

CLASSIFICATION OF BIPARTITE OBSERVABLES II: NIP OBSERVABLES AND THE TODORČEVIĆ TRICHOTOMY

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ABSTRACT. A bipartite observable is stable when its expectation does not depend on which system is idealized first. We show that it is stable exactly when its idealized responses are continuous, and NIP exactly when they are of the first Baire class. They then form a Rosenthal compactum, and the dividing lines of Shelah in model theory, together with the classification of Rosenthal compacta of Todorčević, acquire a physical meaning. We also show that, for observables, mixtures of states make every hereditarily separable compactum of idealized responses metrizable, so that the Todorčević trichotomy becomes a classification into three classes, and that each class contains a physical order: the comparison of photon numbers, the comparison of positions, and the causal order of spacetime, whose idealized responses are not first countable because a light cone can shrink to a point; the comparison of two phases on a circle joins the causal order in the third class. We show that each of the seven minimal families of the heptachotomy, which Argyros, Dodos and Kanellopoulos extracted from the methods of Todorčević, is realized by a tree of preparations of photons, of a particle in a box, or of a spin chain read by a probe, and that the spin chain shows that NIP, unlike its uniform version, is not symmetric between the two parties. Finally, for the Ising model in a transverse field we show that, in the ordered phase, the responses of the two pure phases form split pairs whose midpoints, the responses of the symmetric state, form an uncountable discrete set; the proof uses a Lee–Yang theorem for this model.

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1. INTRODUCTION

Two quantum systems, Alice’s and Bob’s, are prepared independently in states ρ and σ , and a bipartite observable C is measured, with expectation value $\varphi_C(\rho, \sigma) = \text{Tr}((\rho \otimes \sigma)C)$. Physicists routinely idealize preparations: an oscillator excited to infinite energy, a particle localized at a point, a mode with infinitely many photons. Such an idealized preparation is a limit of actual ones; mathematically it is a state on the algebra of bounded operators that no density operator represents. In Part I [DI26] we called C *stable* when its expectation value does not depend on which of the two systems is idealized first, and we showed that stability is equivalent to a list of properties from analysis, topology, probability and model theory. Many physically natural bipartite observables are stable, among them the swap test, Hong–Ou–Mandel coincidences and discrete equality tests; these are even uniformly stable, with bounds that do not depend on the dimension.

Some of the most familiar observables of all are of a different kind: the order comparisons $[A < B]$, the projections onto the event “Bob’s value of B exceeds Alice’s value of A ”. The observables “Bob’s system has more photons than Alice’s” and “Bob’s oscillator has the higher energy” are unstable whenever both quantities have discrete spectra unbounded above: if both energies are sent to infinity, the probability that Bob’s is higher is 1 when Bob’s is idealized first and 0 when Alice’s is. The order of the idealizations matters because an order is built into the observable, and orders are the subject of this paper.

The comparison $[A < B]$ is unstable, but, as we show in Section 7, it is tame. Call a sequence of Alice’s preparations *independent* for C if Bob can read off every finite pattern from it: there are numbers $a < b$ such that, for any two disjoint finite sets of these preparations, a single probe of Bob’s system gives expectation values at most a on the first set and at least b on the second. The observable is *NIP*, short for “not the independence property”, when no independent sequence exists. In Section 7 we show that $[A < B]$ is NIP, with explicit bounds that do not depend on the Hilbert spaces. By contrast, in Proposition 7.13 we show that interference fringes of ever finer spacing have independent sequences: if Alice’s preparations produce fringes of wavenumbers $8, 8^2, 8^3, \dots$ on Bob’s screen, a single position of Bob’s detector can lie on a bright fringe for any chosen finite set of them and on a dark fringe for any other. NIP is also the dividing line of learning theory: by a theorem of Alon, Ben-David, Cesa-Bianchi and Haussler [ABDCBH97], a class of responses is learnable from finitely many samples exactly when its fat-shattering dimension, the quantitative form of NIP, is finite at every margin.

