum_k r_k \lambda_i^k f_j^k = \sum_{i,j} a_i(\omega) c_j(e) M_{ij} = \Theta(e) \cdot \Gamma(\omega)$. □

Theorem 3.2. *There is a univalent simulation of A by $\Delta_{\mathbb{N}}$, with states supported on a common set of at most D points of $\mathbb{N}$, whose error is equal to*

$$\varepsilon(A, \Delta_{\mathbb{N}}) = \varepsilon_\sigma(A) = \varepsilon(A, \Delta_k) \qquad (k \geq D).$$

Proof. Let F be the set of points of $\operatorname{conv} \Psi(X)$ whose last $d+1$ coordinates all equal 1, a compact set. By the first two steps of the proof of Proposition 3.1, the matrix of every univalent simulation of A by a countably additive classical system, followed by the vector $(1, \dots, 1)$, is a point of F; in particular F is nonempty. The continuous function err, applied to the first $(d+1)m$ coordinates, attains its minimum on F at a point p, and $\operatorname{err}(p) \leq \varepsilon_\sigma(A)$. The last two steps of the same proof, applied to p, give a univalent simulation of A by $\Delta_{\mathbb{N}}$, supported on at most D points, with matrix given by p and therefore with error $\operatorname{err}(p)$. Relabelling its support as a subset of $\{1, \dots, k\}$ gives a simulation by Δ_k with the same error, for every $k \geq D$. With the inequalities of Section 2, $\operatorname{err}(p) \leq \varepsilon_\sigma(A) \leq \varepsilon(A, \Delta_{\mathbb{N}}) \leq \varepsilon(A, \Delta_k) \leq \operatorname{err}(p)$. □

Corollary 3.3. *For every $k \geq D$, $\varepsilon(A, \Delta_k) = \varepsilon(A, \Delta_{\mathbb{N}})$. In particular the hypothesis of [1, Remark 18], that $\varepsilon(A, \Delta_m)$ is constant for all sufficiently large m, holds for every finite-dimensional system, and the classical excess can always be computed within Δ_D.*

Remark 3.4. The proofs in this section use the boundedness of Ω_A and E_A but neither their closedness nor a unit effect. If simulations are also required to send a unit effect 1_A of A to the constant function 1, the same proofs apply after two changes: add the closed condition $\sum_j c_j(1_A) f_j = 1$ to the definition of X, which every point $x(t)$ in the proof of Proposition 3.1 satisfies, and define $\Theta'(e)$ off the support of the compressed simulation as its value at a fixed atom, as above; no value of the simulated pairing changes. In the systems of [1] the effects separate the states and the states separate the effects. The first property gives an injective affine map from the affine hull of Ω_A into the dual of the span of E_A, whose image lies in the hyperplane of functionals taking the value 1 at the unit effect; this hyperplane misses 0, so the image spans a space of dimension $d+1$, and $m \geq d+1$. The second property embeds the span of E_A into the space of affine functions on the affine hull of Ω_A, which has dimension $d+1$, so $m \leq d+1$. Hence $m = d+1$, and $D = m(m+1)$.

4 Finite parts determine a single classical model

Simulations of finite subsystems may disagree on shared preparations and effects. An ultraproduct enforces each fixed finite collection of requirements, and the Loeb construction realizes the limiting preparation measures on a common measurable space. We first establish the transfer principle that makes this possible.

A *finitely generated subsystem* F of a system A has as states the convex hull of a nonempty finite set of states of A, as effects the convex hull of a finite set of effects of A containing 0, and as pairing the restriction of the pairing of A. Its mixtures are those of A among its generators. A finitely generated subsystem satisfies the hypotheses of Section 3, and the finitely generated subsystems of A form a set $\mathcal{G}(A)$ directed by inclusion of states and of effects.

We use ultraproducts over an ultrafilter $\mathcal{U}$ on a set I and the $\mathcal{U}$-limit $\lim_{\mathcal{U}} t_i$ of a bounded family of reals, the unique real t such that $\{i : |t_i - t| < \delta\} \in \mathcal{U}$ for every $\delta > 0$. For sets K_i, the ultraproduct $K^* = \prod_i K_i/\mathcal{U}$ is the set of families (x_i) with $x_i \in K_i$, two families being identified when they agree on a set of indices in $\mathcal{U}$; we write $[x_i]$ for the class of (x_i). A family (B_i) with $B_i \subseteq K_i$ determines the subset $[B_i] = \{[x_i] : \{i : x_i \in B_i\} \in \mathcal{U}\}$ of K^*; such subsets are called internal. The ultrafilter $\mathcal{U}$ is *countably incomplete* if there are sets $I_1 \supseteq I_2 \supseteq \cdots$ in $\mathcal{U}$ with empty intersection; every nonprincipal ultrafilter on $\mathbb{N}$ is countably incomplete, with $I_n = \{k : k \geq n\}$. For a countably incomplete ultrafilter, internal sets have the countable intersection property: if $[B_i^1], [B_i^2], \ldots$ are nonempty internal subsets of K^* with $[B_i^1] \supseteq [B_i^2] \supseteq \cdots$, their intersection is nonempty. Indeed, for each i let $n(i)$ be the largest n with $i \in I_n$ and $B_i^1 \cap \cdots \cap B_i^n \neq \emptyset$, or 0 if there is none, a finite number because no index lies in every I_n; choose x_i in that intersection, or arbitrarily if $n(i) = 0$. For each n, the set of $i \in I_n$ with $B_i^1 \cap \cdots \cap B_i^n \neq \emptyset$ lies in $\mathcal{U}$, and on it $n(i) \geq n$, so $[x_i] \in [B_i^n]$.

