TODORČEVIĆ'S TRICHOTOMY FOR QUANTUM DEEP COMPUTATIONS

JOSÉ IOVINO

Abstract. In this paper we extend the classification of deep computations by Todorčević's trichotomy from classical algorithms to quantum channels observed through a bipartite observable. The answers of the chosen probes need not determine what the same probes report after a channel acts, and we therefore close the observations under the channels. The resulting orbit types are the least enrichment on which every channel acts continuously, and on them the extendibility axiom of the classical theory becomes a theorem. The classification then depends on the observations. Local and comparison probes give the same quantum walk different levels; measure-and-prepare instruments on a binary tree realize every class of the trichotomy through families equivalent to canonical ones; and one additional probe can restore first countability. Idealized inputs separate limits that agree on every actual preparation. For countable families without the independence property (NIP) that preserve a bound on the expected energy and are closed under rational mixtures, convexity makes the compactum of limits metrizable whenever it is hereditarily separable. For arbitrary families of probes and channels, the NIP characterizes universal Monte Carlo computability; for countable families, it yields a law of large numbers for idealized random preparations, uniform over the computations at each output coordinate.

Introduction

In this paper we extend the classification of deep computations by Todorčević's trichotomy from classical algorithms to quantum channels, and we show how the observations through which a computation is described govern the complexity of its limits. The continuous action on types that the classical theory requires emerges here from closing the observations under composition with the channels.

Dueñez, Matos-Wiederhold, Salvetti and Tall [13] developed the classical classification for compositional computation structures, whose states are observed through real-valued features. A type records the values of these features, and the theory assumes that the computations extend continuously to the resulting type shards. We call the pointwise limits of these extended maps deep computations; in [13, Section 2.2] they are called ultracomputations. Iteration of one map is one source of such limits, and the definition applies to whole families of computations. For a countable family without the independence property (with the NIP), observed through countably many features, the theorem of Bourgain, Fremlin and Talagrand [6] places the limits on each type shard in a separable Rosenthal compactum. Todorčević's trichotomy [29] then distinguishes these compacta by metrizability, hereditary separability and first countability, and the classical algorithms of [13] realize every alternative.

To observe a quantum channel, Alice prepares her system in a state ρ, the channel Φ acts on it, and Bob evaluates the output with a probe σ through the expectation

$$\varphi_C(\Phi(\rho), \sigma) = \mathrm{Tr}\big((\Phi(\rho) \otimes \sigma)\, C\big)$$

Date: November 17, 2026.

of a fixed bipartite observable C [10, 11, 12]. Here ρ and σ are density operators, and Bob's probes serve as parameters of the binary predicate φ_C. Before the channel acts, the φ-type of ρ records the values $\varphi_C(\rho, \sigma)$ over a chosen countable set of probes.

These values need not determine what the same probes report after a channel has acted. The output observations involve the formulas $\varphi_C(\Phi(x), y)$, and two states with the same φ-type may give them different values. Proposition 2.8 gives an exact criterion: a channel acts on φ-types if and only if its dual preserves the norm-closed real linear span of the identity and the slices of C, the observables on Alice's system that Bob's probes read. For a channel that fails this condition, no action on φ-types is defined, and the observed formulas must be enlarged.

Orbit types. We enlarge the observations by recording the result of every channel followed by every probe. For a monoid Γ of computations, the orbit type of a preparation ρ records the values $\varphi_C(\Phi(\rho), \sigma)$, as Φ ranges over Γ and σ over the chosen probes (Definition 2.5). The enlargement is minimal: equality of orbit types is the coarsest refinement of equality of φ-types that the channels respect (Proposition 2.9). A channel now acts on the record by shifting its coordinates, continuously and with no further hypothesis, and the extendibility axiom of [13] becomes a theorem (Proposition 2.6).

These records also have a model-theoretic realization. In the structure whose sorts are Alice's and Bob's density operators, with the expectation predicate and with the channels as function symbols, the orbit types of the states of Alice's operator algebra are exactly the orbit types of the elements of ultrapowers (Proposition 2.11). The records of idealized states, which no density operator represents, are thus realized by elements of elementary extensions of the structure of physical preparations.

For the classification we work on energy shards: sets of preparations with a prescribed bound on the expected energy, or on the expected number of particles. The closures of their images in the space of orbit types are the type shards, and a family of computations is confined to a shard when each computation preserves it.

Theorem A. Let Γ be a countable monoid of channels, Σ a countable set of probes, and $\Delta \subseteq \Gamma$ a set of channels confined to a family of energy shards that covers the state space.

(1) (Theorem 3.1.) The deep computations of Δ are of the first Baire class on every type shard if and only if, for every $(\Phi, \sigma) \in \Gamma \times \Sigma$, the family $\{\rho \mapsto \varphi_C(\Phi(\Psi(\rho)), \sigma) : \Psi \in \Delta\}$ has the NIP on every shard. Todorčević's trichotomy then classifies them.
(2) (Proposition 2.14.) On a shard that is compact in the trace norm, every set of channels confined to it is stable.

Where the levels live. The second part of Theorem A directs the search for the remaining levels to shards that are not compact in the trace norm. A bound on the expected number of excitations leaves their positions unrestricted, and a single flipped spin on an infinite chain already supplies such a shard. Its hopping motion is the quantum walk on the integers.

The same motion has different limits under different observations. At any fixed site the probability of finding the flip tends to zero, and the walk observed through these local probes is stable. Probes that compare the position of the flip with a marker of Bob's retain the probability that the flip escapes to the right; the limiting computation is then discontinuous at an idealized type, although the compactum of limits remains metrizable. The walk is therefore $\mathrm{NIP}^{\mathrm{met}}$ and not stable for the comparison of positions.

The two ends of the line also constrain more general dynamics. For a countable monoid of channels that preserves the two ends, in a sense made precise in Theorem B through the projections Q_m onto the sites to the right of m, the comparison probes give only $\mathrm{NIP}^{\mathrm{met}}$ families. To realize the lower levels we pass to the richer boundary of a tree.

The infinite binary tree has a Cantor set of branches at infinity, which the primitives of binary search distinguish in different ways. A prefix test asks whether a particle lies below a node; a threshold test compares its position with an address in the lexicographic order. Measuring these tests and recording their answers gives sets of channels at two different levels: the prefix instruments have the NIP without first countability, and the threshold instruments have hereditarily separable, nonmetrizable compacta of limits. Recording a third outcome, or the address as well as the answer to a prefix test, gives first countable compacta that are not hereditarily separable. The realizations also admit exact identifications: suitable indexed subfamilies of the prefix and threshold instruments, and the three-outcome and address families, are equivalent to the families D_5, D_3, D_7 and D_6 of Argyros, Dodos and Kanellopoulos [5].

The prefix instruments also show why the actual preparations cannot replace the full space of types as the domain of a limiting computation. Along a branch x they converge to a deep computation G_x, and along an antichain to a deep computation G_0. Writing τ_0 and τ_1 for the orbit types of the two nodes of depth one, we have

$$G_x(\mathrm{tp}_\Gamma(\rho)) = \tau_0 = G_0(\mathrm{tp}_\Gamma(\rho)) \qquad \text{for every density operator } \rho,$$

yet the idealized orbit type β_x of escape along x separates them: $G_x(\beta_x) = \tau_1$ and $G_0(\beta_x) = \tau_0$. Observations of actual preparations therefore do not determine the extended limits. At any one input, only countably many of the branch limits G_x differ from G_0, so no countable set of inputs distinguishes G_0 from all of them, and first countability fails.

Theorem B. (1) (Proposition 4.5 and Theorem 4.4.) The quantum walk of one flip on an infinite spin chain is stable when observed through local probes, and $\mathrm{NIP}^{\mathrm{met}}$ but not stable when observed through the comparison of positions.

(2) (Theorem 4.6.) Every countable monoid of channels that fixes the two ends of the line, in the sense that the operators $\Phi^*(Q_m) - Q_0$ are compact, is $\mathrm{NIP}^{\mathrm{met}}$ for the comparison of positions.

(3) (Theorems 5.2–5.6.) On the binary tree, the prefix instruments have the NIP and are not $\mathrm{NIP}^{\mathrm{fc}}$; the threshold instruments are $\mathrm{NIP}^{\mathrm{hs}}$ and not $\mathrm{NIP}^{\mathrm{met}}$; and the three-outcome and address instruments are $\mathrm{NIP}^{\mathrm{fc}}$ and not $\mathrm{NIP}^{\mathrm{hs}}$. The indexed prefix and threshold subfamilies selected in Theorem 5.5, and the three-outcome and address families of Theorem 5.6, are equivalent to D_5, D_3, D_7 and D_6, respectively.

(4) (Theorem 5.2.) Distinct deep computations of the prefix instruments can agree at every density operator.

What the observations decide. The dependence on the observations goes beyond the contrast between local and comparison probes. Enlarging the observed family can only increase the complexity as measured by the NIP, metrizability, hereditary separability and stability (Theorem 6.4), but first countability behaves differently. For the address instruments, one additional probe makes a compactum that is not first countable into a first countable one (Proposition 6.5). Forgetting the added probe collapses a Cantor set of limits to one point, whereas forgetting the idealized inputs identifies the limits in pairs and leaves a metrizable compactum. The two losses of information are different quotients of the same family.

Randomization imposes a geometric restriction on the classification. The space of orbit types is compact, as a continuous image of the state space, and convex, because the observation map is affine under mixtures. For a countable family of channels with the NIP, confined to a shard and closed under rational mixtures, the compactum of limits is convex as well, and hereditary separability forces metrizability (Proposition 6.3). Hypotheses on operators impose restrictions of another kind: compact probes make every set of unitary channels stable (Theorem 7.4),

whereas a unitary dynamics with pure point spectrum, observed through a single projection of infinite rank, can fail the NIP (Proposition 7.5).

The logical scope of these results has two independent aspects. The gateway theorem fixes the channel and the probe that follow a computation, and varies the computation; letting the observations vary together gives the strictly stronger condition of Proposition 6.2. Stability of a set of computations likewise does not imply the uniform NIP: the stable set of channels of Proposition 8.12 has an observed family of infinite fat-shattering dimension at a fixed scale. Separately, the NIP in a model concerns patterns realized by actual states, whereas the NIP in the theory excludes independent patterns in elementary extensions as well. Khanaki and Pillay showed that this second distinction is genuine [21], and the topological classification here concerns the model, not the whole theory.

Measure. The measure-theoretic extension begins where the observations are no longer countable, and the space of orbit types need not be metrizable. For the translations of a particle on the line, this space contains the Sorgenfrey line (Proposition 8.7). In this general setting the NIP is equivalent to the universal measurability of every limit, and Luzin's theorem and polynomial approximation give universal Monte Carlo computability (Theorem 8.2).

With countably many channels and probes, the NIP also has an empirical meaning. For each fixed output coordinate it is equivalent to the Glivenko–Cantelli property for every Radon law on the type shard, so empirical mixtures approximate the barycenter uniformly over the computations (Theorem 8.8). For the laws of preparations by density operators, Mourier's law of large numbers already gives this uniformity for every set of channels (Proposition 8.9); the NIP characterizes its extension to all laws on the type shard, including those that charge idealized types.

Earlier work. The framework of [13] is the starting point of this paper; its extendibility axiom, assumed there, is proved here for orbit types. Kerr and Li [22] study independence and tameness for automorphisms of C*-algebras, while Glasner and Megrelishvili [17] bring Todorčević's trichotomy into topological dynamics. Their injectivity theorem [15] supplies the commutative case (Proposition 7.1). Here channels, which need not preserve products, act on the records of a prescribed family of observations, and the completion of these records distinguishes limits that agree on every actual preparation.

The classification of bipartite observables of Dueñez and Iovino [10, 11, 12] provides the physical setting, but it concerns a fixed observable without computations. Its response compacta include idealized probes, while the φ-types used here record responses to actual probes (Proposition 2.3).

For the measure-theoretic part, Khanaki [20] relates the NIP of a formula in a model to Glivenko–Cantelli classes. Theorem 8.8 gives the corresponding statement for the real-valued families of a bipartite computation structure, for every Radon law on a type shard, idealized preparations included; its proof combines the theory of [13, Section 5] with Talagrand's characterization of Glivenko–Cantelli classes [27].

Organization of the paper. Part I is the core of the paper, and the later parts depend on it. After the topological preliminaries, Section 2 introduces orbit types, and Section 3 proves the gateway theorem. Sections 4 and 5 follow the flip on the line and the particle on the tree. Part II studies what the observations decide, and Part III turns to measure. In Section 9 we compare the framework with tame dynamics, observability and quantum learning; the open questions are collected in Section 10.

Part I. Orbit types and the classification

A limiting computation can be classified only after the space of types on which it acts has been fixed. Closing the observations under composition with the channels supplies a continuous action on that space, and the NIP then places the limits within Todorčević's trichotomy. The motion of a particle on a line and the measurements of binary search on a tree give this passage its concrete forms.

1. Topological preliminaries

Section 1 of [13] concerns families of real-valued functions on topological spaces and uses nothing about where the functions come from; it transfers verbatim. We list what is used below.

(1) The NIP for a family A of real functions on a set X: no sequence (f_n) in A and reals $a < b$ such that, for all finite disjoint $I, J \subseteq \mathbb{N}$, some $x \in X$ satisfies $f_n(x) \le a$ for $n \in I$ and $f_n(x) \ge b$ for $n \in J$ [13, Definition 1.5].

(2) Rosenthal's dichotomy and the Bourgain–Fremlin–Talagrand dichotomy: for a compact metrizable K and a pointwise bounded $A \subseteq C_p(K)$, the closure of A lies in $\mathrm{B}_1(K)$ if and only if A has the NIP [26, 6].

(3) The reduction of $\mathbb{R}^{\mathcal{P}}$-valued maps to real-valued functions, for a countable index set $\mathcal{P}$, through $(\mathbb{R}^{\mathcal{P}})^X \cong \mathbb{R}^{X\times\mathcal{P}} \cong (\mathbb{R}^X)^{\mathcal{P}}$, and the resulting form of the Bourgain–Fremlin–Talagrand dichotomy for maps into $\mathbb{R}^{\mathcal{P}}$ [13, Section 1.3 and Theorem 1.9].

(4) The NIP passes from a set to its closure: for $A \subseteq C_p(X)$ and $L \subseteq X$, the restriction $A|_L$ has the NIP if and only if $A|_{\overline{L}}$ has it [13, Lemma 1.10].

(5) Rosenthal compacta, the five canonical compacta $A(2^{\mathbb{N}})$, $\hat{A}(2^{\mathbb{N}})$, $S(2^{\mathbb{N}})$, $D(2^{\mathbb{N}})$ and $\hat{D}(S(2^{\mathbb{N}}))$, Todorčević's trichotomy [29], and the heptachotomy of Argyros, Dodos and Kanellopoulos [5], with Lemma 19 of [5]: two tree-indexed families of isolated points are equivalent as soon as the same sequences of indices converge in both.

2. Bipartite computation structures

2.1. **The binary predicate and φ-types.** Let H_1 and H_2 be separable Hilbert spaces, the spaces of Alice's and Bob's systems, and let C be a bounded self-adjoint operator on $H_1 \otimes H_2$, a bipartite observable. Write $\mathfrak{D}(H)$ for the set of density operators on a Hilbert space H and $\mathcal{S}(B(H))$ for the set of states of $B(H)$, with the weak* topology; the states that no density operator represents are the *idealized* states [11]. The binary predicate of the theory is

$$\varphi_C(\rho,\sigma) = \mathrm{Tr}\big((\rho\otimes\sigma)\,C\big), \qquad \rho \in \mathfrak{D}(H_1),\ \sigma \in \mathfrak{D}(H_2).$$

For each probe σ, the *slice* $S_C(\sigma) = \mathrm{Tr}_2\big((I\otimes\sigma)C\big)$, obtained by the partial trace over Bob's system, is the bounded self-adjoint operator on H_1 with $\varphi_C(\rho,\sigma) = \mathrm{Tr}(\rho\, S_C(\sigma))$ for every ρ. Since $|\,\mathrm{Tr}((\tau\otimes\sigma)C)| \le \|C\|\,\|\tau\|_1\|\sigma\|_1$ for trace-class operators τ and σ, the slice map $\sigma \mapsto S_C(\sigma)$ extends to a linear map from the trace class of H_2 into $B(H_1)$ with

$$\|S_C(\sigma) - S_C(\sigma')\| \le \|C\|\,\|\sigma-\sigma'\|_1,$$

and φ_C is bounded by $\|C\|$ and $\|C\|$-Lipschitz in each variable for the trace norm. In [11] the opposite slice $T_C(\rho) = \mathrm{Tr}_1\big((\rho\otimes I)C\big)$, an operator on H_2, is the *effective observable* of Alice's preparation ρ, and the unital C*-algebra $\mathfrak{A}_C$ generated by these operators is the *probe algebra*; the slices $S_C(\sigma)$ play the same role on Alice's side.

Fix a countable set $\Sigma \subseteq \mathfrak{D}(H_2)$ of Bob's probes.

Definition 2.1. The *φ-type* of a state $\rho \in \mathfrak{D}(H_1)$ over Σ is

$$\mathrm{tp}_\varphi(\rho/\Sigma) = \big(\varphi_C(\rho,\sigma)\big)_{\sigma\in\Sigma} \in [-\|C\|,\|C\|]^\Sigma.$$

The closure $\mathcal{L}$ of the set of φ-types of density operators, in the topology of pointwise convergence, is the *space of φ-types*; its members that are not φ-types of density operators are *unrealized*.

This is the computation states structure of [13, Section 2.1] with the features $P_\sigma(\rho) = \varphi_C(\rho,\sigma)$, $\sigma \in \Sigma$: the features are the instances of one binary predicate, with Bob's probes as parameters. Since Σ is countable, the space $\mathcal{L}$ is a compact metrizable subspace of the cube $[-\|C\|,\|C\|]^\Sigma$.

Proposition 2.2. *The map $\omega \mapsto (\omega(S_C(\sigma)))_{\sigma\in\Sigma}$ is continuous from $\mathcal{S}(B(H_1))$ onto $\mathcal{L}$. Thus every φ-type, realized or not, is the φ-type of a state of $B(H_1)$, and the unrealized φ-types are φ-types of idealized states.*

Proof. The map is continuous for the weak* topology, and $\mathcal{S}(B(H_1))$ is compact, so its image is compact. The density operators are weak* dense in $\mathcal{S}(B(H_1))$, so the image is the closure of the set of realized φ-types, which is $\mathcal{L}$. □

For the responses to actual probes, the countability of Σ costs nothing. Let $\mathcal{L}^{\mathrm{act}}$ be the image of $\mathcal{S}(B(H_1))$ under the map $\omega \mapsto (\omega(S_C(\sigma)))_{\sigma\in\mathfrak{D}(H_2)}$, the compact space of responses to all of Bob's actual probes; restriction to Σ maps it onto $\mathcal{L}$, by Proposition 2.2.

Proposition 2.3. *If Σ is dense in $\mathfrak{D}(H_2)$ for the trace norm, then restriction to Σ is a homeomorphism of $\mathcal{L}^{\mathrm{act}}$ onto $\mathcal{L}$. Such sets Σ exist, since $\mathfrak{D}(H_2)$ is separable in the trace norm.*

Proof. Restriction is continuous, and it is a map between compact Hausdorff spaces, so it suffices to show that it is injective. For every state ω, the function $\sigma \mapsto \omega(S_C(\sigma))$ is $\|C\|$-Lipschitz for the trace norm, by the estimate on slices above, and is therefore determined by its values on the dense set Σ. The trace class of a separable Hilbert space is separable, and so is its subset $\mathfrak{D}(H_2)$. □

The response compactum of C in [11] is larger than $\mathcal{L}^{\mathrm{act}}$. Its members are functions on the probe space, the state space of the probe algebra, which contains idealized probes as well as the records of actual ones, and restriction to actual probes maps it onto $\mathcal{L}^{\mathrm{act}}$, in general without being injective. For the comparison of positions on the line, the two one-sided idealized responses at a point agree at every actual probe and are told apart only by idealized probes that charge the point [11]. The φ-types of this paper are responses to actual probes.

The topological theory of Section 3 uses countably many features and needs Σ to be countable; by Proposition 2.3, a countable dense set of probes sees as much as all actual probes. The measure-theoretic theory of Section 8 needs no countability.

2.2. **Channels and orbit types.**

Definition 2.4. A *bipartite computation structure* consists of a bipartite observable C, a countable set Σ of Bob's probes, and a countable monoid Γ of quantum channels of Alice's system, that is, of completely positive trace-preserving maps of the trace class $\mathcal{T}(H_1)$, containing the identity. The members of Γ are the *computations*.

This is the compositional computation structure of [13, Definition 2.4], with the state space $\mathfrak{D}(H_1)$ and the features P_σ. In [13] every computation is required to extend continuously to the type shards (the Extendibility Axiom). Here a channel Φ acts on all states of $B(H_1)$

through its Heisenberg dual Φ^*, the unital completely positive map of $B(H_1)$ with $\mathrm{Tr}(\Phi(\rho)X) = \mathrm{Tr}(\rho\,\Phi^*(X))$, by $\omega \mapsto \omega \circ \Phi^*$, and

$$\varphi_C(\Phi(\rho), \sigma) = \mathrm{Tr}\big(\rho\,\Phi^*(S_C(\sigma))\big).$$

The output feature $P_\sigma \circ \Phi$ is therefore given by the operator $\Phi^*(S_C(\sigma))$, which in general is not a slice $S_C(\sigma')$ of any probe: a feature followed by a computation is no longer an instance of $\varphi_C(x, \sigma')$, but an instance of the formula $\varphi_C(\Phi(x), y)$, in which the computation enters as a function symbol. The types adapted to this situation record all such instances.

Definition 2.5. The *orbit type* of a state $\omega \in \mathcal{S}(B(H_1))$ is

$$\mathrm{tp}_\Gamma(\omega) = \big(\omega(\Phi^*(S_C(\sigma)))\big)_{(\Phi,\sigma)\in\Gamma\times\Sigma} \in [-\|C\|, \|C\|]^{\Gamma\times\Sigma};$$

for a density operator ρ, its coordinates are the values $\varphi_C(\Phi(\rho), \sigma)$. The *space of orbit types* $\mathcal{L}_\Gamma$ is the set of orbit types of all states of $B(H_1)$.

Closing the observations under composition lets each channel act on orbit types by a shift of coordinates.