The division of theories by properties such as stability and NIP is the subject of Shelah’s classification theory in model theory [She90]. Part I and the present paper apply it to bipartite observables: Part I treats the first dividing line, stability, and this paper treats the next one, NIP. In model theory, an order is the prototype of an unstable but tame structure. The rational order $(\mathbb{Q}, <)$ is unstable, yet it is NIP: over a finite set it has only polynomially many types. The same holds for the ordered field of real numbers and, more generally, for o-minimal structures, the best-known unstable NIP structures.¹

¹Other classical examples are Presburger arithmetic $(\mathbb{Z}, +, <)$ and, more generally, ordered abelian groups [GS84], and the fields of p -adic numbers and algebraically closed valued fields (see [Sim15a, Chapter 2]). For o-minimal structures see [PS86, KPS86]; the real field with the exponential function is o-minimal

These structures, like most applications of NIP, are classical: their formulas take the values true and false, even when, as for the real field, the underlying set is a continuum. Here the formulas are real-valued, being expectation values, and the examples come from physics rather than from algebra or geometry. Real-valued forms of NIP have met quantum theory before, in finite dimensions, as the fat-shattering dimension that governs the learnability of quantum states and measurements [Aar07, CHY15]. To our knowledge, however, the structure theory of NIP, and with it the theory of Rosenthal compacta, has not previously been brought to bear on physical observables.

Related work. Model theory has met operator algebras and quantum theory along other routes. Farah, Hart and Sherman developed the continuous model theory of C^* -algebras and von Neumann algebras and studied stability in that setting [FHS13]; there the structures are the algebras themselves, and stability is a property of their theories, whereas here the object is a single observable together with its idealized responses. Zilber gave a model-theoretic semantics of the canonical commutation relation [Zil16]. On the side of real-valued NIP, Ben Yaacov showed that NIP is preserved under randomization [Ben09], so that averages of NIP relations over random parameters are again NIP; Proposition 7.3 is an elementary, quantitative instance of this phenomenon for relations between quantities measured by Alice and by Bob. Hrushovski and Pillay studied invariant measures in NIP theories [HP11], Simon connected NIP with Rosenthal compacta [Sim15b], and Glasner and Megrelishvili used the Todorćević trichotomy to classify tame dynamical systems [GM22]. To our knowledge, none of these works considers the orders of limits of a bipartite observable, the compactum of its idealized responses, or the place of physical observables in the Todorćević and Argyros–Dodos–Kanellopoulos classifications, which are the subject of Part I and the present paper.

Contributions of the paper. For mathematicians, the news is that the dividing lines of Shelah in model theory and the classification of Rosenthal compacta of Todorćević, with its refinement by Argyros, Dodos and Kanellopoulos, developed for purely mathematical reasons, describe natural phenomena. In Sections 7 and 8 we show that the classes of the Todorćević trichotomy, in the form they take for observables, and all seven minimal families of the hepatichotomy are inhabited by physical systems: photons, particles in a box, light cones, spin chains and interference fringes, each displaying what distinguishes its class from the others. For physicists, the news is a fine scale on which orders of limits can be placed and compared, with an operational meaning at each level, and a new theorem that the scale brings to light, which we prove in Section 9 (see Theorem 9.6): in the Ising model in a transverse field, a Hamiltonian whose terms do not commute, the classification sees the phase transition. Figure 1 in Section 2 is a map of the whole picture, Figure 2 shows the seven minimal families at work, and Tables 1 and 2 give the details.

The dictionary. Fix an observable C , and regard the expectation value of an idealized preparation of Alice’s system as a function of Bob’s probe, idealized probes included; we call it an *idealized response*. In Section 5 we show that the idealized responses form a compact set of functions on a compact metrizable space of probes, and in Theorem 6.1 that Shelah’s classification theory singles out the right regularity of these functions at each level:

- the observable is stable exactly when every idealized response is continuous;
- it is NIP exactly when every idealized response is of the first Baire class, that is, a pointwise limit of a sequence of continuous functions; the idealized responses then form a *Rosenthal*

by a theorem of Wilkie [Wil96]. O-minimality is at the heart of the proof of the André–Oort conjecture by Pila, Shankar and Tsimerman [PST21].

compactum, and every one of them is the limit of a sequence of responses of actual preparations (see [BFT78, Theorem 3F]);

- otherwise, some sequence of actual preparations is independent, and its idealized responses contain a copy of $\beta\mathbb{N}$ (see Fact 3.4).