The following lemma is the only use of Loeb's construction in the paper. For finite signed measures, $\|\cdot\|$ denotes the total variation norm.

Lemma 4.1 (Loeb limits of classical models)**.** *Let A be a system and $\mathcal{U}$ a countably incomplete ultrafilter on a set I, and for each $i \in I$ let $(\Omega_i, \mathcal{F}_i)$ be a measurable space. Suppose that to each state ω of A and each $i \in I$ there is assigned a probability measure $\gamma_i(\omega)$ on $\mathcal{F}_i$, and to each effect e of A and each $i \in I$ an $\mathcal{F}_i$-measurable function $\theta_i(e) \colon \Omega_i \to [0,1]$, such that, for all states ω, ω', effects e, e' and reals $t \in [0,1]$,*

$$\lim_{\mathcal{U}} \|\gamma_i(t\omega + (1-t)\omega') - t\gamma_i(\omega) - (1-t)\gamma_i(\omega')\| = 0,$$
$$\lim_{\mathcal{U}} \|\theta_i(te + (1-t)e') - t\theta_i(e) - (1-t)\theta_i(e')\|_\infty = 0, \qquad \lim_{\mathcal{U}} \|\theta_i(0)\|_\infty = 0.$$

Then there are a measurable space $(\Omega^*, \mathcal{F})$ and a univalent simulation (Γ, Θ) of A by $\Delta_{(\Omega^*, \mathcal{F})}$ such that

$$\Theta(e) \cdot \Gamma(\omega) = \lim_{\mathcal{U}} \theta_i(e) \cdot \gamma_i(\omega)$$

for every state ω and every effect e of A.

Proof. Let $\Omega^* = \prod_i \Omega_i/\mathcal{U}$, and let $\mathcal{F}^*$ be the algebra of internal sets $[B_i]$ with $B_i \in \mathcal{F}_i$ for every i. For each state ω, the formula $\gamma_\omega([B_i]) = \lim_{\mathcal{U}} \gamma_i(\omega)(B_i)$ defines a finitely additive probability on $\mathcal{F}^*$; it does not depend on the representative, because $[B_i] = [C_i]$ implies $B_i = C_i$ for a set of indices in $\mathcal{U}$. It is countably additive on $\mathcal{F}^*$: if a member $[B]$ of $\mathcal{F}^*$ is the union of a disjoint sequence of members $[B^1], [B^2], \ldots$, the internal sets $[B] \setminus ([B^1] \cup \cdots \cup [B^n])$ decrease to the empty set, so by the countable intersection

property one of them is empty, and all but finitely many of the sets $[B^n]$ are empty. By Carathéodory's extension theorem, which in this setting is Loeb's construction [2], γ_ω extends uniquely to a countably additive probability measure $\Gamma(\omega)$ on the σ-algebra $\mathcal{F}$ generated by $\mathcal{F}^*$, a σ-algebra that does not depend on ω.

For each effect e define $\Theta(e)([x_i]) = \lim_{\mathcal{U}} \theta_i(e)(x_i)$, a function $\Omega^* \to [0,1]$. It is $\mathcal{F}$-measurable, because $\{\Theta(e) > s\} = \bigcup_n [\{\theta_i(e) > s + 1/n\}]$, and approximating each $\theta_i(e)$ uniformly by functions with values in a fixed finite grid gives $\int \Theta(e)\, d\Gamma(\omega) = \lim_{\mathcal{U}} \theta_i(e) \cdot \gamma_i(\omega)$.

By the first hypothesis, the measures $\Gamma(t\omega + (1-t)\omega')$ and $t\Gamma(\omega) + (1-t)\Gamma(\omega')$ agree on $\mathcal{F}^*$, which is closed under finite intersections and generates $\mathcal{F}$, so they agree on $\mathcal{F}$. By the other two hypotheses, Θ preserves mixtures pointwise and $\Theta(0) = 0$. □

When the hypotheses hold exactly, for a set of indices in $\mathcal{U}$, they hold in the limit form stated. When the spaces Ω_i are finite, Ω^* is a hyperfinite set, and the simulation of the lemma is in the most literal sense a limit of finite classical models.

The lemma is used through the notion of a logical limit of systems, which applies to families indexed by an arbitrary set.

Definition 4.2. Let $(A_i)_{i \in I}$ be systems and $\mathcal{U}$ a countably incomplete ultrafilter on I. A system A is a *logical limit* of the systems A_i along $\mathcal{U}$ if there are maps $\omega \mapsto [\omega_i]$ from Ω_A into $\prod_i \Omega_{A_i}/\mathcal{U}$ and $e \mapsto [e_i]$ from E_A into $\prod_i E_{A_i}/\mathcal{U}$ that preserve mixtures with real coefficients, send 0 to [0], and satisfy $e \cdot \omega = \lim_{\mathcal{U}} e_i \cdot \omega_i$ for every state ω and effect e of A.