Proposition 2.6. *The space $\mathcal{L}_\Gamma$ is compact and metrizable, the map $\omega \mapsto \mathrm{tp}_\Gamma(\omega)$ is continuous and affine on $\mathcal{S}(B(H_1))$, and the orbit types of density operators are dense in $\mathcal{L}_\Gamma$. Every $\Psi \in \Gamma$ induces a continuous map $\tilde{\Psi}\colon \mathcal{L}_\Gamma \to \mathcal{L}_\Gamma$, given by*

$$\tilde{\Psi}(\mathbf{v})_{(\Phi,\sigma)} = \mathbf{v}_{(\Phi\circ\Psi,\,\sigma)},$$

with $\tilde{\Psi}(\mathrm{tp}_\Gamma(\omega)) = \mathrm{tp}_\Gamma(\omega \circ \Psi^*)$ for every state ω; in particular $\tilde{\Psi}(\mathrm{tp}_\Gamma(\rho)) = \mathrm{tp}_\Gamma(\Psi(\rho))$ for density operators. Thus the Extendibility Axiom holds for the orbit types, with no condition on the channels.

Proof. The map $\omega \mapsto \mathrm{tp}_\Gamma(\omega)$ is affine and weak* continuous on the compact space $\mathcal{S}(B(H_1))$, so its image $\mathcal{L}_\Gamma$ is compact, and it is metrizable as a subspace of a countable product of intervals. Density operators are weak* dense among states, so their orbit types are dense in $\mathcal{L}_\Gamma$. For $\Psi \in \Gamma$ and a state ω, the coordinate (Φ, σ) of $\mathrm{tp}_\Gamma(\omega \circ \Psi^*)$ is $\omega(\Psi^*(\Phi^*(S_C(\sigma)))) = \omega((\Phi \circ \Psi)^*(S_C(\sigma)))$, the coordinate $(\Phi \circ \Psi, \sigma)$ of $\mathrm{tp}_\Gamma(\omega)$, and $\Phi \circ \Psi \in \Gamma$. Hence $\tilde{\Psi}$ is well defined on $\mathcal{L}_\Gamma$, and it is continuous because each of its coordinates is a coordinate projection. □

States with the same orbit type are indistinguishable by every computation followed by every probe, so the theory can proceed without requiring the probes to separate states. We work directly on the compact space $\mathcal{L}_\Gamma$ of orbit types and call the members of

$$K = \overline{\{\tilde{\Psi} : \Psi \in \Gamma\}} \subseteq \mathcal{L}_\Gamma^{\mathcal{L}_\Gamma}$$

the *deep computations*, with the closure taken for pointwise convergence. The deep computations of a subset $\Delta \subseteq \Gamma$ form the closure of $\{\tilde{\Psi} : \Psi \in \Delta\}$. Both closures are compact, since $\mathcal{L}_\Gamma$ is compact.

This convention retains the action on idealized types from the outset. In [13], deep computations are first defined on actual states, and the limits of the extended maps on the type shards are called ultracomputations (see [13, Section 2.2 and Lemma 2.7]); our deep computations are these extended limits. Restriction to actual inputs can lose information: in Theorem 5.2, two distinct deep computations agree at every density operator and differ at an idealized orbit type.

Continuous time. Channels often come in continuous time, and Definition 2.4 asks for a countable monoid. Rational times lose nothing when the semigroup is continuous.

Proposition 2.7. *Let $(\Phi_t)_{t\geq 0}$ be a semigroup of channels, with $\Phi_0 = \mathrm{id}$ and $\Phi_s \circ \Phi_t = \Phi_{s+t}$, such that $t \mapsto \Phi_t^*(X)$ is continuous in norm for every $X \in B(H_1)$, and let $\Gamma = \{\Phi_q : q \in \mathbb{Q},\ q \geq 0\}$. For every real $t \geq 0$, the formula $\tilde{\Phi}_t(\mathrm{tp}_\Gamma(\omega)) = \mathrm{tp}_\Gamma(\omega \circ \Phi_t^*)$ defines a continuous map $\tilde{\Phi}_t \colon \mathcal{L}_\Gamma \to \mathcal{L}_\Gamma$, and $t \mapsto \tilde{\Phi}_t$ is continuous from $[0,\infty)$ into the space of continuous maps of $\mathcal{L}_\Gamma$ into itself, with the topology of uniform convergence. In particular, every $\tilde{\Phi}_t$ is a deep computation of Γ.*

Proof. Let $t \geq 0$ and let (q_n) be rationals with $q_n \to t$. The coordinate (Φ_s, σ) of $\mathrm{tp}_\Gamma(\omega \circ \Phi_t^*)$ is $\omega(\Phi_{s+t}^*(S_C(\sigma)))$, and

$$\left|\omega(\Phi_{s+t}^*(S_C(\sigma))) - \omega(\Phi_{s+q_n}^*(S_C(\sigma)))\right| \leq \left\|\Phi_{s+t}^*(S_C(\sigma)) - \Phi_{s+q_n}^*(S_C(\sigma))\right\| \longrightarrow 0,$$

uniformly in the state ω. The numbers $\omega(\Phi_{s+q_n}^*(S_C(\sigma)))$ are coordinates of $\mathrm{tp}_\Gamma(\omega)$, so the coordinate (Φ_s, σ) of $\tilde{\Phi}_t(\mathrm{tp}_\Gamma(\omega))$ is a function of $\mathrm{tp}_\Gamma(\omega)$, and, as a uniform limit of coordinate projections, a continuous one. The same estimate, with t' in place of q_n, gives the continuity of $t \mapsto \tilde{\Phi}_t$ for uniform convergence in each coordinate, hence for the uniform structure of the metrizable space $\mathcal{L}_\Gamma$. Uniform convergence implies pointwise convergence, so $\tilde{\Phi}_t$ is the limit of the maps $\tilde{\Phi}_{q_n}$ and lies in K. □

φ-types and orbit types compared. The φ-types over Σ are the images of the orbit types under the projection $\pi \colon \mathcal{L}_\Gamma \to \mathcal{L}$ onto the coordinates (id, σ), $\sigma \in \Sigma$, which is continuous; through this projection the theory meets the responses of [11], restricted to actual probes. The next two propositions say exactly when channels act on φ-types, and why orbit types are the least enrichment that makes them act.

Proposition 2.8. *Let V be the norm closure of the real linear span of the identity and the slices $S_C(\sigma)$, $\sigma \in \Sigma$, a subspace of the self-adjoint operators on H_1, and let Φ be a channel. The following are equivalent:*

(i) *there is a map $\hat{\Phi} \colon \mathcal{L} \to \mathcal{L}$ with $\hat{\Phi}(\pi(\mathrm{tp}_\Gamma(\omega))) = \pi(\mathrm{tp}_\Gamma(\omega \circ \Phi^*))$ for every state ω of $B(H_1)$;*
(ii) *$\Phi^*(V) \subseteq V$.*

When they hold, $\hat{\Phi}$ is continuous: the Extendibility Axiom holds for Φ on φ-types.

Proof. (i)⇒(ii). Since Φ^* is unital and norm continuous, it suffices to show that $X = \Phi^*(S_C(\sigma))$ lies in V for every $\sigma \in \Sigma$. If not, the Hahn–Banach theorem in the real Banach space of self-adjoint operators gives a bounded real functional f that vanishes on V and satisfies $f(X) \neq 0$; extended complex-linearly, f is a bounded hermitian functional on $B(H_1)$. By the Jordan decomposition, $f = f_+ - f_-$ with positive functionals $f_\pm$, and $f_+(I) = f_-(I) = m$ because $f(I) = 0$; moreover $m > 0$, since $f \neq 0$. The states $\omega_\pm = f_\pm/m$ agree on V, and so have the same φ-type, but $\omega_+(X) - \omega_-(X) = f(X)/m \neq 0$, so the outputs $\omega_\pm \circ \Phi^*$ have different φ-types, contrary to (i).

(ii)⇒(i), and continuity. A state ω satisfies $\omega(I) = 1$, so its φ-type determines ω on the span of the identity and the slices, and, since $|\omega(X)| \leq \|X\|$, on the norm closure V. Let $\sigma \in \Sigma$ and $\varepsilon > 0$. As $\Phi^*(S_C(\sigma)) \in V$, there are a real c_0, probes $\sigma_1, \ldots, \sigma_k \in \Sigma$ and reals $c_1, \ldots, c_k$ with $\|\Phi^*(S_C(\sigma)) - Y\| < \varepsilon$, where $Y = c_0 I + \sum_j c_j S_C(\sigma_j)$. Then $|\omega(\Phi^*(S_C(\sigma))) - (c_0 + \sum_j c_j\, \omega(S_C(\sigma_j)))| < \varepsilon$ for every state ω. By Proposition 2.2, every member of $\mathcal{L}$ is the φ-type of a state; hence the coordinate σ of $\hat{\Phi}$ is a well-defined function on $\mathcal{L}$, and it lies uniformly

within ε of the continuous function $\mathbf{v} \mapsto c_0 + \sum_j c_j \mathbf{v}_{\sigma_j}$. A uniform limit of continuous functions is continuous. □

When Σ is dense in $\mathfrak{D}(H_2)$, the space V is the norm closure of the span of the identity and all the slices of C, and the condition of Proposition 2.8 is intrinsic to the observable and the channel. Without it, the φ-type of the output of a channel is not a function of the φ-type of the input, and orbit types are the remedy, in the following precise sense.

Proposition 2.9. *Let W be the norm closure of the real linear span of the identity and the operators $\Phi^*(S_C(\sigma))$, $(\Phi, \sigma) \in \Gamma \times \Sigma$.*

(1) *W is the least norm-closed subspace of the self-adjoint operators that contains V and is invariant under Ψ^* for every $\Psi \in \Gamma$, and two states have the same orbit type if and only if they agree on W.*
(2) *Equality of orbit types is the coarsest equivalence relation $\sim$ on the states of $B(H_1)$ that refines equality of φ-types and is invariant under the channels, in the sense that $\omega \sim \omega'$ implies $\omega \circ \Psi^* \sim \omega' \circ \Psi^*$ for every $\Psi \in \Gamma$.*

Proof. (1) Since $\Psi^*\Phi^* = (\Phi \circ \Psi)^*$ and $\Phi \circ \Psi \in \Gamma$, the dual Ψ^* maps the generators of W to generators of W, and W is invariant; it contains the generators of V, those with $\Phi = \mathrm{id}$. A norm-closed subspace that contains V and is invariant under every Ψ^* contains every $\Phi^*(S_C(\sigma))$, hence W. Two states have the same orbit type exactly when they agree on the generators of W, hence, by linearity and continuity, on W.

(2) Equality of orbit types refines equality of φ-types, the coordinates with $\Phi = \mathrm{id}$, and it is invariant by Proposition 2.6. If $\sim$ is invariant and refines equality of φ-types, and $\omega \sim \omega'$, then $\omega \circ \Phi^* \sim \omega' \circ \Phi^*$ for every $\Phi \in \Gamma$, so these states have the same φ-type, and the coordinates (Φ, σ) of the orbit types of ω and ω' agree. □

2.3. **The monster model.** Orbit types are types in the sense of model theory, and the orbit types of idealized states are realized in elementary extensions. Let $\mathcal{M}_{C,\Gamma}$ be the two-sorted structure with Alice's states $\mathfrak{D}(H_1)$ and Bob's probes $\mathfrak{D}(H_2)$ as sorts, each with the metric $d(\tau, \tau') = \frac{1}{2}\|\tau - \tau'\|_1$, of diameter 1, with the binary predicate φ_C and with a function symbol for each channel in Γ, acting on Alice's sort.

Proposition 2.10. *The structure $\mathcal{M}_{C,\Gamma}$ is a structure of continuous logic, and of Keisler's real-valued logic* [19]*: the sorts are complete bounded metric spaces, the predicate φ_C takes values in $[-\|C\|, \|C\|]$ and is $2\|C\|$-Lipschitz in each variable, and every function symbol is 1-Lipschitz.*

Proof. The set $\mathfrak{D}(H)$ is closed in the trace class, which is complete. The Lipschitz bound for φ_C is the estimate of the first subsection, with $\|\tau - \tau'\|_1 = 2d(\tau, \tau')$, and channels are contractions of the trace norm: a positive trace-preserving map Φ satisfies $\|\Phi(\tau)\|_1 \leq \operatorname{Tr}\Phi(\tau_+) + \operatorname{Tr}\Phi(\tau_-) = \|\tau\|_1$ for self-adjoint $\tau = \tau_+ - \tau_-$. □

We fix a monster model $\mathbb{M}$ of the theory of $\mathcal{M}_{C,\Gamma}$, with the probes of Σ named by constants: a κ-saturated and κ-strongly homogeneous elementary extension of $\mathcal{M}_{C,\Gamma}$, for a cardinal κ larger than the continuum, in which every type over a set of fewer than κ parameters is realized. For an element a of Alice's sort of $\mathbb{M}$, its *orbit type* is the list $\operatorname{tp}_\Gamma(a) = (\varphi_C^{\mathbb{M}}(\Phi(a), \sigma))_{(\Phi,\sigma)\in\Gamma\times\Sigma}$, the restriction of its type over Σ to the instances of the formulas $\varphi_C(\Phi(x), y)$.

Proposition 2.11. *The orbit types of the elements of Alice's sort of $\mathbb{M}$ are exactly the members of $\mathcal{L}_\Gamma$. More precisely:*

(1) *if $\mathcal{U}$ is an ultrafilter on a set I, $(\rho_i)_{i\in I}$ are density operators, and ω is the weak* limit of the states ρ_i along $\mathcal{U}$, then the element $a = [\rho_i]_\mathcal{U}$ of the ultrapower satisfies $\operatorname{tp}_\Gamma(a) = \operatorname{tp}_\Gamma(\omega)$;*

(2) *every member of* $\mathcal{L}_\Gamma$ *is the orbit type of an element of the ultrapower of* $\mathcal{M}_{C,\Gamma}$ *by a nonprincipal ultrafilter on* $\mathbb{N}$*;*
(3) *for every element* a *of Alice's sort of* $\mathbb{M}$ *and every* $\Psi \in \Gamma$*, the orbit type* $\mathrm{tp}_\Gamma(a)$ *lies in* $\mathcal{L}_\Gamma$*, and* $\mathrm{tp}_\Gamma(\Psi(a)) = \tilde{\Psi}(\mathrm{tp}_\Gamma(a))$.

Proof. (1) By Łoś's theorem for ultraproducts of metric structures,

$$\varphi_C(\Phi(a), \sigma) = \lim_{i,\mathcal{U}} \mathrm{Tr}\big(\rho_i \, \Phi^*(S_C(\sigma))\big) = \omega\big(\Phi^*(S_C(\sigma))\big).$$

(2) Let $\mathbf{v} \in \mathcal{L}_\Gamma$. The orbit types of density operators are dense in the metrizable space $\mathcal{L}_\Gamma$, so there are density operators ρ_n with $\mathrm{tp}_\Gamma(\rho_n) \to \mathbf{v}$. For a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, Łoś's theorem gives $\mathrm{tp}_\Gamma([\rho_n]_\mathcal{U}) = \lim_{n,\mathcal{U}} \mathrm{tp}_\Gamma(\rho_n) = \mathbf{v}$. The ultrapower has cardinality at most the continuum, so it embeds elementarily in $\mathbb{M}$ over $\mathcal{M}_{C,\Gamma}$, and the image of $[\rho_n]_\mathcal{U}$ is an element of $\mathbb{M}$ with orbit type $\mathbf{v}$.

(3) Let $(\Phi_1, \sigma_1), \ldots, (\Phi_k, \sigma_k)$ be coordinates and $\varepsilon > 0$, and let $r_j = \varphi_C^{\mathbb{M}}(\Phi_j(a), \sigma_j)$. The sentence $\inf_x \max_j |\varphi_C(\Phi_j(x), \sigma_j) - r_j|$, with the constants σ_j, has the value 0 in $\mathbb{M}$, hence in $\mathcal{M}_{C,\Gamma}$, and some density operator ρ satisfies $|\varphi_C(\Phi_j(\rho), \sigma_j) - r_j| < \varepsilon$ for all j. Thus $\mathrm{tp}_\Gamma(a)$ lies in the closure of the orbit types of density operators, which is $\mathcal{L}_\Gamma$. Finally, the coordinate (Φ, σ) of $\mathrm{tp}_\Gamma(\Psi(a))$ is $\varphi_C^{\mathbb{M}}(\Phi(\Psi(a)), \sigma)$, and the function symbols of Φ, Ψ and $\Phi \circ \Psi$ satisfy $\Phi(\Psi(x)) = (\Phi \circ \Psi)(x)$ in $\mathcal{M}_{C,\Gamma}$, hence in $\mathbb{M}$; this is the coordinate $(\Phi \circ \Psi, \sigma)$ of $\mathrm{tp}_\Gamma(a)$. □

The analytic and model-theoretic descriptions therefore give the same orbit types over Σ, and restriction to the formulas $\varphi_C(x, \sigma)$ recovers the φ-types. When Γ or Σ is uncountable, the monster model realizes the types over each small set of parameters, and Section 8.1 treats the corresponding observation families without a countability assumption.

Two notions of NIP must be kept apart [21]. A formula has the NIP *in the model* $\mathcal{M}_{C,\Gamma}$ if no sequence of its instances with actual parameters is independent on the actual states, and it has the NIP *in the theory* if no such sequence exists in $\mathbb{M}$. The gateway theorem of Section 3, like [13, Theorem 3.1], concerns the NIP in the model: its finitary condition quantifies over actual states of a shard, where the Bourgain–Fremlin–Talagrand dichotomy applies. The NIP in the theory is stronger, and the example of Khanaki and Pillay shows that the two differ.

2.4. **Energy shards.** In [13], shards bound the magnitudes of the features. Here every feature is bounded by $\|C\|$, so the single sizer with all bounds equal to $\|C\|$ covers every state, and sizers play no role. The shards of the quantum theory bound instead a physical quantity that is not among the features: a physical computation commits in advance to a bound on its expected energy, or on its expected number of particles, as a numerical computation commits to a bound on the magnitudes it handles.

Definition 2.12. Let H be a positive self-adjoint operator on H_1, possibly unbounded, the *energy*. For a density operator ρ let $\mathrm{Tr}(\rho H) = \mathrm{Tr}(H^{1/2} \rho H^{1/2}) \in [0, \infty]$. For $E \in [0, \infty]$, the *energy shard* L_E is the set of density operators ρ with $\mathrm{Tr}(\rho H) \le E$, and its *type shard* $\mathcal{L}_E$ is the closure in $\mathcal{L}_\Gamma$ of the set of orbit types of members of L_E. A set $\Delta \subseteq \Gamma$ is *confined* to L_E if every member of Δ maps L_E into itself.

The shard L_∞ is the whole of $\mathfrak{D}(H_1)$, and its type shard is $\mathcal{L}_\Gamma$. The number of particles of a system of many particles, or the number of excitations of a field, is an energy in this sense, and its shards bound the expected number of particles; a state of finite expected number may have components with arbitrarily many particles. When the energy is unbounded, the finite-energy shards do not cover $\mathfrak{D}(H_1)$, since some density operators have infinite expected energy, and a family of shards that covers $\mathfrak{D}(H_1)$, as in Theorem 3.1, contains L_∞. The energy shards are

convex, since $\rho \mapsto \mathrm{Tr}(\rho H)$ is affine, and the shards are used only through their sets of types: no definability of the shards in $\mathbb{M}$ is needed.

Proposition 2.13. *The type shards are compact and convex, and if Δ is confined to L_E, then $\tilde{\Psi}(\mathcal{L}_E) \subseteq \mathcal{L}_E$ for every $\Psi \in \Delta$.*

Proof. The type shard $\mathcal{L}_E$ is the closure of the image of the convex set L_E under the affine map tp_Γ, hence closed and convex in $\mathcal{L}_\Gamma$. If $\Psi(L_E) \subseteq L_E$, then $\tilde{\Psi}$ maps the orbit types of members of L_E to orbit types of members of L_E, by Proposition 2.6, and by continuity it maps their closure $\mathcal{L}_E$ into $\mathcal{L}_E$. □

Proposition 2.14 below directs the search to energies without compact resolvent: for an energy with compact resolvent, every set of computations confined to a finite-energy shard is stable on it. An energy without compact resolvent has nonempty essential spectrum: continuous spectrum, an eigenvalue of infinite multiplicity, or eigenvalues accumulating at a finite value. The number of flips of an infinite chain in Section 4, each of whose nonzero eigenvalues has infinite multiplicity, is of this kind.

2.5. **Stability.** Each channel acts continuously on orbit types, but a pointwise limit of channels may lose continuity at an idealized input. A set Δ confined to L_E is *stable* on L_E when every member of the closure of $\{\tilde{\Psi}|_{\mathcal{L}_E} : \Psi \in \Delta\}$ remains continuous [13, Definition 2.8], and Grothendieck's double limit condition characterizes this persistence [3]. On a shard that is compact in the trace norm, the contraction property of channels prevents the loss.

Proposition 2.14. *Let Δ be a set of channels confined to a set $L \subseteq \mathfrak{D}(H_1)$ that is compact in the trace norm. Then every orbit type in the closure of $\mathrm{tp}_\Gamma(L)$ is the orbit type of a member of L, and Δ is stable on L: for every $(\Phi, \sigma) \in \Gamma \times \Sigma$, the family $\{\rho \mapsto \varphi_C(\Phi(\Psi(\rho)), \sigma) : \Psi \in \Delta\}$ is relatively compact in $C(L)$ with the uniform norm, and every deep computation of Δ is continuous on the closure of $\mathrm{tp}_\Gamma(L)$. In particular, this holds on every finite-energy shard L_E, $E < \infty$, of an energy with compact resolvent, and on all of $\mathfrak{D}(H_1)$ when H_1 is finite-dimensional.*

Proof. For density operators ρ, ρ' and all channels Φ and Ψ,

$$|\varphi_C(\Phi(\Psi(\rho)), \sigma) - \varphi_C(\Phi(\Psi(\rho')), \sigma)| \leq \|C\| \, \|\Phi(\Psi(\rho - \rho'))\|_1 \leq \|C\| \, \|\rho - \rho'\|_1,$$

since channels are contractions of the trace norm. In particular every coordinate of tp_Γ is continuous on L with the trace norm, so $\mathrm{tp}_\Gamma(L)$ is compact, hence closed: every orbit type in its closure is realized in L. The family of the statement is uniformly bounded and equicontinuous on the compact metric space L, so by the Arzelà–Ascoli theorem it is relatively compact in $C(L)$, and its pointwise limits are $\|C\|$-Lipschitz. Let ξ be a deep computation of Δ, the limit of $\tilde{\Psi}_i$ along an ultrafilter $\mathcal{U}$, with $\Psi_i \in \Delta$. Each coordinate of $\xi \circ \mathrm{tp}_\Gamma$ on L is the limit along $\mathcal{U}$ of the functions $\rho \mapsto \varphi_C(\Phi(\Psi_i(\rho)), \sigma)$, hence continuous, and $\xi \circ \mathrm{tp}_\Gamma$ is continuous from L into $\mathcal{L}_\Gamma$. The map $\mathrm{tp}_\Gamma \colon L \to \mathrm{tp}_\Gamma(L)$ is a continuous surjection of a compact space onto a Hausdorff space, hence a quotient map, and therefore ξ is continuous on $\mathrm{tp}_\Gamma(L)$. For an energy H with compact resolvent, the set L_E is compact in the trace norm for every finite E [18]; in finite dimensions, $\mathfrak{D}(H_1)$ is compact. □

3. Todorčević's trichotomy classifies quantum deep computations

3.1. **The gateway theorem.** The following is [13, Theorem 3.1], read for bipartite computation structures. Its proof uses two properties of the type shards: they are compact and metrizable, and the computations extend to them continuously. Both hold by Propositions 2.6 and 2.13, and we give the proof in the present notation.