The fine classification. Separable Rosenthal compacta form a vast class, and a priori there is no reason why their failure to be metrizable should be witnessed by anything simple. The Todorćević trichotomy [Tod99] shows that a simple witness always exists: according to which of the properties of metrizability, hereditary separability and first countability fail, a non-metrizable compactum contains one of a handful of canonical spaces, namely the split interval, the Alexandroff duplicate of the Cantor set or its analogue for the split Cantor set, or the one-point compactification of a discrete set of size continuum. In this the trichotomy resembles Rosenthal’s ℓ_1 theorem [Ros74], which detects a single canonical obstruction in a Banach space. The heptachotomy of Argyros, Dodos and Kanellopoulos [ADK08], which they extracted from the methods of Todorćević’s paper, refines the trichotomy to the level of families: every family of functions on a Polish space, indexed by the dyadic tree and relatively compact in the first Baire class, contains, along a dyadic subtree, a subfamily equivalent to one of exactly seven minimal families $D_1, \dots, D_7$. For an observable, a family indexed by the dyadic tree is a tree of preparations, each refining its predecessor, such as successively sharper localizations of a particle, and the heptachotomy says that, up to passing to a subtree, there are exactly seven ways in which such a tree can approach its idealized limits. Through the connection among NIP, VC dimension and Rosenthal compacta, the trichotomy also classifies NIP formulas, and hence NIP theories [DIMW⁺26, Section 4.2].

Quantum mechanics gives this classification a structure of its own. Alice can mix two preparations, and the responses of the mixture are the corresponding mixtures of responses, so the idealized responses form a convex set (see the proof of Corollary 6.8). For convex compacta, hereditary separability implies metrizability (see Theorem 6.6, a special case of a known result on locally connected Rosenthal compacta, for which we give a direct proof), so that observables fall into three Todorćević classes (see Theorem 6.9). In Section 7 we show that each is inhabited by a physical order: the comparison of photon numbers, the comparison of positions or of quadratures of light, and the causal order of spacetime, which is not first countable because a light cone can shrink to a point. In Theorem 7.12 we show that the comparison of two phases on a circle joins the causal order in the last class: on a circle, the jumps at the two ends of a half-turn can cancel. In Theorem 8.5 we show that, at the level of trees of preparations, all seven minimal families appear (see Figure 2).

The classifications are theorems of ZFC, but the Todorćević trichotomy is proved by the method of forcing [Tod99]: the required configuration is first found in a generic extension of the set-theoretic universe, and the conclusion is then brought back by an absoluteness argument, since it is a statement about countable codes that does not change between the two. To our knowledge no proof of the trichotomy that avoids forcing is known [DIMW⁺26]. Through the trichotomy, our results on the fine structure of observables inherit this feature. Orders of limits and the quantum Ising model. Many central notions of physics are defined by an order of limits. Spontaneous magnetization appears when the volume of a ferromagnet becomes infinite before the external field is switched off, and it disappears in the other order (see the proof of Theorem 9.2). The classification gives such phenomena a scale with an operational meaning: the levels NIP_1 , NIP_2 and NIP_3 of NIP, which we define in Section 4, have readings in terms of Bob’s probes; for instance, a NIP family is NIP_3 exactly when countably many probes tell all its idealized responses apart (see Proposition 4.5). In Theorem 9.2 we

show that the magnetization is stable exactly when the spontaneous magnetization vanishes, and that an offset field that moves the transition point places it at the level NIP_1 , the level of positions in a box: the responses of the two pure phases form a split pair, and the response of the symmetric state is their midpoint, exactly as mixtures act in the proof of Theorem 6.6. For the Ising model in a transverse field this is a new theorem (see Theorem 9.6). Its key step is the analyticity of the pressure away from zero field, which we derive from a Lee–Yang theorem for the transverse-field model: the partition function does not vanish off the imaginary axis of the complex longitudinal field (see Proposition 9.4). The Lee–Yang theorems for quantum spin systems of Asano [Asa70] and of Suzuki and Fisher [SF71] exclude zeros, in the presence of a transverse field, only in a smaller half-plane (see the remarks after Proposition 9.4). We include a proof for the whole half-plane, which combines standard tools, because we have not found it in the literature. The Banks–Casher relation for the chiral condensate [BC80] and the criterion of Scalapino, White and Zhang for superconductivity [SWZ93] become, in the same way, statements of stability (see Theorems 10.3 and 10.4).