Theorem 4.3. *Let A be a logical limit of systems $(A_i)_{i \in I}$ along a countably incomplete ultrafilter $\mathcal{U}$. If each A_i has a univalent simulation with error at most $r_i \in [0,1]$ by a countably additive classical system, then A has a univalent simulation with error at most $\lim_{\mathcal{U}} r_i$ by a countably additive classical system. In particular,*

$$\varepsilon_\sigma(A) \leq \lim_{\mathcal{U}} \varepsilon_\sigma(A_i).$$

Proof. Let (Γ_i, Θ_i) be the given simulations, fix representatives (ω_i) and (e_i) of the images of the states and effects of A, and put $\gamma_i(\omega) = \Gamma_i(\omega_i)$ and $\theta_i(e) = \Theta_i(e_i)$. For each fixed mixture identity and for the null-effect identity, the maps of Definition 4.2 enforce the corresponding equality on a set of indices in $\mathcal{U}$. On that set the simulations preserve the equality, so each norm defect in Lemma 4.1 tends to zero along $\mathcal{U}$. The lemma gives a univalent simulation (Γ, Θ) of A by a countably additive classical system with

$$|e \cdot \omega - \Theta(e) \cdot \Gamma(\omega)| = \lim_{\mathcal{U}} |e_i \cdot \omega_i - \Theta_i(e_i) \cdot \Gamma_i(\omega_i)| \leq \lim_{\mathcal{U}} r_i.$$

For the last assertion, let $I_1 \supseteq I_2 \supseteq \cdots$ be sets in $\mathcal{U}$ with empty intersection, let $\delta_i = 1/n$ for $i \in I_n \setminus I_{n+1}$ and $\delta_i = 1$ for $i \notin I_1$, so that $\lim_{\mathcal{U}} \delta_i = 0$, and apply the first assertion with $r_i = \min\{1, \varepsilon_\sigma(A_i) + \delta_i\}$. □

Every system A is a logical limit of its finitely generated subsystems. Let I be the set of pairs (F, k) with $F \in \mathcal{G}(A)$ and k a positive integer. The sets $\{(F,k) : F \supseteq F_0,\ k \geq k_0\}$, for $F_0 \in \mathcal{G}(A)$ and $k_0 \geq 1$, have the finite intersection property because $\mathcal{G}(A)$ is directed; let $\mathcal{U}$ be an ultrafilter on I containing all of them, which is countably incomplete because the sets $\{(F,k) : k \geq n\}$ belong to $\mathcal{U}$ and have empty intersection. For $i = (F,k)$ put $A_i = F$, and send a state ω of A to the class of the family whose term at (F,k) is ω when ω is a state of F and a fixed state of F otherwise, and an effect e to the class of the family whose term is e when e is an effect of F and 0 otherwise. Finitely many states

and effects of A all belong to every finitely generated subsystem containing a fixed one, so every mixture identity, the identity $0 \mapsto [0]$ and every equality $e \cdot \omega = e_i \cdot \omega_i$ hold for a set of indices in $\mathcal{U}$.

Proof of Theorem A. Suppose (i). The given finite simulations are countably additive and each has error at most $\min\{r, 1\}$. Apply Theorem 4.3 to A as the logical limit of its finitely generated subsystems. This gives a countably additive classical model with error at most r on the hyperfinite sample space $\prod_i K_i/\mathcal{U}$, where K_i is the sample space of the simulation of the ith subsystem. Suppose (ii). The restriction of the simulation to a finitely generated subsystem F has error at most r, and Proposition 3.1 replaces it by a simulation of F with the same simulated pairing on at most D points. □

Corollary 4.4. *For every system A,*

$$\varepsilon_\sigma(A) = \sup_{F \in \mathcal{G}(A)} \varepsilon(F, \Delta_{\mathbb{N}}),$$

and there is a univalent simulation of A by a countably additive classical system whose error equals $\varepsilon_\sigma(A)$.

Proof. By Theorem 3.2, a finitely generated subsystem F has a simulation with error at most r by a classical system with finitely many outcomes exactly when $r \geq \varepsilon(F, \Delta_{\mathbb{N}})$. By Theorem A, the reals r for which A has a univalent simulation with error at most r by a countably additive classical system are therefore exactly those with $r \geq \sup_F \varepsilon(F, \Delta_{\mathbb{N}})$, and the supremum is one of them. □

A system A is the *increasing union* of systems $A_1, A_2, \ldots$ if all the state spaces lie in one real vector space and all the effect spaces in another, with $\Omega_{A_n} \subseteq \Omega_{A_{n+1}}$ and $E_{A_n} \subseteq E_{A_{n+1}}$, $\Omega_A = \bigcup_n \Omega_{A_n}$ and $E_A = \bigcup_n E_{A_n}$, and each pairing is the restriction of the pairing of A.