Theorem 3.1. *Let $\mathscr{E} \subseteq [0,\infty]$ be a set of energy bounds whose shards cover $\mathfrak{D}(H_1)$, and let $\Delta \subseteq \Gamma$ be confined to L_E for every $E \in \mathscr{E}$. The following are equivalent:*

(i) *for every $E \in \mathscr{E}$, every member of the closure of $\{\tilde{\Psi}|_{\mathcal{L}_E} : \Psi \in \Delta\}$ is of the first Baire class;*

(ii) *for every $E \in \mathscr{E}$ and every $(\Phi,\sigma) \in \Gamma \times \Sigma$, the family $\{\rho \mapsto \varphi_C(\Phi(\Psi(\rho)),\sigma) : \Psi \in \Delta\}$ has the NIP on L_E.*

If these conditions hold, then, for every $E \in \mathscr{E}$, every deep computation of Δ is, on $\mathcal{L}_E$, the pointwise limit of a sequence of computations of Δ.

Proof. Fix $E \in \mathscr{E}$. The maps $\tilde{\Psi}|_{\mathcal{L}_E}$, $\Psi \in \Delta$, are continuous on the compact metrizable space $\mathcal{L}_E$, by Proposition 2.13. For a fixed coordinate (Φ,σ), their scalar profiles are the continuous functions

$$\mathbf{v} \mapsto \tilde{\Psi}(\mathbf{v})_{(\Phi,\sigma)} = \mathbf{v}_{(\Phi\circ\Psi,\sigma)}, \qquad \Psi \in \Delta.$$

By the Bourgain–Fremlin–Talagrand dichotomy, this family has the NIP on $\mathcal{L}_E$ exactly when its pointwise closure consists of functions of the first Baire class. The reduction of item (3) of Section 1, applied to all the coordinates, shows that condition (i) holds for E exactly when every such family has the NIP on $\mathcal{L}_E$. At the orbit type of a density operator ρ, the displayed function takes the value $\varphi_C(\Phi(\Psi(\rho)),\sigma)$. Since $\mathrm{tp}_\Gamma(L_E)$ is dense in $\mathcal{L}_E$, item (4) of Section 1 identifies this NIP condition with condition (ii).

For the last statement, the same reduction turns the compactum of limits into a compact set of functions of the first Baire class on the Polish space $\mathcal{L}_E \times (\Gamma \times \Sigma)$, with the second factor discrete. By the Bourgain–Fremlin–Talagrand theorem, every member of this compactum is the limit of a sequence from the original family. □

Condition (ii) is the NIP of each instance of the formula $\varphi_C(\Phi(\Psi(x)),\sigma)$ separately, as in [13]; Section 6.2 compares it with the NIP of the whole observed family.

3.2. **Levels of NIP.** The levels $\mathrm{NIP}^{\mathrm{met}}$, $\mathrm{NIP}^{\mathrm{hs}}$ and $\mathrm{NIP}^{\mathrm{fc}}$ of a set of computations satisfying condition (ii) of Theorem 3.1 are defined verbatim by the metrizability, hereditary separability and first countability of the compacta $K_E = \overline{\{\tilde{\Psi}|_{\mathcal{L}_E} : \Psi \in \Delta\}}$ [13, Definition 3.2]. They form the chain $\mathrm{NIP}^{\mathrm{met}} \Rightarrow \mathrm{NIP}^{\mathrm{hs}} \Rightarrow \mathrm{NIP}^{\mathrm{fc}} \Rightarrow \mathrm{NIP}$, and a stable set is $\mathrm{NIP}^{\mathrm{met}}$, by the argument of [13, Section 3.2]. Under the NIP, each K_E is a separable Rosenthal compactum, by Theorem 3.1, and Todorčević's trichotomy applies to it.

Theorem 3.2 (Todorčević [29]; see [13, Theorem 3.3])**.** *Let K be a separable Rosenthal compactum.*

(i) *If K is hereditarily separable and not metrizable, then the split Cantor set $S(2^{\mathbb{N}})$ embeds into K.*

(ii) *If K is first countable and not hereditarily separable, then the Alexandroff duplicate $D(2^{\mathbb{N}})$ of the Cantor set or the extended duplicate $\hat{D}(S(2^{\mathbb{N}}))$ of the split Cantor set embeds into K.*

(iii) *If K is not first countable, then the Alexandroff compactification $A(2^{\mathbb{N}})$ of a discrete set of the cardinality of the continuum embeds into K.*

A set of computations with the NIP is therefore $\mathrm{NIP}^{\mathrm{met}}$, or one of its compacta contains one of the canonical compacta of Theorem 3.2, according to which of the properties of metrizability, hereditary separability and first countability the compactum has.

The description of the levels by tests [13, Proposition 3.4] transfers. Here a *test* is a pair $(\mathbf{v},(\Phi,\sigma))$ of an input orbit type $\mathbf{v} \in \mathcal{L}_E$ and a coordinate: the test applies the computation Φ and then Bob's probe σ to the output of a deep computation on the input $\mathbf{v}$, and its value at

Computations and probes	Level	Where
Push-forwards of a tame cascade	that of $E(X,\gamma)$	Prop. 7.1
Any confined set, compact shard or finite dimension	stable	Prop. 2.14
Unitary channels, compact probes	stable	Thm. 7.4
Quantum walk, local probes	stable	Prop. 4.5
Quantum walk, comparison of positions	$\mathrm{NIP}^{\mathrm{met}}$, not stable	Thm. 4.4
Channels fixing the two ends of the line	$\mathrm{NIP}^{\mathrm{met}}$	Thm. 4.6
Threshold instruments on the tree (D_3)	$\mathrm{NIP}^{\mathrm{hs}}$, not $\mathrm{NIP}^{\mathrm{met}}$	Thms. 5.3, 5.5
Three-outcome and address instruments (D_7, D_6)	$\mathrm{NIP}^{\mathrm{fc}}$, not $\mathrm{NIP}^{\mathrm{hs}}$	Thm. 5.6
Prefix instruments on the tree (D_5)	NIP, not $\mathrm{NIP}^{\mathrm{fc}}$	Thms. 5.2, 5.5
Pure point spectrum, one infinite-rank probe	not NIP	Prop. 7.5
Translations of the line, all real times	NIP, nonmetrizable types	Prop. 8.7

TABLE 1. Levels reached in this paper.

$\xi \in K_E$ is $\xi(\mathbf{v})_{(\Phi,\sigma)}$. Thus Δ is $\mathrm{NIP}^{\mathrm{met}}$ when countably many tests tell all the deep computations apart, $\mathrm{NIP}^{\mathrm{hs}}$ when no type shard carries a finitely separated set of deep computations of the cardinality of the continuum, and $\mathrm{NIP}^{\mathrm{fc}}$ when each deep computation is told apart from all others by countably many tests of its own.

Table 1 records the levels reached by the examples of Sections 4 and 5 and by the other results of this paper.

Every class of the trichotomy is realized by explicit quantum instruments, each through an exact canonical family of [5].

3.3. **The heptachotomy.** The heptachotomy of Argyros, Dodos and Kanellopoulos, and its reading by binary search in [13, Section 3.3], are statements about tree-indexed families of functions and transfer verbatim to families $\{\tilde{\Psi}_t : t \in 2^{<\mathbb{N}}\}$ of computations, observed through the coordinates of the orbit types. Section 5 measures the primitives of binary search as quantum instruments.

4. Motion on a line: flips on an infinite spin chain

A bound on the number of excitations need not confine their positions. A single flip can occupy arbitrarily distant sites of an infinite spin chain, so even the shard of expected number at most one is not compact in the trace norm (Proposition 4.1). This gives a physical system, read through the scanning and template observables of [11], in which the stability that Proposition 2.14 forces on compact shards can fail.

The chain. Let H_1 be the Hilbert space of an infinite chain of spins, indexed by $\mathbb{Z}$, spanned by the configurations that differ from the configuration with all spins up at finitely many sites, the *flips*; the configurations $|F\rangle$, for finite sets $F \subseteq \mathbb{Z}$, form an orthonormal basis, and H_1 is separable. The *number of flips* is the positive self-adjoint operator N with $N|F\rangle = |F|\,|F\rangle$, and the shards are $L_E = \{\rho \in \mathfrak{D}(H_1) : \mathrm{Tr}(\rho N) \le E\}$. The subspace spanned by the configurations with one flip is a copy of $\ell^2(\mathbb{Z})$.

Proposition 4.1. *For every $E \ge 1$, the shard L_E is not compact in the trace norm. The hopping (XX) dynamics of the chain, generated by $H_{XX} = \sum_{j\in\mathbb{Z}} \big(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+\big)$, and the shift of sites are confined to every shard L_E.*

Proof. The pure states $\rho_n = |\{n\}\rangle\langle\{n\}|$, with one flip at the site n, lie in $L_1 \subseteq L_E$ and satisfy $\|\rho_n - \rho_m\|_1 = 2$ for $n \neq m$, so no subsequence converges in the trace norm. Let $H^{(n)}$ be the span of the configurations with n flips; then $H_1 = \bigoplus_n H^{(n)}$, and N acts on $H^{(n)}$ as multiplication by n. A term $\sigma_j^+\sigma_{j+1}^-$ or $\sigma_j^-\sigma_{j+1}^+$ annihilates a configuration $|F\rangle$ unless exactly one of the sites j and $j+1$ lies in F, in which case it moves that flip to the other site. Hence $H_{XX}|F\rangle = \sum_j |F \triangle \{j, j+1\}\rangle$, the sum over the sites j with $|F \cap \{j, j+1\}| = 1$, of which there are at most $2|F|$. On $H^{(n)}$, the operator H_{XX} is therefore given by a symmetric matrix with entries 0 and 1 and at most $2n$ nonzero entries in each row and column, which is bounded, with norm at most $2n$, by Schur's test. We define H_{XX} as the direct sum of these bounded self-adjoint operators, a self-adjoint operator on H_1 that commutes with N; then $e^{-itH_{XX}}$ is the direct sum of the unitaries $e^{-itH_{XX}|_{H^{(n)}}}$, preserves each $H^{(n)}$, and the channel $\rho \mapsto e^{-itH_{XX}}\rho\, e^{itH_{XX}}$ preserves $\mathrm{Tr}(\rho N)$. The shift of sites maps $|F\rangle$ to $|F+1\rangle$, preserves each $H^{(n)}$, and preserves $\mathrm{Tr}(\rho N)$ for the same reason. □

On the one-flip sector the XX dynamics is the continuous-time quantum walk on $\mathbb{Z}$: in the momentum representation, H_{XX} acts as multiplication by $2\cos k$, $k \in [-\pi, \pi)$, an operator with purely absolutely continuous spectrum, and a flip of momentum k moves with group velocity $-2\sin k$. The same structure occurs for a photon in an infinite array of coupled waveguides, with the number of photons in place of the number of flips and passive linear optics as the confined channels, for a free particle or an atom that can ionize, with energy as the bound, and for a charged particle in a uniform magnetic field, whose Landau levels are infinitely degenerate and are preserved by the magnetic translations.

The quantum walk of one flip. We now restrict to the one-flip sector $\ell^2(\mathbb{Z})$, with orthonormal basis $(|n\rangle)_{n\in\mathbb{Z}}$, $|n\rangle$ the configuration with one flip at the site n. On this sector H_{XX} acts as the adjacency operator $A = S + S^*$ of $\mathbb{Z}$, where $S|n\rangle = |n+1\rangle$ is the shift, and the dynamics is $U_t = e^{-itA}$. We write $\hat{\psi}(k) = \sum_n \psi(n)e^{-ink}$, $k \in [-\pi, \pi)$; then $\psi(n) = \frac{1}{2\pi}\int \hat{\psi}(k)e^{ink}\,dk$ and A acts on $\hat{\psi}$ as multiplication by $2\cos k$. The *group velocity* is $v(k) = -2\sin k$, and P_+ denotes the spectral projection onto $\{k : v(k) > 0\} = (-\pi, 0)$, that is, $\widehat{P_+\psi} = \mathbf{1}_{(-\pi,0)}\hat{\psi}$.

The observable. Bob holds a flip on a chain of his own, and the observable $C = \sum_{a>b} |a\rangle\langle a| \otimes |b\rangle\langle b|$ asks whether Alice's flip lies to the right of Bob's. Its slice at a probe concentrated at the site m is the projection $Q_m = \mathbf{1}_{(m,\infty)}(x)$ onto the sites $n > m$, and we take Σ to be the probes concentrated at sites; thus $\varphi_C(\rho, \delta_m) = \mathrm{Tr}(\rho Q_m)$. The computations are the channels $\rho \mapsto U_t\rho U_t^*$, which we also write U_t, for t in the countable monoid $\Gamma = \{U_t : t \in \mathbb{Q},\ t \geq 0\}$, and we use orbit types, for which the Extendibility Axiom holds by Proposition 2.6. Since A is bounded, $t \mapsto U_t$ is continuous in norm, and so is $t \mapsto U_t^* X U_t$ for every bounded X; by Proposition 2.7, every real time $t \geq 0$ gives a deep computation $\tilde{U}_t$, and $t \mapsto \tilde{U}_t$ is continuous. The whole state space of the sector is the shard $\{\mathrm{Tr}(\rho N) \leq 1\}$ restricted to the sector, which is not compact (Proposition 4.1).

Lemma 4.2. *For every $m \in \mathbb{Z}$, the operators $U_t^* Q_m U_t$ converge strongly to P_+ as $t \to \infty$. Consequently, $\mathrm{Tr}(U_t\rho U_t^* Q_m) \to \mathrm{Tr}(\rho P_+)$ for every density operator ρ on $\ell^2(\mathbb{Z})$.*

Proof. The operators are bounded by 1, so it suffices to prove the convergence on a dense set. Let $D_\pm$ be the set of vectors ψ with $\hat{\psi}$ smooth and supported in a compact subset of $(-\pi, 0)$, respectively of $(0, \pi)$; since $\{0, -\pi\}$ is null, $D_+ \cup D_-$ spans a dense subspace, and $P_+\psi = \psi$ for $\psi \in D_+$, $P_+\psi = 0$ for $\psi \in D_-$. Let $\psi \in D_+$, with $\hat{\psi}$ supported in a compact $K \subseteq (-\pi, 0)$, and let $0 < c \leq 1$ with $v \geq 2c$ on K. Then $(U_t\psi)(n) = \frac{1}{2\pi}\int_K \hat{\psi}(k)e^{i\theta(k)}\,dk$ with phase $\theta(k) = nk - 2t\cos k$ and $\theta'(k) = n - tv(k)$. For $n \leq m$ and $t \geq 2(|m|+1)/c$ we have

$|\theta'(k)| = tv(k) - n \geq 2ct - n \geq c\,(t + |n|)$ on K, while every derivative $\theta^{(j)}$, $j \geq 2$, is bounded by $2t$. Repeated integration by parts (non-stationary phase) gives, for every M, a constant C_M with $|(U_t\psi)(n)| \leq C_M(t + |n|)^{-M}$ for all such n and t, whence

$$\|(1 - Q_m)U_t\psi\|^2 = \sum_{n \leq m} |(U_t\psi)(n)|^2 \leq C_M^2 \sum_{n \in \mathbb{Z}} (t + |n|)^{-2M} \longrightarrow 0.$$

Thus $U_t^* Q_m U_t \psi \to \psi = P_+\psi$. For $\psi \in D_-$ the same argument, with $v \leq -2c$ on the support and $n > m$, gives $\|Q_m U_t \psi\| \to 0$, so $U_t^* Q_m U_t \psi \to 0 = P_+\psi$. Finally, strong convergence of uniformly bounded operators gives $\mathrm{Tr}(\rho X_t) \to \mathrm{Tr}(\rho X)$ for every trace-class ρ. □

Lemma 4.3. *For every $t \in \mathbb{R}$ and $m, m' \in \mathbb{Z}$, the operators $U_t^* Q_m U_t - Q_m$ and $Q_m - Q_{m'}$ are compact. Hence, if ω is a state of $B(\ell^2(\mathbb{Z}))$ that vanishes on the compact operators, then $\omega(U_t^* Q_m U_t) = \omega(Q_0)$ for all t and m.*

Proof. The operator $Q_m - Q_{m'}$ has finite rank. The commutator $[S, Q_m]$ has rank one, so $[A, Q_m]$ has finite rank, and Duhamel's formula $[U_t, Q_m] = -i \int_0^t U_{t-s}[A, Q_m]U_s\, ds$ expresses $[U_t, Q_m]$ as a norm limit of finite-rank operators, since $s \mapsto U_s$ is norm continuous (A is bounded). Hence $U_t^* Q_m U_t - Q_m = U_t^*[Q_m, U_t]$ is compact. □

Theorem 4.4. *For the quantum walk of one flip, observed through the comparison of positions, the set $\Gamma = \{U_t : t \in \mathbb{Q},\ t \geq 0\}$ is $\mathrm{NIP}^{\mathrm{met}}$ and not stable. Its deep computations on the space of orbit types are the maps $\tilde{U}_t$, $t \geq 0$, together with one more map ξ, given at the orbit type of a state $\omega = \lambda\omega_{\mathrm{n}} + (1 - \lambda)\omega_{\mathrm{s}}$, with ω_{n} normal, ω_{s} vanishing on the compact operators and $0 \leq \lambda \leq 1$, by*

$$\xi(\mathrm{tp}_\Gamma(\omega))_{(U_s, \delta_m)} = \lambda\,\omega_{\mathrm{n}}(P_+) + (1 - \lambda)\,\omega_{\mathrm{s}}(Q_0).$$

On the orbit type of a density operator ρ, the deep computation ξ takes the value $\mathrm{Tr}(\rho P_+)$ at every coordinate: the probability that the flip moves to the right.

Proof. Every state of $B(\ell^2(\mathbb{Z}))$ decomposes uniquely as a convex combination of a normal state and a state that vanishes on the compact operators [28, Theorem III.2.14]. For the coordinate (U_s, δ_m) of $\tilde{U}_t(\mathrm{tp}_\Gamma(\omega))$, which is $\omega(U_{s+t}^* Q_m U_{s+t})$, Lemma 4.2 applied to the normal part and Lemma 4.3 applied to the singular part give the limit stated as $t \to \infty$. Hence the maps $\tilde{U}_t$ converge pointwise on the whole space of orbit types to ξ, which is therefore a function of the orbit type. By Proposition 2.7, $t \mapsto \tilde{U}_t$ is continuous into the space of maps with the pointwise topology, and with the value ξ at ∞ it is continuous on the compact metrizable space $[0, \infty]$. Its image is compact, contains the maps $\tilde{U}_q$, $q \in \mathbb{Q}$, as a dense subset, and is therefore the closure of $\{\tilde{U}_q : q \in \mathbb{Q},\ q \geq 0\}$. A continuous image of a compact metrizable space in a Hausdorff space is metrizable, so this closure is compact and metrizable, and ξ, a pointwise limit of a sequence of continuous maps, is of the first Baire class; so Γ is $\mathrm{NIP}^{\mathrm{met}}$.

To see that ξ is not continuous, let $\psi_+ \in D_+$ and $\psi_- \in D_-$ be unit vectors, and let $T_j = S^j$. The translations commute with U_s and with P_+, and $T_j^* Q_m T_j = Q_{m-j}$. The coordinate (U_s, δ_m) of the orbit type of $T_j\psi_\pm$ is $\|Q_{m-j}U_s\psi_\pm\|^2$, which tends to 0 as $j \to -\infty$, since $U_s\psi_\pm \in \ell^2(\mathbb{Z})$. Both sequences of orbit types therefore converge to the orbit type $\mathbf{0}$ whose coordinates all vanish, the type of a flip at $-\infty$. But ξ takes the value $\|P_+T_j\psi_+\|^2 = 1$ along the first sequence and $\|P_+T_j\psi_-\|^2 = 0$ along the second, at every coordinate. So ξ is not continuous at $\mathbf{0}$, and Γ is not stable. □

The walk is the quantum counterpart of Newton's method for square roots [13, Section 2.5]: its iterates converge pointwise, and the limit, the probability of moving right, is discontinuous at the flips at infinity, where the type no longer records the momentum.

Local probes. The scanning observable of [11] reads individual sites. On the one-flip sector its slices are the rank-one projections $R_m = |m\rangle\langle m|$, and $\varphi_C(\rho, \delta_m) = \mathrm{Tr}(\rho R_m)$ is the probability that the flip sits at the site m.

Proposition 4.5. *For the quantum walk of one flip, observed through the probes R_m, $m \in \mathbb{Z}$, the set $\Gamma = \{U_t : t \in \mathbb{Q},\ t \geq 0\}$ is stable. Its deep computations are the maps $\tilde{U}_t$, $t \geq 0$, together with the constant map onto the orbit type $\mathbf{0}$ whose coordinates all vanish.*

Proof. For a vector ψ, we have $\|U_t^* R_m U_t \psi\| = |\langle m, U_t\psi\rangle|$, and

$$\langle m, U_t\psi\rangle = \frac{1}{2\pi}\int \hat{\psi}(k) e^{i(mk - 2t\cos k)}\, dk \longrightarrow 0 \qquad (t \to \infty)$$

by the Riemann–Lebesgue lemma, since the push-forward of the complex measure $\hat{\psi}(k)e^{imk}\,dk$ under $k \mapsto 2\cos k$ is absolutely continuous. Hence $U_t^* R_m U_t \to 0$ strongly, and $\mathrm{Tr}(\rho\, U_t^* R_m U_t) \to 0$ for every density operator ρ. A state that vanishes on the compact operators vanishes at each operator $U_t^* R_m U_t$, which has rank one. Decomposing an arbitrary state into a normal part and a part that vanishes on the compact operators, as in the proof of Theorem 4.4, we obtain $\omega(U_{s+t}^* R_m U_{s+t}) \to 0$ as $t \to \infty$ for every state ω and all s and m: the maps $\tilde{U}_t$ converge pointwise on the space of orbit types to the constant map onto $\mathbf{0}$. As in the proof of Theorem 4.4, the closure of $\{\tilde{U}_t : t \in \mathbb{Q},\ t \geq 0\}$ is the image of $[0, \infty]$, and all its members are continuous: the maps $\tilde{U}_t$ by Proposition 2.6, and the constant map trivially. □

A flip that spreads is eventually invisible to every local probe, and only the probes that compare positions see where it went.