The logical setting. Expectation values are real numbers, known only up to the finite resolution of a measurement; the resolution appears as the gap $b - a$ in the definition of the independence property (see Definition 3.3). We work in the model theory of real-valued structures of Keisler [Kei23], in which no metric is given a priori and each formula carries its own bound, rather than in the continuous logic of Ben Yaacov and Usvyatsov [BYU10], in which each sort carries a metric given in advance and all formulas take values in $[0, 1]$. For the bounded observables of this paper the two agree, but physics forces unbounded quantities: no pair of bounded operators satisfies the canonical commutation relation $QP - PQ = i\hbar I$ [Win47, Wie49]. For comparisons unboundedness is harmless, because $[A < B]$ is a projection, unchanged when A and B are replaced by the same increasing function of themselves (see Proposition 7.9). For the expectation of an unbounded quantity, such as the energy $\rho \mapsto \text{Tr}(\rho H)$, it is not. Such a formula can be brought into $[0, 1]$ only by a nonlinear rescaling or by adjoining a point at infinity [Ben08]; expectations are linear while rescalings are not, and the energy constraints $\text{Tr}(\rho H) \leq E$ of quantum information cannot be stated in rescaled terms. In Keisler’s setting each formula φ takes its values in an interval $[-r_\varphi, r_\varphi]$, whose radius r_φ depends on the formula; an unbounded quantity therefore enters only through its restrictions to confinements, on each of which it is bounded, such as a particle in a box, a cavity with a frequency cutoff or a detector with a finite range, and the confinements play the role of the shards of [DIMW⁺26]. A confinement must be compact in the weak* sense without being compact in norm: on a set of Bob’s states that is compact in the trace norm, every bounded observable is stable, because its expectation is Lipschitz in Bob’s state, uniformly in Alice’s.

The paper is organized as follows. In Section 2 we state the main results. In Sections 3 and 4 we collect the facts about Rosenthal compacta that we need and derive a classification of countable NIP families of continuous functions (see Theorem 4.7). In Sections 5 and 6 we attach to each observable its compactum of idealized responses, prove Theorem 6.1, and show that for observables hereditary separability implies metrizability (see Theorem 6.6). In Sections 7 and 8 we give the examples, in Section 9 we treat spontaneous magnetization, classical and quantum, and in Section 10 two further orders of limits. Section 11 collects open questions. Classical arguments are cited, and sketched where the adaptation is not routine.

2. MAIN RESULTS

Figure 1 is a map of the classification, with the physical systems that inhabit each class and the minimal families that each class contains; Figure 2 shows how each of the seven minimal

families of the heptachotomy approaches its idealized limits, and Tables 1 and 2 give the details. The first table lists the Todorčević classes, the second the minimal families of the heptachotomy, each with a physically natural example and with the feature of the example that the class or family isolates.