Corollary 4.5. *If A is the increasing union of systems $A_1, A_2, \ldots$, then the sequence $\varepsilon_\sigma(A_n)$ is nondecreasing and $\varepsilon_\sigma(A) = \lim_{n\to\infty} \varepsilon_\sigma(A_n)$. If moreover each A_n is finite-dimensional, as in Section 3, then $\varepsilon_\sigma(A) = \lim_{n\to\infty} \varepsilon(A_n, \Delta_{\mathbb{N}})$.*

Proof. Restricting a simulation of A, or of A_{n+1}, to the states and effects of A_n gives a simulation of A_n with no larger error, so the sequence $\varepsilon_\sigma(A_n)$ is nondecreasing and bounded by $\varepsilon_\sigma(A)$. Every finitely generated subsystem of A is a finitely generated subsystem of some A_n, so $\varepsilon_\sigma(A) \leq \sup_n \varepsilon_\sigma(A_n)$ by Corollary 4.4. The last assertion follows from Theorem 3.2, by which $\varepsilon_\sigma(A_n) = \varepsilon(A_n, \Delta_{\mathbb{N}})$. □

Within a classical system with finitely many outcomes the infimum of the errors is attained for a simpler reason, the compactness of the space of simulations itself.

Proposition 4.6. *For every system A and every positive integer k, there is a univalent simulation of A by Δ_k whose error equals $\varepsilon(A, \Delta_k)$.*

Proof. Let P_k be the simplex of probability measures on $\{1, \ldots, k\}$. The univalent simulations of A by Δ_k form a subset $\mathcal{S}$ of the product $P_k^{\Omega_A} \times ([0,1]^k)^{E_A}$, which is compact by Tychonoff's theorem. Preservation of mixtures and the condition $\Theta(0) = 0$ are equations between finitely many coordinates, so $\mathcal{S}$ is closed, hence compact, and it is nonempty. For fixed ω and e, the function $(\Gamma, \Theta) \mapsto |e \cdot \omega - \Theta(e) \cdot \Gamma(\omega)|$ is continuous on $\mathcal{S}$, so the error, the supremum of these functions, is lower semicontinuous, and it attains its infimum on the compact set $\mathcal{S}$. □

The optimal model of Corollary 4.4 lives on an ultraproduct, but for systems with countably many independent effects it can be moved to a compact metrizable space.

Proposition 4.7. *Let A be a system whose effects span a vector space of finite or countable dimension. Then there is a univalent simulation of A with error $\varepsilon_\sigma(A)$ by a classical system $\Delta_{(Y,\mathcal{B})}$, where Y is a compact subset of the Hilbert cube $[0,1]^{\mathbb{N}}$ and $\mathcal{B}$ is its Borel σ-algebra.*

Proof. Let (Γ, Θ) be a univalent simulation of A by $\Delta_{(\Omega,\mathcal{F})}$ with error $\varepsilon_\sigma(A)$, given by Corollary 4.4. Choose effects $e_1, e_2, \dots$ of A forming a basis of the span of E_A; if the basis is finite, or empty, set $\Theta(e_j) = 0$ for the missing indices. By Lemma 2.1, each $\Theta(e)$ is the finite combination $\sum_j c_j(e)\Theta(e_j)$, where $c_j(e)$ are the coordinates of e in this basis. Let $\Phi = (\Theta(e_1), \Theta(e_2), \dots)\colon \Omega \to [0,1]^{\mathbb{N}}$, a measurable map, and let Y be the closure of $\Phi(\Omega)$, a compact metrizable space. For an effect e the function $\Theta'(e) = \sum_j c_j(e) z_j$, where z_j are the coordinate functions, is continuous on Y; it satisfies $\Theta'(e) \circ \Phi = \Theta(e)$, so it takes values in $[0,1]$ on $\Phi(\Omega)$ and, by continuity, on Y. Put $\Gamma'(\omega) = \Phi_*\Gamma(\omega)$, a Borel probability measure on Y. The maps Γ' and Θ' preserve mixtures, $\Theta'(0) = 0$, and $\int_Y \Theta'(e)\, d\Gamma'(\omega) = \int_\Omega \Theta(e)\, d\Gamma(\omega)$, so (Γ', Θ') has the same simulated pairing as (Γ, Θ). □

Proposition 4.7 applies to every increasing union of finite-dimensional systems.

5 The uniformization fails for the countable classical system

The fair-coin preparation on Cantor space can be decomposed into arbitrarily many conditional laws on disjoint cylinders. The first example tests whether a countable model can distinguish all of them. The second keeps the same preparations and restricts the effects so that their variation within small cylinders tends uniformly to zero.

Let $K = \{0,1\}^{\mathbb{N}}$ be Cantor space, with the fair-coin probability p and the ultrametric ρ for which two distinct sequences whose longest common initial segment has length j are at distance 2^{-j}. For an integer $n \geq 0$ let $\mathcal{P}_n$ be the partition of K into the 2^n cylinders of depth n; a cylinder of depth n has diameter at most 2^{-n}, and a point of it and a point outside it are at distance at least $2^{-(n-1)}$. For $C \in \mathcal{P}_n$ let p_C be the conditional law of p on C. Let A_n be the system whose states are the convex combinations of the laws p_C with $C \in \mathcal{P}_n$, whose effects are the functions $K \to [0,1]$ that are constant on each member of $\mathcal{P}_n$, and whose pairing is integration; it is the classical simplex Δ_{2^n} in another guise. The systems A_n increase, because the law of a cylinder is the average of the laws of its two halves and a function constant on the cylinders of depth n is constant on those of depth $n+1$. Let A be their increasing union. Let A'_n be the system with the states of A_n whose effects are the effects of A_n that are 1-Lipschitz for ρ, and let A' be the increasing union of the systems A'_n. Each A'_n is simulated exactly by Δ_{2^n}, and the inclusions simulate A and A' exactly by the classical system of Borel probability measures on K, so that $\varepsilon_\sigma(A) = \varepsilon_\sigma(A') = 0$.