The ends of the line. Theorem 4.4 places the walk at $\mathrm{NIP}^{\mathrm{met}}$. The next theorem shows that no family of channels that preserves the two ends of the line goes further: on the line, the comparison of positions sees only two points at infinity.

Theorem 4.6. *On the one-flip sector, observed through the comparison of positions with the probes δ_m, $m \in \mathbb{Z}$, let Γ be a countable monoid of channels such that, for every $\Phi \in \Gamma$ and every m, the operator $\Phi^*(Q_m) - Q_0$ is compact. Then every subset of Γ is $\mathrm{NIP}^{\mathrm{met}}$.*

Proof. Every state ω of $B(\ell^2(\mathbb{Z}))$ decomposes as $\omega = \lambda\omega_{\mathrm{n}} + (1-\lambda)\omega_{\mathrm{s}}$, with ω_{n} normal, ω_{s} vanishing on the compact operators and $0 \leq \lambda \leq 1$ [28, Theorem III.2.14]. By hypothesis, the coordinate (Φ, δ_m) of the orbit type of ω_{s} is $\omega_{\mathrm{s}}(\Phi^*(Q_m)) = \omega_{\mathrm{s}}(Q_0) =: c$, independent of the coordinate. Writing $\mathbf{c}$ for the orbit type with all coordinates equal to c, and ρ for the density operator of ω_{n}, every orbit type therefore has the form $\lambda\,\mathrm{tp}_\Gamma(\rho) + (1-\lambda)\mathbf{c}$. For $\Psi \in \Gamma$ the coordinates of $\tilde{\Psi}(\mathrm{tp}_\Gamma(\omega))$ are $\omega((\Phi\circ\Psi)^*(Q_m))$, and $\Phi\circ\Psi \in \Gamma$, so

$$\tilde{\Psi}\big(\lambda\,\mathrm{tp}_\Gamma(\rho) + (1-\lambda)\mathbf{c}\big) = \lambda\,\mathrm{tp}_\Gamma(\Psi(\rho)) + (1-\lambda)\mathbf{c}.$$

Let $\Delta \subseteq \Gamma$, let K be the closure of $\{\tilde{\Psi} : \Psi \in \Delta\}$ in the space of maps from orbit types to orbit types, and let $\xi = \lim_{i,\mathcal{U}} \tilde{\Psi}_i \in K$. Passing to the limit in the display, $\xi(\lambda\,\mathrm{tp}_\Gamma(\rho) + (1-\lambda)\mathbf{c}) = \lambda\,\xi(\mathrm{tp}_\Gamma(\rho)) + (1-\lambda)\mathbf{c}$, so ξ is determined by its values at the orbit types of density operators. For each coordinate (Φ, δ_m), the operators $\Psi_i^*(\Phi^*(Q_m))$ lie in the unit ball of $B(\ell^2(\mathbb{Z}))$, which is compact and metrizable in the σ-weak topology, the topology of convergence against trace-class operators; let $X^\xi_{\Phi,m}$ be their σ-weak limit along $\mathcal{U}$. Then $\xi(\mathrm{tp}_\Gamma(\rho))_{(\Phi,\delta_m)} = \mathrm{Tr}(\rho X^\xi_{\Phi,m})$ for every density operator ρ. The map $\xi \mapsto (X^\xi_{\Phi,m})_{(\Phi,m)}$ from K into the countable product of σ-weakly compact unit balls is therefore well defined, injective, by the preceding paragraph, and continuous, since pointwise convergence at the orbit types of density operators is σ-weak convergence. A continuous injection of a compact space into a metrizable space is an embedding,

so K is metrizable. Every member of K is then the pointwise limit of a sequence of maps $\tilde{\Psi}$ with $\Psi \in \Delta$, which are continuous, and is therefore of the first Baire class; so Δ has the NIP by Theorem 3.1, and it is $\mathrm{NIP}^{\mathrm{met}}$. □

Corollary 4.7. *Let Γ be a countable monoid generated by any of the following channels of the one-flip sector: the walk U_t, the translations $S^{\pm 1}$, the boosts $\psi \mapsto e^{i\theta x}\psi$, and the classical coarse-graining $\rho \mapsto \sum_n \langle n|\rho|n\rangle\, |\lfloor n/2\rfloor\rangle\langle\lfloor n/2\rfloor|$. Then every subset of Γ is $\mathrm{NIP}^{\mathrm{met}}$.*

Proof. Each generator Φ satisfies $\Phi^*(Q_m) - Q_{m'}$ compact for some m', and Φ^* maps compact operators to compact operators: for the walk by Lemma 4.3; for the translations, since $S^{-j}Q_m S^j = Q_{m-j}$; for the boosts, which commute with Q_m; and for the coarse-graining, whose dual sends Q_m to Q_{2m+1} and each $|j\rangle\langle j|$ to $|2j\rangle\langle 2j| + |2j+1\rangle\langle 2j+1|$. Since $Q_{m'} - Q_0$ has finite rank, these two properties pass to compositions, and Theorem 4.6 applies. □

Theorem 4.6 isolates the source of the lower levels. The restriction of a deep computation to normal states is a σ-weak limit of operators, and the σ-weak topology on bounded sets is metrizable; a nonmetrizable space of deep computations can arise only from the action of the channels on idealized states. On the line, the comparison of positions sees two idealized directions, the two ends, and the channels of Corollary 4.7 fix them. The lower levels therefore require a geometry whose boundary at infinity is rich enough for the channels to act on it.

The extension of these results from one flip to the whole chain, and to lattices of higher dimension, is the subject of Questions 10.9 and 10.10.

5. Measurements on a tree: binary search as quantum instruments

On the infinite rooted binary tree, escape to infinity has a continuum of possible destinations, the branches of its boundary, which is the Cantor set. The tree is a rooted relative of the Bethe lattice of physics, and its prefix and threshold tests are the primitives of binary search through which [13, Section 3.3] reads the heptachotomy. By measuring one of these primitives and preparing a record of the answer, we obtain channels whose limits retain distinctions that the two ends of the line could not support.

Prefix and threshold tests. *The setting.* Let $T = 2^{<\mathbb{N}}$ be the set of finite binary words, the nodes of the tree, and let $H_1 = \ell^2(T)$, the space of one particle on the tree, with basis $(|u\rangle)_{u\in T}$; the shard of one particle is the whole state space, which is not compact. For $u \in T$ write $\bar{u} = u^\frown 0^\infty \in 2^{\mathbb{N}}$, and order $2^{\mathbb{N}}$ lexicographically. Bob's probes select a node $s \in T$ and test it in one of two ways, through the projections

$$V_s = \sum_{s \sqsubseteq u} |u\rangle\langle u|, \qquad L_s = \sum_{\bar{u} \geq \bar{s}} |u\rangle\langle u|,$$

the *prefix test* (is the particle below the node s?) and the *threshold test* (is the particle at or to the right of $\bar{s}$?). Precisely, let $H_2 = \ell^2(T \times \{V, L\})$, with basis $(|s, V\rangle, |s, L\rangle)_{s\in T}$, and

$$C = \sum_{s\in T} \big(V_s \otimes |s, V\rangle\langle s, V| + L_s \otimes |s, L\rangle\langle s, L|\big),$$

a sum of positive operators with mutually orthogonal ranges on Bob's side, which converges strongly to a projection. The slices of C at the probes $|s, V\rangle\langle s, V|$ and $|s, L\rangle\langle s, L|$ are V_s and L_s, and Σ is the countable set of these probes. For a state ω and a node s we write $\omega_{V_s} = \omega(V_s)$ and $\omega_{L_s} = \omega(L_s)$.

The instruments. For a test P, either V_s or L_s, the *measure-and-prepare* channel

$$M_P(\rho) = \mathrm{Tr}(P\rho)\, |1\rangle\langle 1| + \mathrm{Tr}((1-P)\rho)\, |0\rangle\langle 0|$$

measures the test and prepares the particle at the node 1 or at the node 0 of depth one, according to the answer. It implements one primitive of binary search, a single comparison with its answer recorded in the position of the particle, not the adaptive search itself. Let Γ be the monoid generated by the identity and all the channels M_{V_s} and M_{L_s}, $s \in T$. The orbit types of this structure reduce to the probabilities of the tests. For a bounded operator X, the dual of M_P is

$$M_P^*(X) = \langle 1|X|1\rangle P + \langle 0|X|0\rangle (1-P).$$

Every test X takes the value 0 or 1 at each of the nodes 0 and 1, so $M_P^*(X)$ is one of 0, P, $1-P$ and 1. The same holds for composites: the dual of $M_{P_1} \circ \cdots \circ M_{P_k}$ sends each test to 0, P_k, $1-P_k$ or 1. Hence the numbers ω_{V_s} and ω_{L_s}, $s \in T$, determine the orbit type of a state ω, and, writing τ_0 and τ_1 for the orbit types of the nodes 0 and 1, every channel M_P acts on orbit types by

$$\tilde{M}_P(\mathbf{v}) = \mathbf{v}_P\, \tau_1 + (1 - \mathbf{v}_P)\, \tau_0.$$

The prefix test V_1 takes the value 1 at the node 1 and 0 at the node 0, so $\tau_0 \neq \tau_1$, and $x \mapsto x\tau_1 + (1-x)\tau_0$ is a continuous injection of $[0,1]$ into $\mathcal{L}_\Gamma$. Composition with this injection is a homeomorphism of the closure of a family of scalar functions $\{\mathbf{v} \mapsto \mathbf{v}_P\}$ in $[0,1]^{\mathcal{L}_\Gamma}$ onto the closure of the corresponding family of maps $\tilde{M}_P$. A set of instruments therefore has the level of its family of scalar profiles $\mathbf{v} \mapsto \mathbf{v}_P$, and a test may be read as an orbit type $\mathbf{v}$, with the value $g(\mathbf{v})$ at a limit g of such profiles.

Lemma 5.1. (1) *For every orbit type* $\mathbf{v}$ *and every finite set of pairwise incomparable nodes* $s_1, \ldots, s_k$, *we have* $\sum_i \mathbf{v}_{V_{s_i}} \leq 1$.
(2) *For every orbit type* $\mathbf{v}$, *the function* $s \mapsto \mathbf{v}_{L_s}$ *is nonincreasing in* $\bar{s}$.
(3) *For every* $x \in 2^{\mathbb{N}}$, *the orbit types of the nodes* $x{\restriction}n$ *converge, as* $n \to \infty$, *to an orbit type* β_x, *the type of a particle escaping along the branch* x, *with* $(\beta_x)_{V_s} = 1$ *if* $s \sqsubseteq x$ *and* 0 *otherwise, and* $(\beta_x)_{L_s} = 1$ *if* $\bar{s} \leq x$ *and* 0 *otherwise.*

Proof. (1) and (2) hold for states, since the projections V_{s_i} are pairwise orthogonal and $L_s \geq L_{s'}$ when $\bar{s} \leq \bar{s}'$, and they pass to limits of orbit types. (3) For $s \in T$, the node $x{\restriction}n$ lies below s for all large n exactly when $s \sqsubseteq x$. The points $\overline{x{\restriction}n}$ increase to x; if $\bar{s} < x$ they eventually satisfy $\overline{x{\restriction}n} \geq \bar{s}$, if $\bar{s} > x$ they never do, and if $\bar{s} = x$, then x ends in zeros and $\overline{x{\restriction}n} = x$ for large n. The coordinates of the orbit types are affine in these numbers, so they converge. □

Theorem 5.2. *The set* $\Delta_V = \{M_{V_s} : s \in T\}$ *of instruments that measure prefix tests has the NIP and is not* $\mathrm{NIP}^{\mathrm{fc}}$.

Proof. By the homeomorphism above, it suffices to study the family $\{\mathbf{v} \mapsto \mathbf{v}_{V_s} : s \in T\}$ of functions on orbit types. *NIP.* Suppose that distinct nodes s_n and reals $a < b$ witness an independent sequence. By Ramsey's theorem, a subsequence is either a chain or an antichain in the tree. Along a chain $s_1 \sqsubset s_2 \sqsubset \cdots$ the functions decrease, so the pattern $\mathbf{v}_{V_{s_1}} \leq a$, $\mathbf{v}_{V_{s_2}} \geq b$ is never realized. Along an antichain, Lemma 5.1(1) allows at most $1/b$ of the functions to be at least b at the same orbit type, where $b > 0$ because the functions are nonnegative, so the patterns with more than $1/b$ values at least b are never realized.

Not first countable. Let K be the closure of $\{\tilde{M}_{V_s} : s \in T\}$. For $x \in 2^{\mathbb{N}}$ the values $\mathbf{v}_{V_{x{\restriction}n}}$ decrease in n, so $\tilde{M}_{V_{x{\restriction}n}}$ converges pointwise to a member G_x of K, determined by $g_x(\mathbf{v}) = \lim_n \mathbf{v}_{V_{x{\restriction}n}}$; along an antichain the values tend to 0 by Lemma 5.1(1), so K also contains the member G_0 determined by the function 0. By Lemma 5.1(3), $g_x(\beta_x) = 1$, so $G_x \neq G_0$. Since the projections $V_{x{\restriction}n}$ decrease strongly to 0, the function g_x vanishes at the orbit type of every density operator: G_x and G_0 agree at every actual preparation and differ at the idealized orbit

type β_x. A test, read as an orbit type $\mathbf{v}$, takes the value $g_x(\mathbf{v})$ at G_x and 0 at G_0. For fixed $\mathbf{v}$, Lemma 5.1(1), applied to initial segments of finitely many distinct branches at a depth where they are incomparable, gives $\sum_i g_{x_i}(\mathbf{v}) \leq 1$ for distinct $x_1, \dots, x_k$, so $g_x(\mathbf{v}) \neq 0$ for at most countably many x. Hence every countable set of tests agrees on G_0 and G_x for all but countably many x, and G_0 is not countably determined. By [13, Proposition 3.4(ii)], the compactum K is not first countable. □

Theorem 5.3. *The set $\Delta_L = \{M_{L_s} : s \in T\}$ of instruments that measure threshold tests is* $\mathrm{NIP}^{\mathrm{hs}}$ *and not* $\mathrm{NIP}^{\mathrm{met}}$.

Proof. As before, it suffices to study the functions $\mathbf{v} \mapsto \mathbf{v}_{L_s}$. *NIP.* Every sequence of nodes has a subsequence along which $\bar{s}_n$ is monotone, and along it the functions are monotone by Lemma 5.1(2); for reals $a < b$, a nonincreasing sequence of functions never realizes the pattern $f_1 \leq a$, $f_2 \geq b$, and a nondecreasing one never realizes the pattern $f_1 \geq b$, $f_2 \leq a$, so neither is independent.

Not metrizable. For $x \in 2^{\mathbb{N}}$ that does not end in zeros or in ones, and every orbit type $\mathbf{v}$, the one-sided limits $g_x^-(\mathbf{v}) = \lim_{\bar{s}\uparrow x} \mathbf{v}_{L_s}$ and $g_x^+(\mathbf{v}) = \lim_{\bar{s}\downarrow x} \mathbf{v}_{L_s}$ exist by monotonicity, and every sequence of nodes with $\bar{s}_n$ increasing, respectively decreasing, to x gives them; so the closure K of $\{\tilde{M}_{L_s} : s \in T\}$ contains members G_x^- and G_x^+ determined by g_x^- and g_x^+. By Lemma 5.1(3), $g_x^-(\beta_x) = 1$ and $g_x^+(\beta_x) = 0$, so $G_x^- \neq G_x^+$. A test, read as an orbit type $\mathbf{v}$, separates G_x^- from G_x^+ only if $g_x^-(\mathbf{v}) \neq g_x^+(\mathbf{v})$, that is, only if the nonincreasing function $s \mapsto \mathbf{v}_{L_s}$, with values in $[0,1]$, has a jump at x, which happens for at most countably many x. A countable set of tests therefore leaves uncountably many pairs unseparated, and by [13, Proposition 3.4(i)] the compactum K is not metrizable.

Hereditarily separable. Let F be the closure of $\{g_s : s \in T\}$, with $g_s(\mathbf{v}) = \mathbf{v}_{L_s}$, in $[0,1]^{\mathcal{L}_\Gamma}$; by the homeomorphism above, it suffices to show that F is hereditarily separable. By Lemma 5.1(2), the functions g_s form a chain for the pointwise order. So does F: if $f, g \in F$ satisfied $f(\mathbf{v}) < g(\mathbf{v})$ and $f(\mathbf{w}) > g(\mathbf{w})$, then functions g_s and $g_{s'}$ close enough to f and g at $\mathbf{v}$ and $\mathbf{w}$ would satisfy the same strict inequalities and be incomparable. The pointwise order is closed in $F \times F$, so each ray $\{h : h < g\}$, the complement of the closed set $\{h : g \leq h\}$, is open; the order topology of F is therefore coarser than its compact Hausdorff topology, and, being Hausdorff, equal to it. Thus F is a linearly ordered topological space, separable because $\{g_s : s \in T\}$ is countable and dense, and separability is hereditary in linearly ordered topological spaces, by a theorem of Lutzer and Bennett (see [17, Section 8]). □

The two proofs use different features of the boundary: prefix tests distinguish individual branches, while threshold tests retain their order.

Exact canonical families. The class of a compactum does not identify the indexed family that produces it, and exact indexed equivalence requires agreement of convergent index sequences, not merely of topological classes. The same comparison will identify two further instruments, which realize the class of first countable compacta that are not hereditarily separable. We recall the canonical families of Argyros, Dodos and Kanellopoulos [5, Section 4.3], indexed by T. For $t \in T$, let v_t be the indicator function of the set of branches $b \in 2^{\mathbb{N}}$ with $t \sqsubseteq b$, and, for $a \in 2^{\mathbb{N}}$, let x_a^+ be the indicator function of $\{b : a \leq b\}$. Fix the dyadic subtree $(s_t)_{t \in T}$ given by $s_\emptyset = 01$ and $s_t = 01^\frown t(0)01^\frown \cdots ^\frown t(n-1)01$ for t of length n; its nodes have pairwise distinct points $\bar{s}_t$, and its branches do not end in zeros or in ones. The families

$$D_3 = \{x^+_{\bar{s}_t}\}, \qquad D_5 = \{v_{s_t}\}, \qquad D_6 = \{(v_t, \bar{t})\}, \qquad D_7 = \{(v_{s_t}, x^+_{\bar{s}_t})\}$$

have as closures the extended split Cantor set $\hat{S}_+(2^{\mathbb{N}})$, the extended Alexandroff compactification $\hat{A}(2^{\mathbb{N}})$, the extended duplicate $\hat{D}(2^{\mathbb{N}})$ of the Cantor set, and the extended duplicate $\hat{D}(S(2^{\mathbb{N}}))$ of the split Cantor set [5, Sections 4.3.3–4.3.7]; the last two are first countable and not hereditarily separable. Two families $\{f_t\}$ and $\{g_t\}$ are *equivalent* if $f_t \mapsto g_t$ extends to a homeomorphism of their closures. By [5, Lemma 19], two dense families of isolated points of separable Rosenthal compacta on Polish spaces are equivalent if and only if, for every infinite set L of indices, $(f_t)_{t\in L}$ converges pointwise exactly when $(g_t)_{t\in L}$ does.

Lemma 5.4. *Let $L \subseteq T$ be infinite and let ω be a state of $B(\ell^2(T))$.*

(1) *If $(v_t)_{t\in L}$ converges pointwise on $2^{\mathbb{N}}$, then $(\omega(V_t))_{t\in L}$ converges.*
(2) *If $(x^+_{\bar{s}_t})_{t\in L}$ converges pointwise on $2^{\mathbb{N}}$, then $(\omega(L_{s_t}))_{t\in L}$ converges.*

Conversely, the values at the orbit types β_b of Lemma 5.1(3) are $\beta_b(V_t) = v_t(b)$ and $\beta_b(L_s) = x^+_{\bar{s}}(b)$, so convergence at every state gives convergence on $2^{\mathbb{N}}$.

Proof. (1) An infinite chain of nodes along a branch b makes v_t converge to the indicator of $\{b\}$, and a sequence of nodes without an infinite chain makes it converge to 0, by Ramsey's theorem, since an infinite antichain gives the limit 0. If the limit is the indicator of $\{b\}$, all but finitely many members of L are initial segments of b, since infinitely many others would contain a chain along another branch or an antichain, with a different limit; and $\omega(V_t)$ converges along them, because these values are nonincreasing in the length of t. If the limit is 0, no infinite subset of L is a chain, so every infinite subset contains an infinite antichain, along which $\omega(V_t) \to 0$, since the projections of an antichain are orthogonal and their values under ω have sum at most 1; hence $\omega(V_t) \to 0$ along L.

(2) Let $a_t = \bar{s}_t$. Every limit point c of $(a_t)_{t\in L}$ is a branch of the subtree, which ends neither in zeros nor in ones and differs from every a_t; such a point has no immediate neighbor in the lexicographic order. If (a_t) had two limit points $c < c'$, a branch b with $c < b < c'$ would give $x^+_{a_t}(b)$ the values 1 near c and 0 near c'; so $a_t \to c$. If infinitely many a_t lay below c and infinitely many above, the values $x^+_{a_t}(c)$ would be 1 and 0 infinitely often; so eventually $a_t < c$, say. The function $a \mapsto \omega(L_s)$, $a = \bar{s}$, is nonincreasing, and for every $a < c$ eventually $a < a_t < c$; hence $\omega(L_{s_t})$ converges along L to $\inf_{a<c} \omega(L_s)$, the infimum over nodes s of the subtree with $\bar{s} < c$. The case $a_t > c$ is symmetric. The final assertion is Lemma 5.1(3). □

Theorem 5.5. *The following hold.*

(1) *The family $\{\tilde{M}_{V_{s_t}}\}_{t\in T}$ of prefix instruments is equivalent to D_5, and its closure is homeomorphic to $\hat{A}(2^{\mathbb{N}})$.*
(2) *The family $\{\tilde{M}_{L_{s_t}}\}_{t\in T}$ of threshold instruments is equivalent to D_3, and its closure is homeomorphic to $\hat{S}_+(2^{\mathbb{N}})$.*

Proof. By the homeomorphism of the instruments with their scalar profiles, it suffices to compare the families $f_t(\mathbf{v}) = \mathbf{v}_{V_{s_t}}$ and $f_t(\mathbf{v}) = \mathbf{v}_{L_{s_t}}$ on $\mathcal{L}_\Gamma$ with D_5 and D_3. Their closures are separable Rosenthal compacta on the Polish space $\mathcal{L}_\Gamma$, by Theorems 5.2 and 5.3, and every orbit type is the orbit type of a state; by Lemma 5.4, applied to the subfamily (V_{s_t}) in (1), the two families have the same convergent subsequences as D_5 and D_3. Each f_t is isolated in its closure. Restriction to the branch types β_b, $b \in 2^{\mathbb{N}}$, is a continuous map R of the closure onto the closure of the canonical family, with $R(f_t) = d_t$, the canonical function; a member $g \neq f_t$ of the closure is the limit, along an ultrafilter, of members f_{t_i} with $t_i \neq t$, so $R(g)$ lies in the closure of $\{d_{t'} : t' \neq t\}$, which does not contain the isolated point d_t; hence $R^{-1}(d_t) = \{f_t\}$ is open. Lemma 19 of [5] applies. □

Two further instruments. The class of first countable compacta that are not hereditarily separable is reached by recording more than one bit. Both instruments below act on $H = \ell^2(T) \oplus O$, with a finite-dimensional output register O, and the tests V_t and L_t are extended by zero on O. Lemmas 5.1 and 5.4 hold for the states of $B(H)$, with the same proofs, which use only positivity and the value 1 at the identity.