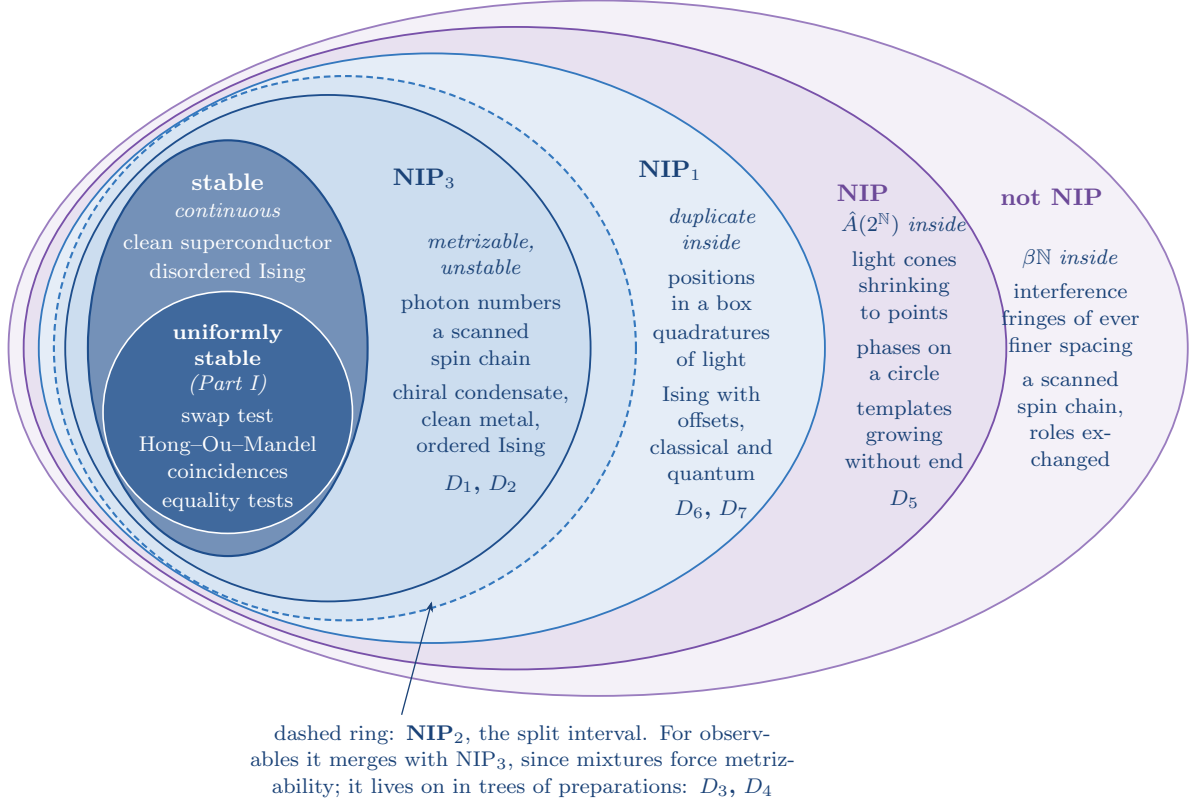


FIGURE 1. The classes of observables and of families of responses, with their physical examples and the minimal families of the heptachotomy that they contain. The regions are nested: $\text{uniformly stable} \subseteq \text{stable} \subseteq \text{NIP}_3 \subseteq \text{NIP}_2 \subseteq \text{NIP}_1 \subseteq \text{NIP}$. The innermost examples are those of Part I [DI26].

Throughout, “Alice’s responses” are the expectation values of Alice’s preparations as functions of Bob’s probes, and the classes refer to their closure.

The results behind the tables are the following:

- (A) *The dictionary.* An observable is stable exactly when its idealized responses are continuous, NIP exactly when they are of the first Baire class, and otherwise has an independent sequence whose idealized responses contain $\beta\mathbb{N}$ (see Theorem 6.1).
- (B) *Mixtures force metrizability.* A separable convex Rosenthal compactum that is hereditarily separable is metrizable (see Theorem 6.6); hence NIP_2 observables are NIP_3 (see Corollary 6.8), and the fine classification of NIP observables has three Todorčević classes (see Theorem 6.9).
- (C) *Natural examples.* The comparison of photon numbers is NIP_3 , the comparison of positions or of quadratures is NIP_1 but not NIP_2 , and the causal order of spacetime and the comparison of two phases are NIP but not NIP_1 (see Theorems 7.8, 7.10 and 7.12, Proposition 7.9); each minimal family is realized by a tree of preparations of one of

Class	Witness	The picture	The physics	
stable	continuous	idealizations commute	swap test; superconductor	10.4
NIP_3	metrizable	countably many probes suffice	photons; spin chains	7.8
$\text{NIP}_2 \setminus \text{NIP}_3$	split interval	mixtures supply the midpoints	merges with NIP_3	6.6
$\text{NIP}_1 \setminus \text{NIP}_2$	duplicate	approach from either side	positions; quadratures; Ising	9.6
$\text{NIP} \setminus \text{NIP}_1$	$\hat{A}(2^{\mathbb{N}})$	a light cone shrinks to a point	causal order; phases on a circle	7.10, 7.12
not NIP	$\beta\mathbb{N}$	ever finer oscillation	interference fringes	7.13

TABLE 1. The Todorčević classes; the last column gives the main result.