Lemma 5.1. *Let (Γ, Θ) be a univalent simulation of A' by $\Delta_{\mathbb{N}}$, let $n \geq 0$ and $N = 2^n$, and for each $C \in \mathcal{P}_n$ let $q_C \in [0,\infty)^{\mathbb{N}}$ satisfy $\sum_C q_C \leq 1$ pointwise. Then for every finite set $J \subseteq \mathbb{N}$,*

$$\frac{1}{N} \sum_{C \in \mathcal{P}_n} q_C \cdot \Gamma(p_C) \leq \frac{|J|}{N} + \Gamma(p)(\mathbb{N} \setminus J).$$

The same holds for simulations of A.

Proof. Since p is the average of the laws p_C, the measure $\Gamma(p)$ is the average of the measures $\Gamma(p_C)$. At a point $j \in J$, each $\Gamma(p_C)(j)$ is at most 1 and $\sum_C q_C(j) \leq 1$, so these points

contribute at most $|J|/N$. At a point $j \notin J$, each $q_C(j)$ is at most 1, so these points contribute at most $\frac{1}{N}\sum_C \Gamma(p_C)(\mathbb{N} \setminus J) = \Gamma(p)(\mathbb{N} \setminus J)$. □

Proof of Theorem B. Part (1). Every finitely generated subsystem of A is a subsystem of some stage A_n, which is simulated exactly by Δ_{2^n}, so its classical excess is 0. Let (Γ, Θ) be a univalent ϵ-simulation of A by $\Delta_{\mathbb{N}}$, let $n \geq 0$, and put $q_C = \Theta(1_C)$ for $C \in \mathcal{P}_n$. The indicators 1_C are effects of A whose average is the effect $2^{-n} \cdot 1$, so by Lemma 2.1 $\sum_C q_C = \Theta(1) \leq 1$. Each $q_C \cdot \Gamma(p_C)$ is at least $1 - \epsilon$, because $1_C \cdot p_C = 1$, so by Lemma 5.1 $1 - \epsilon \leq 2^{-n}|J| + \Gamma(p)(\mathbb{N} \setminus J)$ for every finite set J. Choosing J with $\Gamma(p)(\mathbb{N} \setminus J) < \eta$ and then n with $2^{-n}|J| < \eta$ gives $\epsilon \geq 1 - 2\eta$ for every $\eta > 0$, so $\epsilon \geq 1$; the upper bound 1 holds for every simulation.

Part (2)*, approximate simulations.* Every finitely generated subsystem of A' is a subsystem of some stage A'_n, which is simulated exactly by Δ_{2^n}, so its classical excess is 0. Let $j \geq 0$, identify $\mathcal{P}_j$ with 2^j points of $\mathbb{N}$, and choose a point x_C in each $C \in \mathcal{P}_j$. Put $\Gamma_j(\mu)(C) = \mu(C)$ and $\Theta_j(e)(C) = e(x_C)$. The map Γ_j is affine, Θ_j is the restriction of a linear map with $\Theta_j(0) = 0$, and $\Theta_j(e)$ takes values in $[0, 1]$. Since each $C \in \mathcal{P}_j$ has diameter at most 2^{-j} and every effect of A' is 1-Lipschitz,

$$\Big|\int e\, d\mu - \sum_C \mu(C)e(x_C)\Big| \leq \sum_C \int_C |e - e(x_C)|\, d\mu \leq 2^{-j}$$

for every state μ and effect e of A', so $\varepsilon(A', \Delta_{2^j}) \leq 2^{-j}$ and $\varepsilon(A', \Delta_{\mathbb{N}}) = 0$.

Part (2)*, no exact simulation.* Suppose that (Γ, Θ) is a univalent simulation of A' by $\Delta_{\mathbb{N}}$ with error 0. Let $n \geq 0$ and $c = 2^{-n}$. For $C \in \mathcal{P}_n$ the function $c1_C$ is constant on the cylinders of depth n and is 1-Lipschitz, because a point of C and a point outside C are at distance at least $2^{-(n-1)}$; so it is an effect of A', and so is $c \cdot 1$, the sum of these functions. By Lemma 2.1, the sequences $q_C = \Theta(c1_C)/c$ are nonnegative and $\sum_C q_C = \Theta(1) \leq 1$. Exactness gives $q_C \cdot \Gamma(p_C) = c^{-1}(c1_C) \cdot p_C = 1$ for every C, and Lemma 5.1 gives $1 \leq 2^{-n}|J| + \Gamma(p)(\mathbb{N} \setminus J)$ for every n and every finite set J, which is impossible. □

By Proposition 4.6, each $\varepsilon(A', \Delta_k)$ is attained, and it is strictly positive, since an optimal simulation by Δ_k is a simulation by $\Delta_{\mathbb{N}}$; the classical excess 0 of A' is the limit of these strictly positive minima. In the language of Corollary 4.5, the classical excess is not continuous along increasing unions, whereas ε_σ is.