The *three-outcome instrument* at a node s has $O = \mathbb{C}^3$, with basis $|\ell\rangle, |m\rangle, |r\rangle$, and is the channel

$$J_s(\rho) = \operatorname{Tr}(\rho(I - L_s))\,|\ell\rangle\langle\ell| + \operatorname{Tr}(\rho V_s)\,|m\rangle\langle m| + \operatorname{Tr}(\rho(L_s - V_s))\,|r\rangle\langle r|;$$

since $V_s \le L_s$, the three projections $I - L_s$, V_s and $L_s - V_s$ are orthogonal with sum I, and J_s measures whether the particle lies to the left of $\bar{s}$, below s, or to the right of $\bar{s}$ and not below s, and records the answer. Bob reads the register through the observable $C = A\otimes|0\rangle\langle 0| + B\otimes|1\rangle\langle 1|$ on $H \otimes \mathbb{C}^2$, with $A = |m\rangle\langle m|$ and $B = |m\rangle\langle m| + |r\rangle\langle r|$, whose slices at the two basis probes are A and B; then $J_s^*(A) = V_s$ and $J_s^*(B) = L_s$. The coordinates of the orbit type of a state ω given by J_s followed by these two probes are therefore $\omega(V_s)$ and $\omega(L_s)$, and, as in the binary case, we write them $\mathbf{v}_{V_s}$ and $\mathbf{v}_{L_s}$.

The *address instrument* uses the register $O = \mathbb{C}^2 \otimes \mathbb{C}^2$: it records the answer of the prefix test at a node t in the second qubit, and the padded address $\bar{t} = t^\frown 0^\infty$ of the node in the preparation probability of the first; distinct nodes, such as t and $t^\frown 0$, may have the same padded address. For $t \in T$, let $r_t = \sum_{j\in\mathbb{N}} 2\bar{t}(j)3^{-j-1}$ be the point of the middle-thirds Cantor set with address $\bar{t}$, and let $\tau(r) = (1-r)|0\rangle\langle 0| + r|1\rangle\langle 1|$ for $r \in [0,1]$. The address instrument at t is the channel

$$N_t(\rho) = \tau(r_t) \otimes \big(\operatorname{Tr}(\rho V_t)\,|1\rangle\langle 1| + \operatorname{Tr}(\rho(I - V_t))\,|0\rangle\langle 0|\big).$$

Bob reads it through $C = E \otimes |0\rangle\langle 0| + F \otimes |1\rangle\langle 1|$, where the register projections $E = |1\rangle\langle 1| \otimes I$ and $F = I \otimes |1\rangle\langle 1|$ are extended by zero on $\ell^2(T)$. Then $N_t^*(E) = r_t I$ and $N_t^*(F) = V_t$, so the coordinate given by N_t followed by the probe of F is $\omega(V_t)$, which we again write $\mathbf{v}_{V_t}$.

Theorem 5.6. *Let Γ_J and Γ_N be the monoids generated by the identity and the channels J_{s_t}, respectively N_t, $t \in T$.*

(1) *The family $\{\tilde{J}_{s_t}\}_{t\in T}$ is equivalent to D_7, and its closure is homeomorphic to $\hat{D}(S(2^{\mathbb{N}}))$.*
(2) *The family $\{\tilde{N}_t\}_{t\in T}$ is equivalent to D_6, and its closure is homeomorphic to $\hat{D}(2^{\mathbb{N}})$.*

In particular, both sets of instruments are $\mathrm{NIP}^{\mathrm{fc}}$ *and not* $\mathrm{NIP}^{\mathrm{hs}}$.

Proof. Each instrument measures orthogonal projections with sum I and prepares a state for each outcome, so the channels are completely positive and trace preserving. After one instrument the state lies in the output register, where all the tests vanish. A second instrument therefore prepares a fixed state: $J_s \circ J_{s'}$ prepares $|\ell\rangle$, and $N_t \circ N_{t'}$ prepares $\tau(r_t) \otimes |0\rangle\langle 0|$. In particular, the generated monoids are countable.

For a state ω, let $p = \omega(V_s)$ and $q = \omega(L_s)$. The output state $\omega \circ J_s^*$ is represented by

$$(1-q)\,|\ell\rangle\langle\ell| + p\,|m\rangle\langle m| + (q-p)\,|r\rangle\langle r|,$$

which depends continuously and injectively on (p, q). Its coordinates at the identity, followed by the probes of A and B, are p and q, and all its other coordinates are constant. Hence the closure of $\{\tilde{J}_{s_t}\}$ is homeomorphic, through the indexing, to the closure of the pairs $\mathbf{v} \mapsto (\mathbf{v}_{V_{s_t}}, \mathbf{v}_{L_{s_t}})$. The address instruments give the pairs $(r_t, \mathbf{v}_{V_t})$ and the corresponding homeomorphism for $\{\tilde{N}_t\}$.

A sequence of pairs converges exactly when both components do. By Lemma 5.4, the prefix components have the convergent subsequences of D_5 in both cases, the threshold components those of D_3 in (1), and in (2) the address components r_t converge exactly when $\bar{t}$ does, since

the map $b \mapsto \sum_j 2b(j)3^{-j-1}$ is a homeomorphism of $2^{\mathbb{N}}$ onto the Cantor set. The families have the NIP by the arguments of Theorems 5.2 and 5.3, so their closures are separable Rosenthal compacta, and their members are isolated by the argument of Theorem 5.5. Lemma 19 of [5] gives the equivalences, and the classes of the canonical closures are those of [5, Section 4.3]. □

Part II. What the observations decide

The examples have attached levels to channels together with the observations that distinguish their outputs. Forgetting observations can identify limits that were distinct, and enlarging the observed formulas can separate them. The maps between spaces of types that do this are the subject of Part II.

6. Observations and levels

6.1. **φ-types and orbit types.** The φ-types are the projections $\pi(\mathbf{v})$ of the orbit types onto the coordinates (id, σ) (Section 2). In two situations the two kinds of type lead to the same NIP or the same deep computations. A set $\Delta \subseteq \Gamma$ is a *left ideal* of Γ if $\Phi \circ \Psi \in \Delta$ for all $\Phi \in \Gamma$ and $\Psi \in \Delta$; the monoid Γ itself is one.

Proposition 6.1. *Let $\Delta \subseteq \Gamma$ and let $L \subseteq \mathfrak{D}(H_1)$.*

(1) *If Δ is a left ideal of Γ and, for every $\sigma \in \Sigma$, the family $\{\rho \mapsto \varphi_C(\Psi(\rho), \sigma) : \Psi \in \Delta\}$ has the NIP on L, then so does the family $\{\rho \mapsto \varphi_C(\Phi(\Psi(\rho)), \sigma) : \Psi \in \Delta\}$, for every $(\Phi, \sigma) \in \Gamma \times \Sigma$.*

(2) *If the members of Γ commute, then every deep computation ξ of Δ satisfies*

$$\xi(\mathbf{v})_{(\Phi,\sigma)} = \pi\big(\xi(\tilde{\Phi}(\mathbf{v}))\big)_\sigma \qquad (\mathbf{v} \in \mathcal{L}_\Gamma,\ (\Phi, \sigma) \in \Gamma \times \Sigma),$$

and $\xi \mapsto \pi \circ \xi$ is a homeomorphism of the space of deep computations of Δ onto the closure of $\{\pi \circ \tilde{\Psi} : \Psi \in \Delta\}$ in $\mathcal{L}^{\mathcal{L}_\Gamma}$.

Proof. (1) Since $\Phi \circ \Psi \in \Delta$ for $\Psi \in \Delta$, the family for (Φ, σ) is a subfamily of the family for (id, σ), and the NIP passes to subfamilies.

(2) Let $\xi = \lim_{i,\mathcal{U}} \tilde{\Psi}_i$, with $\Psi_i \in \Delta$, and let ω be a state with $\mathrm{tp}_\Gamma(\omega) = \mathbf{v}$. Since $\Phi \circ \Psi_i = \Psi_i \circ \Phi$,

$$\xi(\mathbf{v})_{(\Phi,\sigma)} = \lim_{i,\mathcal{U}} \omega\big((\Phi \circ \Psi_i)^*(S_C(\sigma))\big) = \lim_{i,\mathcal{U}} (\omega \circ \Phi^*)\big(\Psi_i^*(S_C(\sigma))\big) = \xi\big(\mathrm{tp}_\Gamma(\omega \circ \Phi^*)\big)_{(\mathrm{id},\sigma)},$$

and $\mathrm{tp}_\Gamma(\omega \circ \Phi^*) = \tilde{\Phi}(\mathbf{v})$ by Proposition 2.6. Hence $\pi \circ \xi$ determines ξ, and the map $\xi \mapsto \pi \circ \xi$, which is continuous for the pointwise topologies, is injective; a continuous injection of a compact space into a Hausdorff space is an embedding, and its image is the closure of the image of the dense set $\{\tilde{\Psi} : \Psi \in \Delta\}$. □

For commuting computations, therefore, the classification sees only what the probes of Σ report about the outputs of the computations, read on orbit types: by Proposition 6.1(2), the space of deep computations and its level are those of the family of maps $\mathbf{v} \mapsto \mathbf{v}_{(\Psi,\sigma)}$, $\Psi \in \Delta$.

6.2. **Instances and the whole family.** Condition (ii) of Theorem 3.1 is the NIP of each instance of the formula $\varphi_C(\Phi(\Psi(x)), \sigma)$ separately. The NIP of the whole family, with the computation and the probe varying together, is strictly stronger, and no regularity of the deep computations alone can detect it.

Proposition 6.2. *Let H_1 have the orthonormal basis $(|F\rangle)$ indexed by the finite subsets $F \subseteq \mathbb{N}$, let P_j be the projection onto the span of the vectors $|F\rangle$ with $j \in F$, let $H_2 = \ell^2(\mathbb{N})$, and let $C = \sum_j P_j \otimes |j\rangle\langle j|$, with the probes $\Sigma = \{|j\rangle\langle j| : j \in \mathbb{N}\}$ and $\Gamma = \Delta = \{\mathrm{id}\}$. Then condition (ii) of Theorem 3.1 holds and Δ is stable, but the whole family $\{\rho \mapsto \mathrm{Tr}(\rho P_j) : j \in \mathbb{N}\}$ does not have the NIP on $\mathfrak{D}(H_1)$.*

Proof. The operator C is a sum of projections with mutually orthogonal ranges, hence a projection, and its slices are the projections P_j. Each family of condition (ii) has a single member, and the only deep computation is the identity, which is continuous. For disjoint finite sets $I, J \subseteq \mathbb{N}$, the basis vector $|J\rangle$ satisfies $\langle J|P_j|J\rangle = 0$ for $j \in I$ and 1 for $j \in J$, so the functions $\rho \mapsto \operatorname{Tr}(\rho P_j)$ form an independent sequence. □

What the NIP of the whole family adds to the classification is Question 10.3.

6.3. **Mixing.** Computations can be mixed: one may choose at random which channel to apply. Mixing produces convex combinations of the induced maps, so a set of channels that is closed under mixing has a convex compactum of limits, and convexity restricts its place in the trichotomy. A set Δ of channels is *closed under rational mixtures* if $\lambda\Psi_1 + (1-\lambda)\Psi_2 \in \Delta$ for all $\Psi_1, \Psi_2 \in \Delta$ and rational $\lambda \in [0,1]$.

Proposition 6.3. *Let $\Delta \subseteq \Gamma$ be countable, closed under rational mixtures, confined to L_E, and with the NIP on L_E. Then K_E is convex, and if K_E is hereditarily separable, then it is metrizable. Hence Δ is* $\mathrm{NIP}^{\mathrm{met}}$ *on L_E, or* $\mathrm{NIP}^{\mathrm{fc}}$ *and not* $\mathrm{NIP}^{\mathrm{hs}}$*, or not* $\mathrm{NIP}^{\mathrm{fc}}$*.*

Proof. For $\Psi = \lambda\Psi_1 + (1-\lambda)\Psi_2$ and a state ω, $\omega(\Psi^*(X)) = \lambda\,\omega(\Psi_1^*(X)) + (1-\lambda)\,\omega(\Psi_2^*(X))$ for every operator X, so $\tilde{\Psi} = \lambda\tilde{\Psi}_1 + (1-\lambda)\tilde{\Psi}_2$ on $\mathcal{L}_\Gamma$, every point of which is the orbit type of a state. Convex combinations are continuous for the topology of pointwise convergence, so the closure K_E of the set $\{\tilde{\Psi}|_{\mathcal{L}_E} : \Psi \in \Delta\}$, which is closed under rational convex combinations, is closed under all convex combinations. Through the identification of maps $\mathcal{L}_E \to \mathcal{L}_E$ with functions on $\mathcal{L}_E \times (\Gamma \times \Sigma)$, a Polish space, K_E is a separable convex compact set of functions of the first Baire class, by Theorem 3.1. A separable convex Rosenthal compactum that is hereditarily separable is metrizable, a consequence of results of Todorčević and Gruenhage that is proved for convex compacta in [11]. □

The iterates of a single channel are not closed under mixing, and on commutative algebras they realize every level that tame cascades realize in [13], by Proposition 7.1.

6.4. **Enlarging the observations.** The level of a set of computations depends on the probes through which it is observed and on the computations that may follow it. Enlarging either can only raise the complexity, measured by the NIP, metrizability, hereditary separability and stability; first countability is the exception, and one probe more can restore it (Proposition 6.5).

Let $\Gamma \subseteq \Gamma'$ be monoids of channels and $\Sigma \subseteq \Sigma'$ sets of probes, all countable, and let $\mathcal{L}_\Gamma$ and $\mathcal{L}'_{\Gamma'}$ be the spaces of orbit types over $\Gamma \times \Sigma$ and over $\Gamma' \times \Sigma'$. The projection $\pi\colon \mathcal{L}'_{\Gamma'} \to \mathcal{L}_\Gamma$ onto the coordinates in $\Gamma \times \Sigma$ is continuous and affine, and it carries the orbit type of every state over $\Gamma' \times \Sigma'$ to its orbit type over $\Gamma \times \Sigma$; it therefore maps each type shard $\mathcal{L}'_E$ onto $\mathcal{L}_E$. For $\Psi \in \Gamma$ we write $\tilde{\Psi}$ and $\tilde{\Psi}'$ for the maps that Ψ induces on the two spaces, and K_E and K'_E for the compacta of deep computations of a set $\Delta \subseteq \Gamma$ on the two type shards.

Theorem 6.4. *Let $\Delta \subseteq \Gamma$ be confined to L_E.*

1. *For every $\xi' \in K'_E$ there is a unique map $\xi\colon \mathcal{L}_E \to \mathcal{L}_E$ with $\xi \circ \pi = \pi \circ \xi'$; it belongs to K_E, and the map $\theta\colon \xi' \mapsto \xi$ is a continuous surjection of K'_E onto K_E with $\theta(\tilde{\Psi}'|_{\mathcal{L}'_E}) = \tilde{\Psi}|_{\mathcal{L}_E}$.*
2. *If Δ has, on L_E and over (Γ', Σ'), the NIP, or is* $\mathrm{NIP}^{\mathrm{met}}$*,* $\mathrm{NIP}^{\mathrm{hs}}$ *or stable, then it has the same property over (Γ, Σ).*
3. *If π is injective on $\mathcal{L}'_E$, in particular if the pair (Γ, Σ) is norm-dense in (Γ', Σ') in the sense of Section 8.1, then θ is a homeomorphism, and Δ has the same level over both pairs.*

Proof. (1) For $\Psi \in \Gamma$ and a coordinate $(\Phi, \sigma) \in \Gamma \times \Sigma$, both $\pi(\tilde{\Psi}'(\mathbf{v}'))_{(\Phi,\sigma)}$ and $\tilde{\Psi}(\pi(\mathbf{v}'))_{(\Phi,\sigma)}$ equal $\mathbf{v}'_{(\Phi\circ\Psi,\sigma)}$, since $\Phi \circ \Psi \in \Gamma$; so $\pi \circ \tilde{\Psi}' = \tilde{\Psi} \circ \pi$. Let $\xi' = \lim_{i,\mathcal{U}} \tilde{\Psi}'_i$ on $\mathcal{L}'_E$, with $\Psi_i \in \Delta$, and let $\xi = \lim_{i,\mathcal{U}} \tilde{\Psi}_i$ on $\mathcal{L}_E$, which exists because K_E is compact. Since π is continuous, $\xi \circ \pi = \lim_{i,\mathcal{U}} \tilde{\Psi}_i \circ \pi = \lim_{i,\mathcal{U}} \pi \circ \tilde{\Psi}'_i = \pi \circ \xi'$, and since π maps $\mathcal{L}'_E$ onto $\mathcal{L}_E$, this identity determines ξ. If $\xi'_\alpha \to \xi'$ pointwise, then $\theta(\xi'_\alpha)(\pi(\mathbf{v}')) = \pi(\xi'_\alpha(\mathbf{v}')) \to \pi(\xi'(\mathbf{v}')) = \theta(\xi')(\pi(\mathbf{v}'))$, so θ is continuous. Its image is compact and contains the dense set $\{\tilde{\Psi}|_{\mathcal{L}_E} : \Psi \in \Delta\}$, so θ is onto.

(2) The families of condition (ii) of Theorem 3.1 over (Γ, Σ) are among those over (Γ', Σ'), so the NIP passes. A continuous image of a compact metrizable space in a Hausdorff space is metrizable. If K'_E is hereditarily separable and $Y \subseteq K_E$, then $\theta^{-1}(Y)$ has a countable dense subset, whose image is dense in Y; so K_E is hereditarily separable. If every $\xi' \in K'_E$ is continuous, then $\theta(\xi') \circ \pi = \pi \circ \xi'$ is continuous, and since $\pi\colon \mathcal{L}'_E \to \mathcal{L}_E$ is a continuous surjection of compact Hausdorff spaces, hence a quotient map, $\theta(\xi')$ is continuous.

(3) If π is injective on $\mathcal{L}'_E$, it is a homeomorphism of $\mathcal{L}'_E$ onto $\mathcal{L}_E$, the identity of (1) reads $\xi' = \pi^{-1} \circ \theta(\xi') \circ \pi$, and θ is injective; a continuous bijection of compact Hausdorff spaces is a homeomorphism. Norm density gives injectivity by Proposition 8.6. □

In particular, by Proposition 2.3 and Theorem 6.4(3), the level of a set of computations depends on the probes only through their closure in the trace norm. First countability is not preserved by continuous images in general: collapsing the copy of the Cantor set in the Alexandroff duplicate $D(2^{\mathbb{N}})$, which is first countable, to a point produces the Alexandroff compactification $A(2^{\mathbb{N}})$, which is not. For channels this happens: by Proposition 6.5, adding one probe to the address instruments makes their compactum first countable, and forgetting it collapses a Cantor set of deep computations to a point.

One probe more. The address instruments of Section 5 show that first countability can be gained by observing more. They also separate two ways of losing information: forgetting a probe and forgetting the idealized inputs. Keep the channels N_t, the monoid Γ_N, Bob's observable and the shard, and vary only the set of probes: $\{E\}$, $\{F\}$ or $\{E, F\}$. For $b \in 2^{\mathbb{N}}$, let A_b and B_b be the deep computations over $\{E, F\}$ that correspond to the limits of $(v_t, \bar{t})$ along the initial segments of b and along nodes that depart from b with addresses tending to b; both prepare the address r_b, and A_b records the limit of the prefix answers along b, while B_b records the answer 0.

Proposition 6.5. *Let $\Delta_N = \{N_t : t \in T\}$.*

(1) *Over the probe E alone, every deep computation of Δ_N is a constant map, the compactum of deep computations is metrizable, and Δ_N is stable.*
(2) *Over the probe F alone, Δ_N has the NIP and is not* $\mathrm{NIP}^{\mathrm{fc}}$.
(3) *Over the probes E and F, Δ_N is* $\mathrm{NIP}^{\mathrm{fc}}$ *and not* $\mathrm{NIP}^{\mathrm{hs}}$.
(4) *The deep computations A_b and B_b agree at the orbit type of every density operator, and the restrictions of the deep computations over $\{E, F\}$ to the orbit types of density operators form a metrizable compactum.*

In particular, adding the probe E to the probe F makes the compactum of deep computations first countable, and the continuous surjection of Theorem 6.4, which forgets E, maps the closed set $\{B_b : b \in 2^{\mathbb{N}}\}$, a copy of the Cantor set, to a single point, at which first countability fails.

Proof. The duals satisfy $N_t^*(E) = r_t I$ and $N_t^*(F) = V_t$, the tests vanish on the register, and the composites of two instruments are the reset channels R_t that prepare $\tau(r_t) \otimes |0\rangle\langle 0|$, with $R_t^*(E) = r_t I$ and $R_t^*(F) = 0$.

(1) Over $\{E\}$, every coordinate of an orbit type other than the one at the identity is a constant r_s, and the output of N_t has the coordinate r_t at the identity; so $\tilde{N}_t$ is the constant map with a value determined by r_t, injectively. The closure of these constant maps consists of constant maps, whose values form a compact subset of a metrizable space; constant maps are continuous.

(2) Over $\{F\}$, the coordinates of an orbit type are $\omega(F)$, the numbers $\omega(V_s)$ and zeros, and the output of N_t has the coordinate $\omega(V_t)$ at the identity and 0 elsewhere. Hence $\tilde{N}_t$ is the composite of $\mathbf{v} \mapsto \mathbf{v}_{(N_t,F)} = \omega(V_t)$ with a continuous injection of $[0,1]$, and the proof of Theorem 5.2 applies verbatim: it uses only Lemma 5.1(1) and (3) for the values $\omega(V_s)$, which hold here, the node states $|x\restriction n\rangle$ having orbit types that converge to a type β_x with $\beta_x(V_s) = 1$ exactly when $s \sqsubseteq x$.