Family	Closure	The picture	The physics
D_1	$A(2^{<\mathbb{N}})$	escape to infinity	photon number states
D_2	$2^{\leq\mathbb{N}}$	infinite configurations	a spin chain read site by site
D_3	$\hat{S}_+(2^{\mathbb{N}})$	approach from the right	a particle in a box
D_4	$\hat{S}_-(2^{\mathbb{N}})$	approach from the left	a particle in a box
D_6	$\hat{D}(2^{\mathbb{N}})$	twins over a Cantor set	a spin chain, sites and templates
D_7	$\hat{D}(S(2^{\mathbb{N}}))$	approach from both sides	a particle in a box, half on each side
D_5	$\hat{A}(2^{\mathbb{N}})$	a point-supported limit	light cones; templates

TABLE 2. The seven minimal families, grouped by Todorčević class as in Figure 2 (see Theorem 8.5).

these observables or of a scanned spin chain (see Theorem 8.5). The scanned spin chain is NIP for the chain's responses but not for the probe's, so NIP is not symmetric for observables (see Corollary 8.7), although uniform NIP is (see Remark 3.8).

- (D) *The quantum Ising model.* For the Ising model in a transverse field, the magnetization with offsets is stable and metrizable when the spontaneous magnetization vanishes, and NIP_1 but not NIP_2 when it is positive (see Theorem 9.6). The proof uses the real analyticity of the pressure away from zero field, which we derive from a Lee–Yang theorem for this model: the partition function has no zeros off the imaginary axis of the complex longitudinal field (see Propositions 9.4 and 9.5).
- (E) *Orders of limits.* Spontaneous magnetization, the Banks–Casher relation and the criterion of Scalapino, White and Zhang are statements of stability (see Theorems 9.2, 10.3 and 10.4).

Theorem 6.6 in (B) is a special case of a known result (see Remark 6.7). To our knowledge, its consequences for observables in (B), result (D), the realizations in (C) and the asymmetry of nonuniform NIP are new.