Remark 5.2. Let K be an uncountable set and $\mathcal{F}$ a σ-algebra of subsets of K containing the singletons. The classical system $\Delta_{(K,\mathcal{F})}$, which is noncontextual, has classical excess 1. Indeed, let (Γ, Θ) be a univalent simulation of it by $\Delta_{\mathbb{N}}$ and fix a point n of $\mathbb{N}$. For distinct points $x_1, \ldots, x_r$ of K, the indicator of $\{x_1, \ldots, x_r\}$ is an effect equal to $\sum_i 1_{\{x_i\}}$, so by Lemma 2.1 the nonnegative numbers $\Theta(1_{\{x_i\}})(n)$ have sum at most 1, and $\Theta(1_{\{x\}})(n) > 0$ for at most countably many x. Taking the union over n, we find a point x with $\Theta(1_{\{x\}}) = 0$, while $1_{\{x\}} \cdot \delta_x = 1$.

In finite dimension the excesses within finite targets eventually equal the countable excess by Corollary 3.3. In infinite dimension they can fail even to converge to that value, as the countable classical system itself shows.

Example 5.3. For every positive integer k, $\varepsilon(\Delta_{\mathbb{N}}, \Delta_k) = 1$, whereas $\varepsilon(\Delta_{\mathbb{N}}, \Delta_{\mathbb{N}}) = 0$.

Proof. Let (Γ, Θ) be a univalent ϵ-simulation of $\Delta_{\mathbb{N}}$ by Δ_k, and let N be a positive integer. The indicator of $\{1, \ldots, N\}$ is an effect equal to $\sum_{n \leq N} 1_{\{n\}}$, so by Lemma 2.1 the responses

$\Theta(1_{\{n\}})$, for $n \le N$, have sum at most 1 at each of the k points. Since every coordinate of every $\Gamma(\delta_n)$ is at most 1,

$$N(1-\epsilon) \le \sum_{n \le N} \Theta(1_{\{n\}}) \cdot \Gamma(\delta_n) \le \sum_{j=1}^{k} \sum_{n \le N} \Theta(1_{\{n\}})(j) \le k.$$

Since N is arbitrary, $\epsilon \ge 1$. □

6 Realized and omitted types in dual pairs

Theorem A, and the failure of its analogue for the countable target, have a common logical form. We describe it in the framework of dual pairs, which is due to Eduardo Dueñez [3]. Only the elementary features described here are used, and the general theory is not needed.

A *dual pair* is a structure with a sort V of bounded random variables, playing the role of effects, a sort S of functionals on V, playing the role of states, a pairing $\langle f, \mu \rangle \in [0,1]$ that is 1-Lipschitz in each argument, convex combinations in each sort, and a sort N of *normal* states, interpreted inside S, over which one may quantify. No measure is fixed in advance, as it is in a randomization; each state is a point of the sort S. The *standard dual pair* is

$$\mathcal{C} = \big([0,1]^{\mathbb{N}},\ \text{the states of } \ell^\infty\big), \qquad N = \text{the probability measures on } \mathbb{N},$$

with the ℓ^1 norm on N. Every state of ℓ^∞ is uniquely the sum of a countably additive positive functional, given by a summable sequence, and a purely finitely additive one, which vanishes on every finitely supported sequence [4]; either part may be zero, and N consists of the states whose purely finitely additive part is zero. The system $\Delta_{\mathbb{N}}$ has as states the points of N; the larger sort S belongs to the logical structure, not to the system.

Finite dimension. Let A be as in Section 3, with the coordinates fixed there. For $\epsilon \ge 0$, let Σ_ϵ be the set of conditions, in variables $\mu_0, \dots, \mu_d$ of the sort N and $f_1, \dots, f_m$ of the sort V,

$$P(\bar{\mu}) = \sup_{\omega \in \Omega_A} \sup_{g \in V} \Big(-\sum_i a_i(\omega) \langle g, \mu_i \rangle \Big)^+ = 0,$$

$$R(\bar{f}) = \sup_{e \in E_A} \sup_{\nu \in N} \operatorname{dist}\Big(\sum_j c_j(e) \langle f_j, \nu \rangle, [0,1] \Big) = 0,$$

$$\mathrm{Err}(\bar{\mu}, \bar{f}) = \sup_{\omega \in \Omega_A,\, e \in E_A} \Big| e \cdot \omega - \sum_{i,j} a_i(\omega) c_j(e) \langle f_j, \mu_i \rangle \Big| \le \epsilon.$$