(3) is Theorem 5.6(2).

(4) For a density operator ρ, let D_n be the projection onto the nodes of length at least n; then $\mathrm{Tr}(\rho V_t) \leq \mathrm{Tr}(\rho D_{|t|}) \to 0$ as $|t| \to \infty$. A sequence of distinct nodes has lengths tending to infinity, so every deep computation that is not one of the maps $\tilde{N}_t$ takes, at the orbit type of ρ, the value determined by the pair $(r, 0)$, where r is its address; in particular A_b and B_b agree there. The map that sends a deep computation ξ to its address and to its prefix answers $\xi(\mathrm{tp}_\Gamma(|u\rangle\langle u|))_{(\mathrm{id},F)}$ at the node states, $u \in T$, is continuous into a countable product of intervals, and it is injective on the restrictions: $\tilde{N}_t$ has the answer 1 at the node t and the answer 0 at every node not below t, and the limits have address r_b and answers 0. A continuous injection of a compact space into a metrizable space is an embedding. □

The address is lost in one quotient and kept in the other. Forgetting E identifies the whole Cantor set $\{B_b\}$ of departure limits with a single point, and destroys first countability. Restricting instead to actual inputs keeps the address but identifies each pair A_b, B_b: the two limits agree at every actual preparation, although the idealized type β_b separates them. The resulting compactum is metrizable.

These measure-and-prepare channels act on quantum state spaces, use finite-dimensional registers, and admit classical probabilistic implementations: the realizations are by quantum channels, not a claim of quantum advantage. The exact compactum of the threshold instruments on the whole tree, and coherent dynamics on the tree at the same levels, remain the subjects of Questions 10.11 and 10.12.

7. Commutative channels, pure states and compact probes

The classical theory of [13] is the commutative case of the quantum one, and its points become pure states.

7.1. **Commutative channels.** The definitions of Section 2 use only that the states of $B(H_1)$ form a weak* compact convex set on which the duals of the channels act, and they apply verbatim to a commutative C*-algebra. Let X be a compact metrizable space and $\gamma\colon X \to X$ a continuous map. The states of the commutative C*-algebra $C(X)$ are the Radon probability measures on X, which form the compact metrizable convex set $P(X)$, and the push-forward $\mu \mapsto \gamma_*\mu$ is a channel of $C(X)$, the dual of the unital positive map $f \mapsto f \circ \gamma$. In place of the slices, fix a countable set F of real-valued continuous functions on X whose linear span is dense in the real-valued continuous functions, and let $\Gamma = \{\gamma^n : n \in \mathbb{N}\}$. The orbit type of $\mu \in P(X)$ is $\mathrm{tp}_\Gamma(\mu) = (\int f \circ \gamma^n \, d\mu)_{(n,f)\in\mathbb{N}\times F}$. Its coordinates with $n = 0$ determine μ, since the span of F is dense, so tp_Γ is a homeomorphism of $P(X)$ onto the space of orbit types; we identify the two, and the computations act on $P(X)$ by the push-forwards γ_*^n. Point masses

are the counterparts of the states of [13]: the map $x \mapsto \delta_x$ is a homeomorphism of X onto the closed set of point masses, and $\gamma_*^n \delta_x = \delta_{\gamma^n(x)}$.

The *enveloping semigroup* $E(X,\gamma)$ is the closure of $\{\gamma^n : n \in \mathbb{N}\}$ in X^X, with the topology of pointwise convergence, and the cascade of γ is *tame* if every member of $E(X,\gamma)$ is of the first Baire class [17].

Proposition 7.1. *Let the cascade of γ be tame, and let K be the closure of $\{\gamma_*^n : n \in \mathbb{N}\}$ in $P(X)^{P(X)}$. Then every $\xi \in K$ is the push-forward $\xi(\mu) = p_*\mu$ by the map $p \in E(X,\gamma)$ defined by $\xi(\delta_x) = \delta_{p(x)}$, and the map $\xi \mapsto p$ is a homeomorphism of K onto $E(X,\gamma)$, which carries γ_*^n to γ^n.*

Proposition 7.1 is the case of cascades of the theorem of Glasner and Megrelishvili that tame systems are injective, that is, that restriction from the enveloping semigroup of $P(X)$ to that of X is injective [15, Theorem 8.2]; in their terms, the affine compactification $X \to P(X)$ is E-compatible. We include a short proof by sequential limits and dominated convergence, which also gives the formula $\xi = p_*$.

Proof. The NIP passes from points to measures. Let f be a real-valued continuous function on X. For $p = \lim_{i,\mathcal{U}} \gamma^{n_i}$ in $E(X,\gamma)$, the continuity of f gives $\lim_{i,\mathcal{U}} f \circ \gamma^{n_i} = f \circ p$, so the pointwise closure of $\{f \circ \gamma^n : n \in \mathbb{N}\}$ consists of functions of the first Baire class, and by the Bourgain–Fremlin–Talagrand dichotomy this family has the NIP on X. By Rosenthal's dichotomy, every sequence $(f \circ \gamma^{n_k})_k$ has a subsequence that converges pointwise on X, and, by dominated convergence, the integrals $\int f \circ \gamma^{n_k}\, d\mu$ along that subsequence converge for every $\mu \in P(X)$. An independent sequence has no subsequence that converges at every point, so the family $\{\mu \mapsto \int f \circ \gamma^n\, d\mu : n \in \mathbb{N}\}$ of continuous functions on $P(X)$ has the NIP.

Deep computations are push-forwards. Choose a countable dense subset F' of the unit ball of the real-valued continuous functions on X. The map $\mu \mapsto (\int g\, d\mu)_{g \in F'}$ embeds $P(X)$ in $[-1,1]^{F'}$, and the preceding paragraph gives the NIP of each coordinate family $\{\mu \mapsto \int g \circ \gamma^n\, d\mu : n \in \mathbb{N}\}$. The reduction of item (3) of Section 1, followed by the Bourgain–Fremlin–Talagrand theorem, makes K a compact set of maps of the first Baire class of $P(X)$ into itself, every member of which is the pointwise limit of a sequence of push-forwards γ_*^n. This sequential description permits dominated convergence. Let $\xi \in K$ and $\xi = \lim_k \gamma_*^{n_k}$ pointwise. Then $\delta_{\gamma^{n_k}(x)} \to \xi(\delta_x)$ for every x, and since the point masses form a closed set homeomorphic to X, $\xi(\delta_x) = \delta_{p(x)}$ with $p(x) = \lim_k \gamma^{n_k}(x)$; thus $p \in E(X,\gamma)$, and p is of the first Baire class, hence Borel. For every $f \in C(X)$, the functions $f \circ \gamma^{n_k}$ converge pointwise and boundedly to $f \circ p$, and dominated convergence gives

$$\int f\, d\xi(\mu) = \lim_k \int f \circ \gamma^{n_k}\, d\mu = \int f \circ p\, d\mu = \int f\, d(p_*\mu),$$

so $\xi(\mu) = p_*\mu$ for every $\mu \in P(X)$.

The homeomorphism. The map $R\colon \xi \mapsto p$ is continuous, being restriction to the point masses followed by the inverse of $x \mapsto \delta_x$; it is injective, since $\xi = p_*$; and it carries γ_*^n to γ^n. Its image is compact and contains $\{\gamma^n : n \in \mathbb{N}\}$, so it is $E(X,\gamma)$. A continuous bijection of a compact space onto a Hausdorff space is a homeomorphism. □

The levels of the classification and the canonical families of the heptachotomy are topological properties of the compactum of deep computations and of its indexed dense family. By Proposition 7.1, the push-forwards of a tame cascade of a compact metrizable space realize exactly the levels and families that its iterates realize in $E(X,\gamma)$, and every realization in [13] by a tame cascade of a compact metrizable space is a realization by commutative channels.

7.2. **Pure states and compact probes.** Two features of the state space of $B(H_1)$ bear on the classification. It is not a simplex, and a mixture does not determine its pure components; and a channel need not preserve purity, whereas the maps of [13] act on points. The first result replaces the point masses of Proposition 7.1 by pure states, idealized ones included; the second identifies the probes for which unitary dynamics is always stable.

Proposition 7.2. *Let $\Delta \subseteq \Gamma$ satisfy condition* (ii) *of Theorem 3.1 with $\mathscr{E} = \{\infty\}$, and let $\mathcal{P} \subseteq \mathcal{L}_\Gamma$ be the set of orbit types of pure states of $B(H_1)$. Then every deep computation of Δ is an affine map of $\mathcal{L}_\Gamma$ into itself whose coordinates are of the first Baire class, every deep computation is determined by its restriction to $\mathcal{P}$, and restriction to $\mathcal{P}$ is a homeomorphism of the space of deep computations of Δ onto its image in $\mathcal{L}_\Gamma^{\mathcal{P}}$.*

Proof. The space $\mathcal{L}_\Gamma$ is compact, convex and metrizable, by Proposition 2.6. Each map $\tilde{\Psi}$ permutes and repeats coordinates and is therefore the restriction of a linear map, and pointwise limits of affine maps are affine. By Theorem 3.1, the coordinates of every deep computation ξ are of the first Baire class. An affine function of the first Baire class on a metrizable compact convex set satisfies the barycentric formula, a theorem of Choquet and Mokobodzki [23]: its value at a point is its integral against every Radon probability measure with that point as barycenter. By Choquet's theorem [25], every point of $\mathcal{L}_\Gamma$ is the barycenter of a Radon probability measure carried by the set of extreme points of $\mathcal{L}_\Gamma$, so each coordinate of ξ, and therefore ξ, is determined by its values at the extreme points. Every extreme point $\mathbf{v}$ of $\mathcal{L}_\Gamma$ lies in $\mathcal{P}$: the preimage of $\mathbf{v}$ under the affine continuous surjection $\mathrm{tp}_\Gamma \colon \mathcal{S}(B(H_1)) \to \mathcal{L}_\Gamma$ is a closed face of the state space, which has an extreme point by the Krein–Milman theorem, and an extreme point of a face is a pure state. Hence restriction to $\mathcal{P}$ is injective on the space of deep computations; it is continuous for the pointwise topologies, and a continuous injection of a compact space into a Hausdorff space is an embedding. □

For commutative algebras the pure states are the point masses, and Proposition 7.2 is the part of Proposition 7.1 that identifies deep computations with their restrictions to points. The set $\mathcal{P}$ contains the orbit types of idealized pure states, the pure states of $B(H_1)$ that vanish on the compact operators; at such a state ω, the deep computation ξ of Theorem 4.4 takes the value $\omega(Q_0)$ at every coordinate.

A unitary channel $\rho \mapsto U\rho U^*$ acts on pure states as a dynamical system on the projective space of H_1, with the quadratic features $\psi \mapsto \langle \psi, S_C(\sigma)\psi\rangle$. Classically, a flow representable by isometries of a reflexive Banach space is weakly almost periodic, a theorem of Megrelishvili (see [16]), and weak almost periodicity is Grothendieck's double limit condition for the matrix coefficients of the representation. The quantum counterpart holds when the probes are compact.

Lemma 7.3. *Let K be a compact operator on H_1 and let (U_n) be unitaries such that $U_n^* K U_n$ converges in the weak operator topology to an operator X. Then X is compact.*

Proof. Let $\varepsilon > 0$, and write $K = G + E$ with $G = \sum_{j \le r} |a_j\rangle\langle b_j|$ of finite rank and $\|E\| < \varepsilon$. Pass to a subsequence along which the vectors $U_n^* a_j$ and $U_n^* b_j$ converge weakly, to a_j' and b_j', and $U_n^* E U_n$ converges in the weak operator topology, to E'. For vectors x and y, $\langle x, U_n^*|a_j\rangle\langle b_j|U_n y\rangle = \langle x, U_n^* a_j\rangle\langle U_n^* b_j, y\rangle \to \langle x, a_j'\rangle\langle b_j', y\rangle$, so $X = \sum_{j \le r} |a_j'\rangle\langle b_j'| + E'$, with $\|E'\| \le \varepsilon$ because the unit ball is closed in the weak operator topology. Thus X lies within ε of an operator of finite rank, for every ε, and is compact. □

Theorem 7.4. *Let every member of Γ be a unitary channel, and let every slice $S_C(\sigma)$, $\sigma \in \Sigma$, be compact. Then every set $\Delta \subseteq \Gamma$ is stable on $\mathfrak{D}(H_1)$: all its deep computations are continuous on $\mathcal{L}_\Gamma$.*

Proof. For a unitary channel $\Phi(\rho) = V\rho V^*$, the dual is $\Phi^*(Y) = V^*YV$, which maps compact operators to compact operators; so every coordinate of the orbit types is the value of the state at a compact operator. Every state ω decomposes as $\omega = \lambda\rho + (1-\lambda)\omega_{\mathrm{s}}$, with ρ a density operator, ω_{s} vanishing on the compact operators and $0 \le \lambda \le 1$ [28, Theorem III.2.14], and therefore $\mathrm{tp}_\Gamma(\omega) = \lambda\,\mathrm{tp}_\Gamma(\rho)$.

Fix a coordinate (Φ, σ) and let $K = \Phi^*(S_C(\sigma))$, a compact operator. The functions $f_\Psi(\mathbf{v}) = \tilde{\Psi}(\mathbf{v})_{(\Phi,\sigma)}$, $\Psi \in \Delta$, are continuous on $\mathcal{L}_\Gamma$, and $f_\Psi(\lambda\,\mathrm{tp}_\Gamma(\rho)) = \lambda\,\mathrm{Tr}(\rho\,\Psi^*(K))$. We verify Grothendieck's double limit condition. Let $\Psi_n \in \Delta$ and $\mathbf{v}_m = \lambda_m\,\mathrm{tp}_\Gamma(\rho_m) \in \mathcal{L}_\Gamma$ be sequences for which both iterated limits of $f_{\Psi_n}(\mathbf{v}_m)$ exist. Passing to subsequences preserves both iterated limits. The unit ball of the trace class is weak* compact and metrizable, as the dual ball of the separable space of compact operators; after passing to a subsequence in m, we may therefore assume that $\lambda_m \to \lambda$ and that ρ_m converges weak* on the compact operators to a positive trace-class operator τ. After passing to a subsequence in n, the operators $\Psi_n^*(K)$ also converge in the weak operator topology, to an operator X, which is compact by Lemma 7.3. Since the operators $\Psi_n^*(K)$ and X are compact and the weak operator topology agrees with the σ-weak topology on bounded sets,

$$\lim_n \lim_m f_{\Psi_n}(\mathbf{v}_m) = \lim_n \lambda\,\mathrm{Tr}(\tau\,\Psi_n^*(K)) = \lambda\,\mathrm{Tr}(\tau X) = \lim_m \lambda_m\,\mathrm{Tr}(\rho_m X) = \lim_m \lim_n f_{\Psi_n}(\mathbf{v}_m).$$

By Grothendieck's criterion, the family $\{f_\Psi : \Psi \in \Delta\}$ is relatively compact in the space of continuous functions on $\mathcal{L}_\Gamma$ with the pointwise topology, and every coordinate of every deep computation of Δ is continuous. □

Proposition 4.5 is the instance of Theorem 7.4 for the quantum walk and the rank-one probes R_m. Theorem 4.4 shows that compactness of the probes cannot be dropped: the comparison of positions has the projections Q_m, of infinite rank, as slices, and the walk is not stable for them. Unitary dynamics reaches the nontrivial levels only through probes that are not compact, and the transfer of the tame dynamics of [17] to such probes is Question 10.14.

Spectrum in place of compact probes. A hypothesis on the spectrum of the dynamics does not replace the hypothesis on the probes. Proposition 2.14 has a classical counterpart for single orbits. For a unitary group e^{-itH}, the theorem of Ruelle, Amrein, Georgescu and Enss [7] says that the orbit of a vector in the point spectral subspace of H stays in a norm-compact set, whereas the orbit of a vector in the continuous spectral subspace leaves every compact region on average in time. Compactness of each orbit does not, however, give stability, or even the NIP, on the space of orbit types: the classification concerns the behavior of the computations uniformly over all states, including the idealized ones, and pure point spectrum does not force it.

Proposition 7.5. *Let $H_1 = \ell^2(\mathbb{N}) \otimes \mathbb{C}^2$, with orthonormal basis $|n,0\rangle, |n,1\rangle$, let $A|n,0\rangle = 0$ and $A|n,1\rangle = 2\pi n|n,1\rangle$, and let $U_t = e^{-itA}$. Let Bob's system be one-dimensional, with the observable $P = \sum_n |n,+\rangle\langle n,+|$, where $|n,+\rangle = (|n,0\rangle + |n,1\rangle)/\sqrt{2}$, and let $\Gamma = \{U_t : t \in \mathbb{Q},\ t \ge 0\}$. Then A has pure point spectrum, and the family $\{\rho \mapsto \mathrm{Tr}(U_t\rho U_t^* P) : t \in \mathbb{Q},\ t \ge 0\}$ does not have the NIP on $\mathfrak{D}(H_1)$; so Γ fails condition* (ii) *of Theorem 3.1 on $L_\infty = \mathfrak{D}(H_1)$.*

Proof. The vectors $|n,j\rangle$ are eigenvectors of A. Since $U_t|n,+\rangle = (|n,0\rangle + e^{-2\pi int}|n,1\rangle)/\sqrt{2}$ is orthogonal to $|m,+\rangle$ for $m \ne n$, the state $\rho_n = |n,+\rangle\langle n,+|$ gives

$$\mathrm{Tr}(U_t\rho_n U_t^* P) = \big|\langle n,+|U_t|n,+\rangle\big|^2 = \tfrac{1}{2}\big(1 + \cos(2\pi nt)\big).$$

Let $t_j = 8^{-j}$ for $j \ge 1$, let $I, J \subseteq \{1,2,\dots\}$ be disjoint and finite, and let $n = \sum_{i\in I} 4\cdot 8^{i-1}$. The fractional part of nt_j is $\theta_j = \sum_{i\in I,\, i\le j} 4\cdot 8^{i-1-j}$, and the terms with $i < j$ add up to at

most $4\sum_{k\geq 2} 8^{-k} = \frac{1}{14}$. Hence $\theta_j \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{14}]$ for $j \in I$ and $\theta_j \in [0, \frac{1}{14}]$ for $j \notin I$, and since $\cos(\pi/7) > 0.9$, the value at ρ_n of the function indexed by t_j is below 0.05 for $j \in I$ and above 0.95 for $j \in J$. The functions indexed by the times t_j therefore form an independent sequence, with $a = 0.1$ and $b = 0.9$. □

There is no conflict with Proposition 2.14 and Theorem 7.4: the state space is not compact in the trace norm, and the projection P has infinite rank.

Part III. Measure

With uncountably many observations, the spaces of types remain compact but may cease to be metrizable. We turn from the topology of every limit of computations to its behavior under a law of random preparations.

8. Monte Carlo computability

The topological classification rests on a countable family of observations, through which the type shards become metrizable. With an arbitrary family, compactness remains, but the argument through the first Baire class may not be available. Following [13, Section 5], we replace it by universal measurability; the translations of the line exhibit a nonmetrizable space of orbit types where this is needed (Proposition 8.7).

The measure-theoretic conclusion takes two forms. Under the NIP, a fixed output coordinate is approximated to any accuracy by a polynomial in finitely many input observations, uniformly on a compact set whose measure is as close to 1 as desired, for any prescribed Radon law (Theorem 8.2). Empirical mixtures ask instead for the approximation of a barycenter, uniformly over the computations. For preparations by density operators, the law of large numbers in the trace norm already provides this uniformity, and the NIP governs its extension to all Radon laws on the type shards (Theorem 8.8 and Proposition 8.9).

8.1. **Uncountably many probes and computations.** In this section $\Sigma \subseteq \mathfrak{D}(H_2)$ is an arbitrary set of probes, possibly all of them, and Γ is an arbitrary monoid of channels of Alice's system. Orbit types, the maps $\tilde{\Psi}$, energy shards and their type shards, and deep computations are defined as in Section 2. Propositions 2.6 and 2.13 hold with the same proofs, except that the space $\mathcal{L}_\Gamma \subseteq [-\|C\|, \|C\|]^{\Gamma\times\Sigma}$ of orbit types is compact and convex but need not be metrizable: the maps $\tilde{\Psi}$ are continuous and affine, and they map $\mathcal{L}_E$ into itself when Ψ is confined to L_E. The product $[-\|C\|, \|C\|]^{\Gamma\times\Sigma}$ carries the cylinder σ-algebra, generated by the coordinate projections, and a map into it is *universally measurable* if its composition with every coordinate projection is measurable for the completion of every Radon probability measure on its domain; by [13, Proposition 5.4], this is the universal measurability of the map for the cylinder σ-algebra.

Definition 8.1. A map $\xi\colon \mathcal{L}_\Gamma \to \mathcal{L}_\Gamma$ is *universally Monte Carlo computable* on an energy shard L_E if ξ maps $\mathcal{L}_E$ into itself and its restriction to $\mathcal{L}_E$ is universally measurable.

Theorem 8.2. *Let $\mathscr{E} \subseteq [0,\infty]$ be a set of energy bounds whose shards cover $\mathfrak{D}(H_1)$, and let $\Delta \subseteq \Gamma$ be confined to L_E for every $E \in \mathscr{E}$. The following are equivalent:*

(i) *for every $E \in \mathscr{E}$ and every $(\Phi,\sigma) \in \Gamma \times \Sigma$, the family $\{\rho \mapsto \varphi_C(\Phi(\Psi(\rho)),\sigma) : \Psi \in \Delta\}$ has the NIP on L_E;*

(ii) *for every $E \in \mathscr{E}$, every member of the closure of $\{\tilde{\Psi}|_{\mathcal{L}_E} : \Psi \in \Delta\}$ is universally measurable.*

If these conditions hold, then every deep computation ξ of Δ is universally Monte Carlo computable on every L_E, $E \in \mathscr{E}$, and, for every Radon probability measure μ on $\mathcal{L}_E$, every coordinate (Φ, σ) and all $\varepsilon, \delta > 0$, there are a compact set $F \subseteq \mathcal{L}_E$ with $\mu(\mathcal{L}_E \setminus F) < \varepsilon$ and a polynomial p in finitely many coordinates of the orbit types with $|\xi(\mathbf{v})_{(\Phi,\sigma)} - p(\mathbf{v})| < \delta$ for every $\mathbf{v} \in F$.

Proof. Fix $E \in \mathscr{E}$ and a coordinate (Φ, σ), let K_E be the closure of $\{\tilde{\Psi}|_{\mathcal{L}_E} : \Psi \in \Delta\}$ in $\mathcal{L}_E^{\mathcal{L}_E}$, and let A be the family of functions $\mathbf{v} \mapsto \tilde{\Psi}(\mathbf{v})_{(\Phi,\sigma)} = \mathbf{v}_{(\Phi\circ\Psi,\sigma)}$ on $\mathcal{L}_E$, $\Psi \in \Delta$. The members of A are continuous and bounded by $\|C\|$, and, since composition with a coordinate projection is continuous for the pointwise topologies, the closure of A is the set of coordinates (Φ, σ) of the members of K_E. On the orbit type of a density operator ρ, the members of A take the values $\varphi_C(\Phi(\Psi(\rho)), \sigma)$, and the orbit types of members of L_E are dense in $\mathcal{L}_E$.

Assume (i). By [13, Lemma 1.10], the family A has the NIP on $\mathcal{L}_E$, and hence on every compact subset of $\mathcal{L}_E$. By the theorem of Bourgain, Fremlin and Talagrand [6, Theorem 2F] (see [13, Theorem 5.5]), the closure of A consists of universally measurable functions. As the coordinate (Φ, σ) is arbitrary, every member of K_E is universally measurable, and (ii) holds. Conversely, if (ii) holds, the closure of A consists of universally measurable functions, and the same theorem, applied to the compact space $\mathcal{L}_E$, shows that A has the NIP on $\mathcal{L}_E$; by [13, Lemma 1.10], condition (i) holds.

Let $\xi = \lim_{i,\mathcal{U}} \tilde{\Psi}_i$ be a deep computation of Δ. Each $\tilde{\Psi}_i$ maps the closed set $\mathcal{L}_E$ into itself, by confinement, and so does ξ; the restriction $\xi|_{\mathcal{L}_E}$ belongs to K_E and is universally measurable. Let μ be a Radon probability measure on $\mathcal{L}_E$, and let $\varepsilon, \delta > 0$. By Luzin's theorem for Radon measures, the μ-measurable function $\mathbf{v} \mapsto \xi(\mathbf{v})_{(\Phi,\sigma)}$ is continuous on a compact set $F \subseteq \mathcal{L}_E$ with $\mu(\mathcal{L}_E \setminus F) < \varepsilon$. The coordinate functions separate the points of F, so, by the Stone–Weierstrass theorem, the polynomials in finitely many coordinates are uniformly dense in the continuous functions on F. □

For countable sets of computations, the NIP is universal Talagrand stability, and both routes of [13, Section 5] lead to the same deep computations.

Corollary 8.3. *Let Δ be countable and confined to L_E, and let $(\Phi, \sigma) \in \Gamma \times \Sigma$. The family $\{\rho \mapsto \varphi_C(\Phi(\Psi(\rho)), \sigma) : \Psi \in \Delta\}$ has the NIP on L_E if and only if the family of its continuous extensions $\mathbf{v} \mapsto \mathbf{v}_{(\Phi\circ\Psi,\sigma)}$ to $\mathcal{L}_E$ is Talagrand μ-stable for every Radon probability measure μ on $\mathcal{L}_E$.*

Proof. The extended family is a countable, uniformly bounded family of continuous functions on the compact space $\mathcal{L}_E$, in which the orbit types of members of L_E are dense. By [13, Lemma 1.10], the NIP on L_E is the NIP of the extended family, which is universal Talagrand stability by [13, Proposition 5.14 and Theorem 5.16]. □

Countable reduction. The NIP is preserved by small uniform perturbations, and this makes it a countable condition whenever the coordinates are separable in norm.

Lemma 8.4. *Let $(f_\Psi)_{\Psi\in\Delta}$ be a family of real functions on a set L such that, for every $\varepsilon > 0$, some family $(g_\Psi)_{\Psi\in\Delta}$ with the NIP on L satisfies $|f_\Psi - g_\Psi| \le \varepsilon$ on L for every $\Psi \in \Delta$. Then $(f_\Psi)_{\Psi\in\Delta}$ has the NIP on L.*

Proof. Let (f_{Ψ_n}) and $a < b$ witness an independent sequence, and choose a family (g_Ψ) with the NIP and $|f_\Psi - g_\Psi| \le \varepsilon = (b-a)/3$ for all Ψ. Wherever $f_{\Psi_n} \le a$, we have $g_{\Psi_n} \le a + \varepsilon$, and wherever $f_{\Psi_n} \ge b$, we have $g_{\Psi_n} \ge b - \varepsilon$, with $a + \varepsilon < b - \varepsilon$; so (g_{Ψ_n}) and $a + \varepsilon < b - \varepsilon$ witness an independent sequence, a contradiction. □

We say that a pair of sets $\Gamma_0 \subseteq \Gamma$ and $\Sigma_0 \subseteq \Sigma$ is *norm-dense* if, for every $(\Phi, \sigma) \in \Gamma \times \Sigma$, the operator $\Phi^*(S_C(\sigma))$ lies in the norm closure of $\{\Phi_0^*(S_C(\sigma_0)) : (\Phi_0, \sigma_0) \in \Gamma_0 \times \Sigma_0\}$.

Proposition 8.5. *Let (Γ_0, Σ_0) be norm-dense, and let $\Delta \subseteq \Gamma$ and $L \subseteq \mathfrak{D}(H_1)$. If the family $\{\rho \mapsto \varphi_C(\Phi_0(\Psi(\rho)), \sigma_0) : \Psi \in \Delta\}$ has the NIP on L for every $(\Phi_0, \sigma_0) \in \Gamma_0 \times \Sigma_0$, then the family $\{\rho \mapsto \varphi_C(\Phi(\Psi(\rho)), \sigma) : \Psi \in \Delta\}$ has the NIP on L for every $(\Phi, \sigma) \in \Gamma \times \Sigma$.*

Proof. Let $\varepsilon > 0$ and choose (Φ_0, σ_0) with $\|\Phi^*(S_C(\sigma)) - \Phi_0^*(S_C(\sigma_0))\| \leq \varepsilon$. The dual Ψ^* of a channel is unital and completely positive, hence of norm 1, so

$$\left|\varphi_C(\Phi(\Psi(\rho)), \sigma) - \varphi_C(\Phi_0(\Psi(\rho)), \sigma_0)\right| = \left|\mathrm{Tr}\big(\rho\, \Psi^*\big(\Phi^*(S_C(\sigma)) - \Phi_0^*(S_C(\sigma_0))\big)\big)\right| \leq \varepsilon$$

for every density operator ρ and every $\Psi \in \Delta$, and Lemma 8.4 applies. □

Proposition 8.6. *Let (Γ_0, Σ_0) be norm-dense. Then the projection of $\mathcal{L}_\Gamma$ onto the coordinates in $\Gamma_0 \times \Sigma_0$ is a homeomorphism onto its image. In particular, if Γ_0 and Σ_0 are countable, then $\mathcal{L}_\Gamma$ is metrizable, and Theorem 3.1 applies to every $\Delta \subseteq \Gamma$ confined to the shards, and the levels of Section 3 to every countable such Δ, with the coordinates of all of $\Gamma \times \Sigma$.*

Proof. The projection is continuous, and $\mathcal{L}_\Gamma$ is compact, so it suffices to show that the projection is injective. For every state ω and every (Φ, σ), the coordinate $\omega(\Phi^*(S_C(\sigma)))$ is the limit of coordinates $\omega(\Phi_0^*(S_C(\sigma_0)))$ with $(\Phi_0, \sigma_0) \in \Gamma_0 \times \Sigma_0$, since $|\omega(X)| \leq \|X\|$. For countable Γ_0 and Σ_0, the image is a subspace of a countable product of intervals, and the proof of Theorem 3.1 uses only that the type shards are compact and metrizable and that the computations extend continuously to them. □

Since $\|\Phi^*(S_C(\sigma)) - \Phi_0^*(S_C(\sigma_0))\| \leq \|\Phi^*(S_C(\sigma)) - \Phi_0^*(S_C(\sigma))\| + \|C\|\, \|\sigma - \sigma_0\|_1$, by the estimate on slices in Section 2, the pair (Γ_0, Σ_0) is norm-dense when Σ_0 is dense in Σ for the trace norm and every member of Γ is a limit of members of Γ_0 for the pointwise norm convergence of the duals; in particular, for a semigroup of channels whose duals are continuous in norm, with Γ_0 the rational times. The examples of Sections 4 and 5 are of this kind when their probes are enlarged to all of $\mathfrak{D}(H_2)$, and for them the topological classification already sees all the probes and all the computations.

Translations of the line. When the operators of the coordinates are not separable in norm, the space of orbit types need not be metrizable, and universal measurability takes the place of the first Baire class. The basic example is a particle on the line, translated by all real distances and observed through the comparison of positions.

Let $H_1 = L^2(\mathbb{R})$, let Q_c be the projection onto the functions supported in (c, ∞), for $c \in \mathbb{R}$, and let Bob hold a particle on the integers, with $H_2 = \ell^2(\mathbb{Z})$ and the observable $C = \sum_{m \in \mathbb{Z}} Q_m \otimes |m\rangle\langle m|$, a projection, whose slice at the probe $\delta_m = |m\rangle\langle m|$ is Q_m; let $\Sigma = \{\delta_m : m \in \mathbb{Z}\}$. The translations $(T_s\psi)(x) = \psi(x - s)$ satisfy $T_s^* Q_m T_s = Q_{m-s}$, and we let $\Gamma = \{T_s : s \in \mathbb{R}\}$, acting by the unitary channels $\rho \mapsto T_s \rho T_s^*$.

Proposition 8.7. *For the translations of the line observed through the comparison of positions:*

(1) *the space $\mathcal{L}_\Gamma$ of orbit types is not metrizable, and no countable pair (Γ_0, Σ_0) is norm-dense;*
(2) *every set $\Delta \subseteq \Gamma$ satisfies the equivalent conditions of Theorem 8.2 on the single shard $L_\infty = \mathfrak{D}(H_1)$, and all its deep computations are universally Monte Carlo computable.*

Proof. The coordinate (T_s, δ_m) of the orbit type of a state ω is $\omega(Q_{m-s})$, so the orbit type of ω is determined by, and determines, the function $c \mapsto \omega(Q_c)$ on $\mathbb{R}$, which is nonincreasing because $Q_c \geq Q_d$ for $c \leq d$.

(1) The map sending an orbit type $\mathbf{v}$ to the function $c \mapsto \mathbf{v}_{(T_{-c}, \delta_0)}$ is continuous and, by the preceding paragraph, injective on the compact space $\mathcal{L}_\Gamma$, so it is a homeomorphism onto its

image in $[0,1]^{\mathbb{R}}$. For $c \in \mathbb{R}$, let ω_c be a weak* cluster point of the vector states of unit vectors supported in $(c, c+1/n)$, as $n \to \infty$. For $d \le c$ these vector states take the value 1 at Q_d, and for $d > c$ they eventually take the value 0; hence $\omega_c(Q_d) = \mathbf{1}_{d \le c}$, and the image contains every function $\mathbf{1}_{(-\infty,c]}$. In the topology of pointwise convergence, the conditions imposed by finitely many points on a function $\mathbf{1}_{(-\infty,c']}$ near $\mathbf{1}_{(-\infty,c]}$ are $c' \ge d$ for some points $d \le c$ and $c' < d$ for some points $d > c$; so the sets $\{\mathbf{1}_{(-\infty,c']} : c \le c' < d\}$, $d > c$, form a base of neighborhoods, and the subspace $\{\mathbf{1}_{(-\infty,c]} : c \in \mathbb{R}\}$ is homeomorphic to the Sorgenfrey line, which is not metrizable. Hence $\mathcal{L}_\Gamma$ is not metrizable, and by Proposition 8.6 no countable pair (Γ_0, Σ_0) is norm-dense; directly, $\|Q_c - Q_d\| = 1$ for $c \ne d$.

(2) For each coordinate (T_r, δ_m), the family $\{\rho \mapsto \operatorname{Tr}(\rho\, T_s^* Q_{m-r} T_s) : s \in \mathbb{R}\}$ consists of the functions $\rho \mapsto \operatorname{Tr}(\rho\, Q_{m-r-s})$, which are nonincreasing in $-s$, and hence monotone in s. Every sequence of reals has a monotone subsequence, along which these functions form a monotone sequence, and for reals $a < b$, a nonincreasing sequence of functions never realizes the pattern $f_1 \le a$, $f_2 \ge b$, and a nondecreasing one never realizes the pattern $f_1 \ge b$, $f_2 \le a$, so neither is independent. So every subfamily has the NIP, and Theorem 8.2 applies. □

The translations by rational distances form a countable monoid, to which the topological theory applies, but its orbit types record the positions Q_c only at countably many points c; the measure-theoretic theory sees all of them.

Random preparations. A *random preparation* of Alice's system is a Borel probability measure λ on $\mathfrak{D}(H_1)$ with the trace norm, a Polish space. Its law on orbit types is the push-forward $\mu = (\mathrm{tp}_\Gamma)_* \lambda$, a Radon probability measure on $\mathcal{L}_\Gamma$, since tp_Γ is continuous from the trace norm and Borel probability measures on Polish spaces are Radon; if λ is carried by L_E, then μ is carried by $\mathcal{L}_E$. Every Radon probability measure on $\mathcal{L}_E$, including those that charge the orbit types of idealized states, is a weak* limit of laws of random preparations carried by L_E, because the measures with finite support in the dense set $\mathrm{tp}_\Gamma(L_E)$ are weak* dense.

Theorem 8.2 has the following reading, whose proof is that of the theorem, carried out on a single shard. Fix a shard L_E and a random preparation λ carried by L_E, and let $\Delta \subseteq \Gamma$ be confined to L_E and satisfy condition (i) of Theorem 8.2 for E. Let ξ be a deep computation of Δ, let $(\Phi, \sigma) \in \Gamma \times \Sigma$ be an output coordinate, and let $\varepsilon, \delta > 0$. There are finitely many pairs $(\Phi_j, \sigma_j) \in \Gamma \times \Sigma$, $1 \le j \le k$, and a polynomial p such that, with λ-probability at least $1 - \varepsilon$ over the preparation ρ, the value $\xi(\mathrm{tp}_\Gamma(\rho))_{(\Phi,\sigma)}$ differs by less than δ from

$$p\big(\varphi_C(\Phi_1(\rho), \sigma_1), \ldots, \varphi_C(\Phi_k(\rho), \sigma_k)\big).$$

The arguments of the polynomial are expectation values, obtained by applying finitely many computations and probes to the input: up to an event of small probability, the outcome of a deep computation is computed from the statistics of finitely many computations followed by probes. The pairs and the polynomial depend on the law, on the deep computation and on the output coordinate, and the theorem asserts their existence without giving a procedure for finding them. In Section 8.2 we turn to a uniformity of another kind, that of empirical averages over all the computations at once.

8.2. **Glivenko–Cantelli classes.** Through Talagrand's characterization of Glivenko–Cantelli classes [27], the NIP of a bipartite computation structure becomes a law of large numbers for random preparations of Alice's system. A uniformly bounded family $\mathcal{F}$ of measurable functions on a probability space (X, μ) is a *Glivenko–Cantelli class* for μ if, for $\mu^{\mathbb{N}}$-almost every sequence (x_i) of independent samples, $\sup_{f \in \mathcal{F}} |\frac{1}{n} \sum_{i<n} f(x_i) - \int f \, d\mu| \to 0$.

Theorem 8.8. *Let Γ and Σ be countable, and let $\Delta \subseteq \Gamma$ be confined to the energy shard L_E. For every pair $(\Phi, \sigma) \in \Gamma \times \Sigma$, the following are equivalent:*

(i) *the family $\{\rho \mapsto \varphi_C(\Phi(\Psi(\rho)), \sigma) : \Psi \in \Delta\}$ has the NIP on L_E;*
(ii) *the family of its extensions to the type shard $\mathcal{L}_E$ is a Glivenko–Cantelli class for every Radon probability measure on $\mathcal{L}_E$.*

Proof. By the Extendibility Axiom, which holds for orbit types by Proposition 2.6, the extended family is a countable family of continuous functions on the compact space $\mathcal{L}_E$, bounded by $\|C\|$, and by [13, Lemma 1.10] condition (i) is the NIP of the extended family on $\mathcal{L}_E$. The family Δ is countable, so, by Corollary 8.3, this is the Talagrand μ-stability of the extended family for every Radon probability measure μ on $\mathcal{L}_E$. For a uniformly bounded family of functions on a complete probability space, Talagrand's theorem [27] (see [14, Theorem 465M]) shows that stability is equivalent to the measurability of the members together with the Glivenko–Cantelli property. The members of the extended family are continuous, and a countable family is a Glivenko–Cantelli class for μ exactly when it is one for the completion of μ. □

Mixtures. The functions of Theorem 8.8 are affine, and the law of large numbers that the theorem gives is a statement about mixtures. Let μ be a Radon probability measure on $\mathcal{L}_E$. Its *barycenter* is the point $\bar{\mathbf{v}}$ of $\mathcal{L}_E$ with $\bar{\mathbf{v}}_{(\Phi,\sigma)} = \int \mathbf{v}_{(\Phi,\sigma)}\, d\mu(\mathbf{v})$ for every coordinate; it lies in $\mathcal{L}_E$ because $\mathcal{L}_E$ is compact and convex, by Proposition 2.13, and it is the orbit type of a state $\bar{\omega}$, by Proposition 2.6. For independent samples $\mathbf{v}_0, \mathbf{v}_1, \dots$ of μ, the *empirical mixture* is $\hat{\mathbf{v}}_n = \frac{1}{n}\sum_{i<n} \mathbf{v}_i$; if $\mathbf{v}_i = \mathrm{tp}_\Gamma(\omega_i)$, then $\hat{\mathbf{v}}_n$ is the orbit type of the mixture $\frac{1}{n}\sum_{i<n} \omega_i$, since tp_Γ is affine.

Proposition 8.9. *Let $\Delta \subseteq \Gamma$ be confined to L_E, let μ be a Radon probability measure on $\mathcal{L}_E$, and let (Φ, σ) be a coordinate.*

(1) *The family of Theorem 8.8 is a Glivenko–Cantelli class for μ if and only if, for $\mu^{\mathbb{N}}$-almost every sequence of samples,*

$$\sup_{\Psi \in \Delta} \left| \tilde{\Psi}(\hat{\mathbf{v}}_n)_{(\Phi,\sigma)} - \tilde{\Psi}(\bar{\mathbf{v}})_{(\Phi,\sigma)} \right| \longrightarrow 0 :$$

the empirical mixture simulates the barycenter, uniformly over the computations followed by the probe.

(2) *If μ is the law $(\mathrm{tp}_\Gamma)_*\lambda$ of a random preparation λ carried by L_E, then the barycenter is the orbit type of the density operator $\bar{\rho} = \int \rho\, d\lambda(\rho)$, and the convergence in* (1) *holds for every set Δ of channels, with or without the NIP.*

Proof. (1) The function $f_\Psi(\mathbf{v}) = \tilde{\Psi}(\mathbf{v})_{(\Phi,\sigma)} = \mathbf{v}_{(\Phi\circ\Psi,\sigma)}$ is the restriction of a coordinate projection, which is linear; hence $\int f_\Psi\, d\mu = f_\Psi(\bar{\mathbf{v}})$ by the definition of the barycenter, and $\frac{1}{n}\sum_{i<n} f_\Psi(\mathbf{v}_i) = f_\Psi(\hat{\mathbf{v}}_n)$.

(2) The trace class of H_1 is a separable Banach space in which the density operators form a bounded closed convex set, so $\bar{\rho}$ is a Bochner integral, a density operator, and the continuous affine map tp_Γ carries it to the barycenter of μ. By the strong law of large numbers in separable Banach spaces, due to Mourier [24], the empirical means $\hat{\rho}_n = \frac{1}{n}\sum_{i<n} \rho_i$ of independent samples of λ converge to $\bar{\rho}$ in the trace norm almost surely. Channels are contractions of the trace norm, so $|\varphi_C(\Phi(\Psi(\hat{\rho}_n)), \sigma) - \varphi_C(\Phi(\Psi(\bar{\rho})), \sigma)| \le \|C\|\, \|\hat{\rho}_n - \bar{\rho}\|_1$ for every Ψ, and the supremum over $\Psi \in \Delta$ tends to 0. □

For random preparations by density operators, the trace-norm law of large numbers already gives uniformity over every set of channels. Theorem 8.8 characterizes the extension to every Radon law on a type shard: for each fixed output coordinate, empirical mixtures simulate the

barycenter uniformly over the computations if and only if the corresponding observed family has the NIP. In [10], stability of the observable instead gives simulation of idealized preparations by finite mixtures of actual ones, uniformly over Bob's probes, idealized or not, and Maurey's empirical method bounds the number of terms.

Uniformity over the coordinates. Theorem 8.8 concerns one coordinate at a time, and countably many coordinates give uniformity in the metric of the type shard; uniformity in the supremum over all coordinates is the NIP of the whole family. Enumerate $\Gamma \times \Sigma$ as $(c_k)_{k\in\mathbb{N}}$, and let $d(\mathbf{v},\mathbf{w}) = \sum_k 2^{-k}|\mathbf{v}_{c_k} - \mathbf{w}_{c_k}|$, a metric for the topology of $\mathcal{L}_\Gamma$.

Proposition 8.10. *Let Γ and Σ be countable, let $\Delta \subseteq \Gamma$ be confined to L_E, and let μ be a Radon probability measure on $\mathcal{L}_E$.*

(1) *If the conditions of Theorem 8.8 hold for every coordinate, then, for $\mu^{\mathbb{N}}$-almost every sequence of samples, $\sup_{\Psi\in\Delta} d(\tilde{\Psi}(\hat{\mathbf{v}}_n), \tilde{\Psi}(\bar{\mathbf{v}})) \to 0$.*

(2) *The whole family $\{\rho \mapsto \varphi_C(\Phi(\Psi(\rho)),\sigma) : \Psi \in \Delta,\ (\Phi,\sigma) \in \Gamma\times\Sigma\}$ has the NIP on L_E if and only if the family of its extensions to $\mathcal{L}_E$ is a Glivenko–Cantelli class for every Radon probability measure on $\mathcal{L}_E$, that is, if and only if, almost surely, $\sup_{\Psi\in\Delta}\sup_k |\tilde{\Psi}(\hat{\mathbf{v}}_n)_{c_k} - \tilde{\Psi}(\bar{\mathbf{v}})_{c_k}| \to 0$ for every such measure.*

Proof. (1) The events of Proposition 8.9(1), one for each of the countably many coordinates, have probability 1, and so does their intersection. On it, for every K,

$$\sup_{\Psi\in\Delta} d(\tilde{\Psi}(\hat{\mathbf{v}}_n), \tilde{\Psi}(\bar{\mathbf{v}})) \le \sum_{k\le K} 2^{-k} \sup_{\Psi\in\Delta} \left|\tilde{\Psi}(\hat{\mathbf{v}}_n)_{c_k} - \tilde{\Psi}(\bar{\mathbf{v}})_{c_k}\right| + 2^{1-K}\cdot 2\|C\|,$$

and the right side tends to $2^{2-K}\|C\|$, which is arbitrarily small.