The suprema over $g \in V$ and $\nu \in N$ are quantifiers of the language, while the suprema over ω and e range over parameters: the numbers $a_i(\omega)$, $c_j(e)$ and $e \cdot \omega$ enter as real constants through continuous connectives. The suprema over ω and e are suprema over the parameter vectors $(a(\omega), c(e), e \cdot \omega)$, which range over a bounded subset of $\mathbb{R}^{d+m+2}$; since the pairing takes values in $[0,1]$, the expressions inside these suprema are uniformly continuous functions of the parameter vector, in every model of the theory of $\mathcal{C}$, and total boundedness makes each supremum a uniform limit of maxima over finite sets of parameters. Hence P, R and Err are definable predicates, and their values are preserved by elementary extensions of $\mathcal{C}$. In $\mathcal{C}$ a tuple satisfies Σ_ϵ exactly when it is the tuple $(\Gamma(v_i))_i, (\Theta(e_j))_j$ of a univalent simulation (Γ, Θ) of A by $\Delta_{\mathbb{N}}$ with error at most ϵ: the condition $P = 0$ makes each $\sum_i a_i(\omega)\mu_i$ a probability measure, the condition $R = 0$, tested on point masses, makes each $\sum_j c_j(e) f_j$ a point of $[0,1]^{\mathbb{N}}$, and Lemma 2.1 gives the converse.

Proposition 6.1. *Let A be as in Section 3, let $\epsilon \geq 0$, and let $\mathcal{C}^*$ be an elementary extension of $\mathcal{C}$. The following are equivalent:*

(i) $\epsilon \geq \varepsilon(A, \Delta_{\mathbb{N}})$;

(ii) *the set of conditions Σ_ϵ is realized in $\mathcal{C}^*$;*

(iii) *the set of conditions Σ_ϵ is realized in $\mathcal{C}$.*

Proof. That (iii) implies (ii) holds because $\mathcal{C}$ is an elementary substructure of $\mathcal{C}^*$, that (iii) implies (i) holds by the characterization of the tuples satisfying Σ_ϵ in $\mathcal{C}$, and that (i) implies (iii) follows from Theorem 3.2.

Suppose (ii). The sentence $\inf_{\bar{\mu},\bar{f}} \max(P, R, (\mathrm{Err} - \epsilon)^+)$ has value 0 in $\mathcal{C}^*$, hence in $\mathcal{C}$, so for each $k \geq 1$ there is a tuple $(\bar{\mu}^k, \bar{f}^k)$ in $\mathcal{C}$ with $P \leq 1/k$, $R \leq 1/k$ and $\mathrm{Err} \leq \epsilon + 1/k$. Let $\tilde{X}$ be the compact set defined like X but without the condition $\sum_i a_i(\omega)\lambda_i \geq 0$ and with the range condition relaxed to $-1 \leq \sum_j c_j(e) f_j \leq 2$. The construction in the first step of the proof of Proposition 3.1, applied to $(\bar{\mu}^k, \bar{f}^k)$, gives a probability measure ν_k on $\tilde{X}$ with $\int \lambda_i \, d\nu_k = 1$ for every i, whose moment vector $\int \Psi \, d\nu_k$ has error at most $\epsilon + 1/k$, whose atoms satisfy the range condition up to $1/k$, and such that $\int (\sum_i a_i(\omega)\lambda_i)^- \, d\nu_k \leq 1/k$ for every state ω, this integral being the mass of the negative part of $\sum_i a_i(\omega)\mu_i^k$. A subsequence of (ν_k) converges weak* to a probability measure ν on the compact metrizable space $\tilde{X}$, and every integrand above is continuous. Hence $\int \lambda_i \, d\nu = 1$, the moment vector of ν has error at most ϵ, the range condition holds exactly on the support of ν, by the portmanteau theorem applied to the closed sets on which it holds up to $1/k$, and each continuous nonnegative function $(\sum_i a_i(\omega)\lambda_i)^-$ has ν-integral 0 and so vanishes on the support of ν. The support of ν is therefore contained in X, and the Carathéodory step of the proof of Proposition 3.1 gives a simulation with error at most ϵ, whose tuple realizes Σ_ϵ in $\mathcal{C}$. □

Arbitrary systems. The finite-dimensional formulation uses basis coordinates to express approximate admissibility. For an arbitrary system we instead assign a variable to each preparation and effect and impose their mixture identities directly. For a system A and $\epsilon \geq 0$, let $\Sigma_\epsilon(A)$ be the set of conditions, in variables μ_ω of the sort N for $\omega \in \Omega_A$ and f_e of the sort V for $e \in E_A$, which say that

$$\mu_{t\omega+(1-t)\omega'} = t\mu_\omega + (1-t)\mu_{\omega'}, \quad f_{te+(1-t)e'} = tf_e + (1-t)f_{e'}, \quad f_0 = 0, \quad |e\cdot\omega - \langle f_e, \mu_\omega\rangle| \leq \epsilon,$$

for all states ω, ω', effects e, e' and reals $t \in [0,1]$. An equation $\mu = \mu'$ between states is expressed by the condition $\sup_{g \in V} |\langle g, \mu\rangle - \langle g, \mu'\rangle| = 0$, and an equation $f = f'$ between effects by $\sup_{\nu \in N} |\langle f, \nu\rangle - \langle f', \nu\rangle| = 0$. In $\mathcal{C}$ a realization of $\Sigma_\epsilon(A)$ is exactly a univalent simulation of A by $\Delta_{\mathbb{N}}$ with error at most ϵ, since in $\mathcal{C}$ the effects are tested by the point masses. Only the sorts V and N occur in these conditions; the sort S is not used. Ultraproducts, internal sets and countably incomplete ultrafilters are as in Section 4.

Theorem 6.2. *Let A be a system and $\epsilon \geq 0$. The following are equivalent:*

(i) $\epsilon \geq \varepsilon_\sigma(A)$;

(ii) *the set of conditions $\Sigma_\epsilon(A)$ is finitely approximately satisfiable in $\mathcal{C}$;*

(iii) *the set of conditions $\Sigma_\epsilon(A)$ is realized in an ultrapower of $\mathcal{C}$ over a countably incomplete ultrafilter.*

Moreover, $\Sigma_\epsilon(A)$ is realized in $\mathcal{C}$ if and only if A has a univalent simulation by $\Delta_{\mathbb{N}}$ with error at most ϵ.

Proof. The last assertion was observed above.

Suppose (i), and let Φ be a finite subset of $\Sigma_\epsilon(A)$. By Corollary 4.4 there is a univalent simulation of A by a countably additive classical system with error at most ϵ. Let F be the subsystem generated by the states and effects whose variables occur in Φ, together with the zero effect and, if no state variable occurs, one arbitrary state of A. Every mixture named in Φ belongs to F. By Proposition 3.1, the restriction of the simulation to F may be replaced by a univalent simulation of F by $\Delta_{\mathbb{N}}$ with the same simulated pairing, and its values at the states and effects occurring in Φ satisfy every condition of Φ exactly in $\mathcal{C}$, which gives (ii).

Suppose (ii). Let I be the set of pairs (Φ, n) with Φ a finite subset of $\Sigma_\epsilon(A)$ and n a positive integer, and for each (Φ, n) choose a tuple in $\mathcal{C}$ that satisfies every condition of Φ up to $1/n$, arbitrary in the variables that do not occur in Φ. The sets $\{(\Phi', n') : \Phi' \supseteq \Phi,\ n' \geq n\}$ have the finite intersection property; let $\mathcal{U}$ be an ultrafilter on I containing all of them, which is countably incomplete because the sets $\{(\Phi', n') : n' \geq n\}$ belong to $\mathcal{U}$ and have empty intersection. By Łoś's theorem for metric structures [5], the class of the chosen tuples realizes $\Sigma_\epsilon(A)$ in $\mathcal{C}^I/\mathcal{U}$.

Suppose (iii), with a realization (μ^*_ω, f^*_e) in $\mathcal{C}^I/\mathcal{U}$, and choose families $(\mu_{\omega,i})$ of probability measures on $\mathbb{N}$ and $(f_{e,i})$ of points of $[0,1]^{\mathbb{N}}$ representing them; two families represent the same element when their distance tends to 0 along $\mathcal{U}$. For probability measures μ, μ' on $\mathbb{N}$, the supremum of $|\langle g, \mu\rangle - \langle g, \mu'\rangle|$ over $g \in [0,1]^{\mathbb{N}}$ is half the ℓ^1 distance between them, and for $f, f' \in [0,1]^{\mathbb{N}}$ the supremum of $|\langle f, \nu\rangle - \langle f', \nu\rangle|$ over the point masses ν is $\|f - f'\|_\infty$; by Łoś's theorem, the equations of $\Sigma_\epsilon(A)$ therefore say that the corresponding distances between representatives tend to 0 along $\mathcal{U}$. These are the hypotheses of Lemma 4.1, with $\Omega_i = \mathbb{N}$, $\gamma_i(\omega) = \mu_{\omega,i}$ and $\theta_i(e) = f_{e,i}$, and the lemma gives a univalent simulation (Γ, Θ) of A by a countably additive classical system with $\Theta(e)\cdot\Gamma(\omega) = \lim_{\mathcal{U}} \langle f_{e,i}, \mu_{\omega,i}\rangle = \langle f^*_e, \mu^*_\omega\rangle$. Its error is at most ϵ, and (i) holds. □

Theorem 6.2 is Theorem A in logical form. The ultrafilter on the pairs (Φ, n) in the proof of (ii)⇒(iii) plays the role of the ultrafilter on the pairs (F, k) that presents A as the logical limit of its finitely generated subsystems, Łoś's theorem takes the place of the verification of each requirement on all sufficiently large subsystems, and both proofs end with Lemma 4.1.

Corollary 6.3. *For the system A' of Theorem B, the set of conditions $\Sigma_0(A')$ is finitely approximately satisfiable in $\mathcal{C}$ and is omitted by $\mathcal{C}$. For the system A of Theorem B, and for the classical system $\Delta_{(K,\mathcal{F})}$ of an uncountable set K with a σ-algebra $\mathcal{F}$ containing its singletons, every set $\Sigma_\epsilon(A)$ and $\Sigma_\epsilon(\Delta_{(K,\mathcal{F})})$ with $\epsilon < 1$ is finitely approximately satisfiable in $\mathcal{C}$ and is omitted by $\mathcal{C}$.*

Proof. The three systems have exact countably additive models, so $\varepsilon_\sigma = 0$ for each of them, and Theorem 6.2 gives the approximate satisfiability. The omission follows from the last assertion of Theorem 6.2, with Theorem B and Remark 5.2. □

Acknowledgments

The framework of dual pairs used in Section 6 is due to Eduardo Dueñez.

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