(2) The whole family is countable, bounded by $\|C\|$, and continuous on $\mathcal{L}_E$; the proof of Theorem 8.8 applies to it verbatim, and Proposition 8.9(1) translates the conclusion. □

By Proposition 6.2, the NIP of each instance does not imply the NIP of the whole family; by Proposition 8.10, uniformity in the metric of the type shard is therefore strictly weaker than uniformity over all the coordinates.

Rates. A family $\mathcal{F}$ of functions on a set X *γ-shatters* a finite set $\{x_1,\dots,x_d\} \subseteq X$ if there are reals $r_1,\dots,r_d$ such that, for every $I \subseteq \{1,\dots,d\}$, some $f \in \mathcal{F}$ satisfies $f(x_i) \ge r_i + \gamma$ for $i \in I$ and $f(x_i) \le r_i - \gamma$ for $i \notin I$. The *fat-shattering dimension* $\mathrm{fat}_\gamma(\mathcal{F})$ is the supremum of the cardinalities of the γ-shattered sets, and $\mathcal{F}$ has the *uniform NIP* if $\mathrm{fat}_\gamma(\mathcal{F})$ is finite for every $\gamma > 0$. Alon, Ben-David, Cesa-Bianchi and Haussler proved that a uniformly bounded family has the uniform NIP if and only if it is a Glivenko–Cantelli class uniformly over all probability measures, with rates of convergence that depend only on the fat-shattering dimensions [2].

Proposition 8.11. *Let Γ and Σ be countable, let $\Delta \subseteq \Gamma$ be confined to L_E, and let (Φ,σ) be a coordinate.*

(1) *For $0 < \gamma' < \gamma$, the fat-shattering dimensions of the family $\{\rho \mapsto \varphi_C(\Phi(\Psi(\rho)),\sigma) : \Psi \in \Delta\}$ on L_E and of its extensions to $\mathcal{L}_E$ satisfy $\mathrm{fat}_\gamma(\text{on } \mathcal{L}_E) \le \mathrm{fat}_{\gamma'}(\text{on } L_E) \le \mathrm{fat}_{\gamma'}(\text{on } \mathcal{L}_E)$. Hence the family has the uniform NIP on L_E if and only if its extensions have the uniform NIP on $\mathcal{L}_E$, and then the convergence of Proposition 8.9(1) is uniform over all Radon probability measures on $\mathcal{L}_E$, idealized random preparations included, with the rates of [2].*

(2) *If H_1 has finite dimension D, then $\mathrm{fat}_\gamma \le D^2$ for every $\gamma > 0$ and every set of channels.*

Proof. (1) The functions on L_E factor through tp_Γ, so the members of a shattered set of density operators have distinct orbit types, which are shattered at the same scale; this gives the second

inequality. For the first, let $\mathbf{v}_1, \dots, \mathbf{v}_d \in \mathcal{L}_E$ be γ-shattered with witnesses r_i, and choose, for each of the 2^d patterns I, a function f_I of the family that realizes it. The finitely many functions f_I are continuous, and the orbit types of members of L_E are dense in $\mathcal{L}_E$, so there are density operators $\rho_i \in L_E$ with $|f_I(\mathrm{tp}_\Gamma(\rho_i)) - f_I(\mathbf{v}_i)| \leq \gamma - \gamma'$ for all i and I; the states ρ_i are then γ'-shattered with the same witnesses, and they are distinct because the patterns separate them. The families are countable families of continuous functions, which satisfy the measurability conditions of [2].

(2) Every function of the family has the form $\rho \mapsto \mathrm{Tr}(\rho X)$ for a self-adjoint operator X, and is the restriction of an affine function on the real affine space of self-adjoint operators of trace 1, of dimension $D^2 - 1$. A family of affine functions on a real affine space of dimension m has pseudo-dimension at most $m+1$, and the fat-shattering dimension at every scale is at most the pseudo-dimension. □

In Aaronson's theorem on the learnability of quantum states [1], the roles are exchanged: the points are measurements, the functions are indexed by states, and the fat-shattering dimension on n qubits is $O(n/\gamma^2)$, logarithmic in the dimension $D = 2^n$. In the direction of Proposition 8.11, with states as points and effects as functions, no bound logarithmic in D holds in general: the D basis states are $\frac{1}{2}$-shattered by the diagonal operators with entries 0 and 1, with all witnesses equal to $\frac{1}{2}$. The bound in Proposition 8.11(2) is uniform in the computations but grows with the dimension, and in infinite dimensions even stability does not give the uniform NIP.

Proposition 8.12. *Let H_1 have an orthonormal basis consisting of two vectors $|0\rangle, |1\rangle$ and the vectors $|n, w\rangle$, for $n \geq 1$ and $w \in \{0,1\}^n$. For $1 \leq i \leq n$, let $P_{n,i}$ be the projection onto the span of the vectors $|n, w\rangle$ with $w_i = 1$, and let $M_{n,i}(\rho) = \mathrm{Tr}(\rho P_{n,i})|1\rangle\langle 1| + (1 - \mathrm{Tr}(\rho P_{n,i}))|0\rangle\langle 0|$. Let Bob's system be one-dimensional, with the observable $|1\rangle\langle 1|$, let $\Delta = \{M_{n,i}\}$, and let Γ be the monoid generated by Δ. Then Δ is stable on $\mathfrak{D}(H_1)$, and the family $\{\rho \mapsto \mathrm{Tr}(\rho P_{n,i})\}$ of its outputs at the coordinate of the identity has infinite fat-shattering dimension at the scale $\frac{1}{2}$.*

Proof. The projections $P_{n,i}$ vanish on $|0\rangle$ and $|1\rangle$, so every composite of two members of Δ is the reset channel $\rho \mapsto |0\rangle\langle 0|$, and the coordinates of the orbit type of a state ω are $\omega(|1\rangle\langle 1|)$, the numbers $\omega(P_{n,i})$, and 0. Hence $\tilde{M}_{n,i}(\mathrm{tp}_\Gamma(\omega))$ has the coordinate $\omega(P_{n,i})$ at the identity and 0 elsewhere. Let B_n be the projection onto the span of the vectors $|n, w\rangle$; the projections B_n are mutually orthogonal, so $\sum_n \omega(B_n) \leq 1$ for every state ω, and $0 \leq \omega(P_{n,i}) \leq \omega(B_n) \to 0$. Since each n carries finitely many indices i, every sequence of distinct members of Δ has $n \to \infty$, and the maps $\tilde{M}_{n,i}$ converge pointwise to the constant map onto the orbit type of $|0\rangle$; the closure of $\{\tilde{M}_{n,i}\}$ consists of these maps and the constant map, all continuous. For $d \geq 1$, let $n = 2^d$, index the coordinates $1, \dots, n$ by the subsets S of $\{1, \dots, d\}$, and for $k \leq d$ let w^k be the word with $w^k_S = 1$ exactly when $k \in S$. For every S, the function $\rho \mapsto \mathrm{Tr}(\rho P_{n,S})$ takes the value 1 at $|n, w^k\rangle$ for $k \in S$ and 0 for $k \notin S$, so these d pure states are $\frac{1}{2}$-shattered, with all witnesses equal to $\frac{1}{2}$. □

Which of the sets of computations of Sections 4 and 5 have the uniform NIP is Question 10.8.

9. Comparison with the literature

9.1. **Observable orbits and tame dynamics.** The closest dynamical antecedents are the independence theory of Kerr and Li [22] and the representation theory of Glasner and Megrelishvili [16, 15, 17]. For an action α of a group G by automorphisms of a unital C*-algebra A and a self-adjoint $a \in A$, both theories examine the functions $\omega \mapsto \omega(\alpha_g(a))$, $g \in G$, on the state space $S(A)$. Kerr and Li call the system tame when no element has an infinite ℓ_1-isomorphism

set in its orbit; their tensor independence, defined through completely bounded estimates for multiplication maps on tensor powers of finite-dimensional operator systems, involves in addition the multiplicative structure of the algebra. Here the observables are the chosen slices and their channel orbits, the Heisenberg maps need not preserve products, the inputs can be restricted by an energy bound, and the NIP is required of the observed family at each fixed coordinate.

Glasner and Megrelishvili represent tame metrizable systems on separable Banach spaces not containing ℓ_1, and their affine theory also treats semigroup actions by contractions. Their injectivity theorem [15, Theorem 8.2] identifies the enveloping semigroup of a tame system with that of the induced action on probability measures; Proposition 7.1 is its specialization to cascades. The quantum state space is already an affine compact space, but its states are observed through a prescribed fragment of the operator algebra. Propositions 2.6 and 2.9 identify the invariant fragment needed for the channels to act on the records, and Proposition 2.11 identifies its completed records with orbit types realized in ultrapowers.

The hierarchy in [17] is the direct dynamical antecedent of the classification by metrizability, hereditary separability and first countability. When $\Delta = \Gamma$ acts on an invariant type shard, its compactum of limits is an enveloping semigroup; a selected set $\Delta \subseteq \Gamma$ need not have this additional structure. Failure of first countability under factors is already known in tame dynamics. Proposition 6.5 gives a concrete measurement realization of that phenomenon: recording one more probe changes the quotient and restores first countability. The selected tree-indexed families of instruments give exact indexed realizations of the canonical families of the heptachotomy within this hierarchy.

9.2. **Observability and channel asymptotics.** The invariant observable space has a finite-dimensional antecedent in quantum observability. For controlled unitary systems, D'Alessandro [9] generates an observability space by repeated commutators with the control Lie algebra and characterizes indistinguishable states by agreement on the resulting observables. Our space W is generated by the Heisenberg orbits of all the chosen slices, under an arbitrary channel monoid. Both constructions organize preparations by the observations available after allowed evolutions. Proposition 2.9 gives the least invariant observation space, while Proposition 2.8 characterizes when the original fragment already suffices. The classification then concerns the compactum of induced maps on the completed records of those observations.

Finite-dimensional channel asymptotics supplies a useful baseline, but addresses a different limit. The peripheral structure of the powers of one channel is described by unitary blocks together with possible permutations of blocks; the resulting asymptotic map need not be unitary [30, 4]. Proposition 2.14 applies instead to arbitrary confined families on a trace-norm compact set of preparations. Its proof uses contraction and Arzelà–Ascoli, without a spectral decomposition of a single channel. Beyond compact shards, the observations become decisive: compact probes give stability for unitary dynamics (Theorem 7.4), whereas the pure-point example of Proposition 7.5 shows that spectral recurrence alone does not prevent independent responses to a noncompact probe, since the classification quantifies over all preparations and their idealized orbit types.

9.3. **Learning states and learning observations.** There are two complementary learning directions. Aaronson [1] treats states as hypotheses evaluated on measurements; Cheng, Hsieh and Yeh [8] treat measurements as hypotheses evaluated on states. The latter is the direction of Proposition 8.11. For a fixed final coordinate $(\Phi, \sigma) \in \Gamma \times \Sigma$, the operator $K = \Phi^*(S_C(\sigma))$ determines the hypotheses $\rho \mapsto \operatorname{Tr}(\rho\Psi^*(K))$, indexed by the preceding computation Ψ. If $M = \|C\\| > 0$, the operators $(\Psi^*(K) + MI)/(2M)$ are effects, so these hypotheses form, after

normalization, a dynamically selected subclass of the measurement class. The case $M = 0$ is constant. The finite-dimensional estimate is thus a bound for a subclass of the measurement class studied in [8].

The extension here concerns infinite dimension and completed type shards. The empirical-mixture theorem concerns random preparations and their barycenter, at a fixed output coordinate and uniformly over the chosen computations. For actual random density operators, the trace-norm law of large numbers in Proposition 8.9 already gives this uniformity for every family of channels. The distinguishing conclusion of Theorem 8.8 concerns every Radon law on the type shard, including laws charging unrealized types. Distribution-free rates require the uniform NIP, through scale-sensitive dimensions [2]; the stable example following Proposition 8.11 shows why qualitative stability alone cannot supply it.

9.4. **Local NIP and empirical convergence.** Khanaki [20] relates the NIP of a formula in a model to Glivenko–Cantelli classes and states a local equivalence for Boolean formulas. Theorem 8.8 concerns real-valued families: at each fixed coordinate (Φ, σ), the computations Ψ index a countable, uniformly bounded family of continuous functions on a compact type shard, the preparations supply their arguments, and the laws range over all Radon probability measures on the shard, idealized preparations included. Its proof rests on Talagrand's characterization of Glivenko–Cantelli classes by stability [27] and on the theory of [13, Section 5]. The almost-sure convergence for each fixed law is distinct from the distribution-free rates of Proposition 8.11, which require the uniform NIP.

10. Questions

The setting.

Question 10.1. Find a set of computations whose level on orbit types differs from its level computed on φ-types alone. By Proposition 6.1(2), such a set is not contained in a commuting monoid; for a left ideal, Proposition 6.1(1) gives the NIP on both kinds of type, but not necessarily the same level.

Question 10.2. The first paper on bipartite observables [10] relates stability of an observable to the exchange of two orders of idealization: idealizing Alice's state first and Bob's probe second, or the reverse. For a set of computations, iterating the channels is a limit in Alice's variable, and idealizing the probe is a limit in Bob's. Is stability of a set of computations equivalent to the exchange of these two limits, uniformly over the computations?

The classification.

Question 10.3. By Proposition 6.2, the NIP of the whole family $\{\rho \mapsto \varphi_C(\Phi(\Psi(\rho)), \sigma) : \Psi \in \Delta,\ (\Phi, \sigma) \in \Gamma \times \Sigma\}$ is strictly stronger than condition (ii) of Theorem 3.1, and the deep computations alone cannot detect it. Is there a form of Theorem 3.1 for the whole family, in terms of a compactum that records the computations and the probes together, such as the closure of the maps $(\mathbf{v}, \sigma) \mapsto \tilde{\Psi}(\mathbf{v})_{(\mathrm{id}, \sigma)}$?

Question 10.4. By Proposition 6.5, enlarging the observations can make a set of channels first countable. Which enrichments of the observed formulas preserve the level of a set of computations, and is there a characterization of the sets of channels whose first countability depends on the probes?

Question 10.5. The channels act only on Alice's variable, and the NIP of an observable in a model need not be symmetric in its two variables [21]. Is the NIP of a set of computations symmetric, in a suitable sense, in Alice's and Bob's variables?

Question 10.6. The deep computations take values in type shards, whose projections to φ-types are responses to actual probes, the restrictions of the members of the response compactum of C, which [11] places in one of three classes when C is NIP. How does the level of a set of computations relate to the class of the observable? In particular, can a set of computations of an observable with metrizable response compactum be $\mathrm{NIP}^{\mathrm{hs}}$ without being $\mathrm{NIP}^{\mathrm{met}}$?

Question 10.7. The trees of preparations of [11] realize the seven minimal families of the heptachotomy through Alice's states, and the trees of computations of [13] realize them through the computations. How do the two kinds of tree combine for bipartite computation structures?

Question 10.8. Which of the sets of computations of Sections 4 and 5 have the uniform NIP of Proposition 8.11, with fat-shattering dimensions finite at every scale? Proposition 8.12 shows that stability alone does not give it.

The examples.

Question 10.9. Extend the comparison observable, Theorem 4.4 and Theorem 4.6 from the one-flip sector to the whole chain, with the position of the first flip, or the number of flips to the right of Bob's site, in place of the position of the flip.

Question 10.10. On the lattice $\mathbb{Z}^2$, observed through half-plane probes, a particle escaping to infinity remembers its direction. Do rotations and boosts reach a level below $\mathrm{NIP}^{\mathrm{met}}$ there?

Question 10.11. Theorem 5.5 identifies the closure of the threshold instruments along the regular subtree (s_t). Over the whole tree, nodes with the same point $\bar{s}$, such as s and $s^\frown 0$, give the same instrument, and branches ending in zeros or ones have immediate neighbors. Is the closure K of Theorem 5.3 the set of the maps $\tilde{M}_{L_s}$ and $G_x^{\pm}$, and which compactum is it?

Question 10.12. The instruments of Section 5 measure in the basis of nodes, break entanglement and admit a classical probabilistic description, and the boundary types β_x have classical counterparts. Does a coherent dynamics on the tree, such as the quantum walk on the tree, whose limit should be the distribution of the point at which the particle escapes, reach the same levels?

Single channels and unitary dynamics.

Question 10.13. Does a single channel on $B(H_1)$, confined to an energy shard, realize the split Cantor set?

Question 10.14. A unitary channel acts on the projective space of H_1, which is not compact when H_1 is infinite-dimensional and whose closure in the type topology contains idealized states. Transfer the tame dynamics of [13, Section 4.1] and of Glasner and Megrelishvili [17] to unitary channels observed through probes that are not compact. Which alternatives of Todorčević's trichotomy are realized by unitary dynamics, and which require dissipation or measurement?

Question 10.15. A channel that is not unitary need not preserve purity, and then has no dynamics on pure states. Measurement followed by classical feedback is the quantum counterpart of the rounding in Bresenham's algorithm [13], which reaches the split Cantor set. Which levels do such channels reach, and is any level reached by channels and not by commutative computations?

Question 10.16. Which physical dynamics realize each level: channels that relax to equilibrium, repeated measurement with feedback, thermalizing many-body dynamics such as the transverse-field Ising model of [11], dispersive dynamics read through comparisons of positions, and quantum chaotic maps, which must act in infinite dimensions, since every set of channels on a finite-dimensional space is stable (Proposition 2.14)?

Question 10.17. Is there a single channel, confined to an energy shard, that is NIP but not $\mathrm{NIP}^{\mathrm{fc}}$? This is the quantum form of [13, Question 6.2], open for classical cascades.

References

[1] S. Aaronson, The learnability of quantum states, *Proc. R. Soc. A* 463 (2007), 3089–3114.
[2] N. Alon, S. Ben-David, N. Cesa-Bianchi and D. Haussler, Scale-sensitive dimensions, uniform convergence, and learnability, *J. ACM* 44 (1997), 615–631.
[3] S. Alva, E. Dueñez, J. Iovino and C. Walton, Approximability of deep computations, *Math. Structures Comput. Sci.*, to appear; arXiv:2409.06064.
[4] D. Amato, P. Facchi and A. Konderak, Asymptotics of quantum channels, *J. Phys. A: Math. Theor.* 56 (2023), 265304; arXiv:2210.17513.
[5] S. A. Argyros, P. Dodos and V. Kanellopoulos, A classification of separable Rosenthal compacta and its applications, *Dissertationes Math.* 449 (2008), 1–52.
[6] J. Bourgain, D. H. Fremlin and M. Talagrand, Pointwise compact sets of Baire-measurable functions, *Amer. J. Math.* 100 (1978), 845–886.
[7] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger Operators, with Application to Quantum Mechanics and Global Geometry*, Springer, 1987.
[8] H.-C. Cheng, M.-H. Hsieh and P.-C. Yeh, The learnability of unknown quantum measurements, *Quantum Inf. Comput.* 16 (2016), 615–656; arXiv:1501.00559.
[9] D. D'Alessandro, On quantum state observability and measurement, *J. Phys. A: Math. Gen.* 36 (2003), 9721–9735; arXiv:quant-ph/0307127.
[10] E. Dueñez and J. Iovino, Classification of bipartite observables I: stable and uniformly stable observables, arXiv:2610.11002.
[11] E. Dueñez and J. Iovino, Classification of bipartite observables II: NIP observables and Todorčević's trichotomy, arXiv:2610.12908.
[12] E. Dueñez and J. Iovino, Classification of bipartite observables III: simplicity, mixtures and quantum error correction, arXiv:2610.22205.
[13] E. Dueñez, T. Matos-Wiederhold, L. Salvetti and F. D. Tall, Complexity of deep computations via topology of function spaces, arXiv:2601.00528.
[14] D. H. Fremlin, *Measure Theory*, Vol. 4: *Topological Measure Spaces*, Torres Fremlin, 2003.
[15] E. Glasner and M. Megrelishvili, Banach representations and affine compactifications of dynamical systems, in: *Asymptotic Geometric Analysis*, Fields Inst. Commun. 68, Springer, 2013, 75–144; arXiv:1204.0432.
[16] E. Glasner and M. Megrelishvili, Representations of dynamical systems on Banach spaces not containing ℓ_1, *Trans. Amer. Math. Soc.* 364 (2012), 6395–6424.
[17] E. Glasner and M. Megrelishvili, Todorčević' trichotomy and a hierarchy in the class of tame dynamical systems, *Trans. Amer. Math. Soc.* 375 (2022), 4513–4548.
[18] A. S. Holevo, *Quantum Systems, Channels, Information*, 2nd ed., De Gruyter, 2019.
[19] H. J. Keisler, Model theory for real-valued structures, in *Beyond First Order Model Theory*, Vol. II, CRC Press, 2023.
[20] K. Khanaki, Glivenko–Cantelli classes and NIP formulas, *Arch. Math. Logic* (2024), doi:10.1007/s00153-024-00932-7; arXiv:2103.10788.
[21] K. Khanaki and A. Pillay, Remarks on the NIP in a model, *Math. Log. Q.* 64 (2018), 429–434.
[22] D. Kerr and H. Li, Independence in topological and C^*-dynamics, *Math. Ann.* 338 (2007), 869–926.
[23] J. Lukeš, J. Malý, I. Netuka and J. Spurný, *Integral Representation Theory: Applications to Convexity, Banach Spaces and Potential Theory*, De Gruyter, 2010.
[24] E. Mourier, Éléments aléatoires dans un espace de Banach, *Ann. Inst. H. Poincaré* 13 (1953), 161–244.
[25] R. R. Phelps, *Lectures on Choquet's Theorem*, 2nd ed., Lecture Notes in Math. 1757, Springer, 2001.
[26] H. P. Rosenthal, A characterization of Banach spaces containing ℓ^1, *Proc. Nat. Acad. Sci. USA* 71 (1974), 2411–2413.
[27] M. Talagrand, The Glivenko–Cantelli problem, *Ann. Probab.* 15 (1987), 837–870.
[28] M. Takesaki, *Theory of Operator Algebras I*, Springer, 1979.
[29] S. Todorčević, Compact subsets of the first Baire class, *J. Amer. Math. Soc.* 12 (1999), 1179–1212.
[30] M. M. Wolf, Quantum channels and operations: guided tour, lecture notes, 2012.