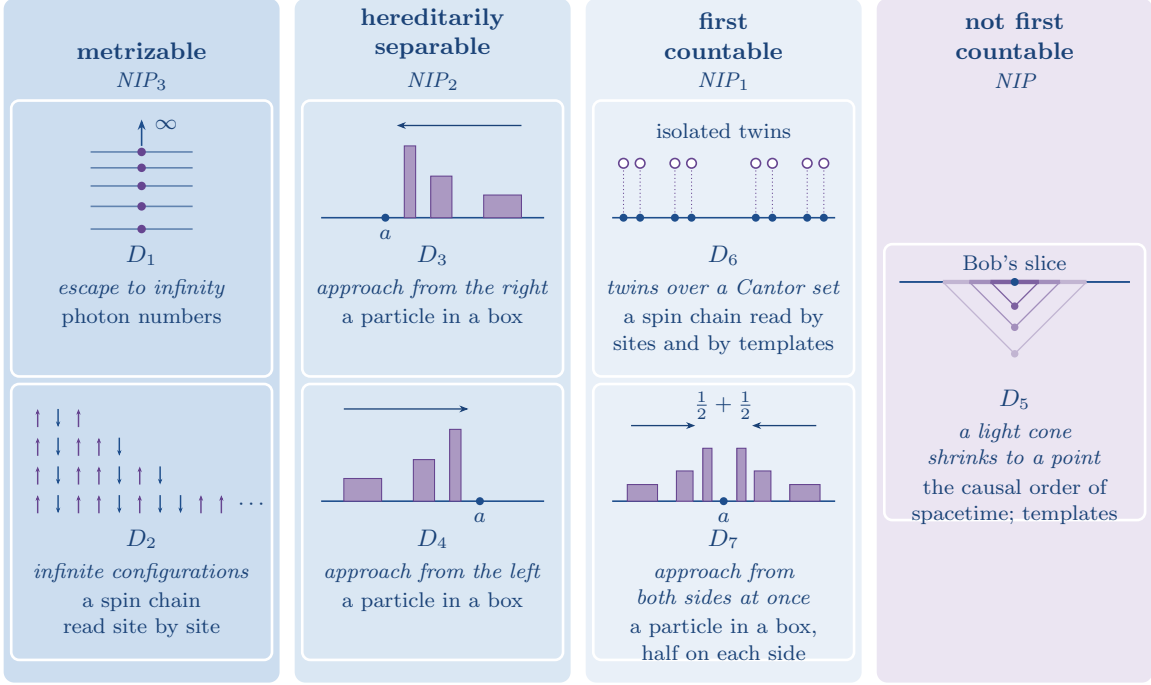


FIGURE 2. The seven minimal families of the heptachotomy, arranged by the class of the Todorčević trichotomy to which their closures belong, each with the physical picture of its realization (see Theorem 8.5): how a tree of preparations approaches its idealized limits.

3. A FAMILY OF CONTINUOUS FUNCTIONS IS NIP EXACTLY WHEN ITS CLOSURE CONSISTS OF BAIRE-1 FUNCTIONS

The goal of this section is to state the topological dichotomy on which the paper rests: a pointwise bounded family of continuous functions on a compact metrizable space either has the independence property or has its pointwise closure inside the functions of the first Baire class. Throughout the paper the functions will be responses of Alice's preparations and the points will be Bob's probes; accordingly, we write Y for the underlying space and $\mathcal{R}$ for families of functions.

3.1. Functions of the first Baire class. For a topological space Y , let $C_p(Y)$ be the set of continuous functions $Y \rightarrow \mathbb{R}$ with the topology of pointwise convergence. A function $f: Y \rightarrow \mathbb{R}$ is of the *first Baire class*, or *Baire-1*, if it is the pointwise limit of a sequence of continuous functions, and we write $B_1(Y)$ for the set of these functions, again with the topology of pointwise convergence. For $\mathcal{R} \subseteq \mathbb{R}^Y$, we write $\overline{\mathcal{R}}$ for the closure of $\mathcal{R}$ in $\mathbb{R}^Y$, and we call $\mathcal{R}$ *pointwise bounded* if $\sup_{r \in \mathcal{R}} |r(y)| < \infty$ for every $y \in Y$.

We use Baire's characterization of Baire-1 functions on Polish spaces.

Fact 3.1 (see [Tod97, Section 10]). *Let Y be a Polish space and $f: Y \rightarrow \mathbb{R}$ a function. The following are equivalent:*

- (1) *the function f is Baire-1;*
- (2) *for every open $U \subseteq \mathbb{R}$, the set $f^{-1}[U]$ is F_σ ;*
- (3) *for every closed $F \subseteq Y$, the restriction $f|_F$ has a point of continuity.*

Moreover, if f is not Baire-1, then there are countable sets $D_0, D_1 \subseteq Y$ with $\overline{D_0} = \overline{D_1}$ and reals $a < b$ such that $f \leq a$ on D_0 and $f \geq b$ on D_1 .

The last clause of Fact 3.1 lets us test membership in $B_1(Y)$ on countable sets, which is what makes pointwise closures manageable.

Lemma 3.2. *Let Y be a Polish space and $\mathcal{R} \subseteq B_1(Y)$ a pointwise bounded family. The following are equivalent:*

- (i) *the family $\mathcal{R}$ is relatively compact in $B_1(Y)$;*
- (ii) *every countable subset of $\mathcal{R}$ has an accumulation point in $B_1(Y)$;*
- (iii) *we have $\overline{\mathcal{R}} \subseteq B_1(Y)$.*

Sketch. This is the argument of Bourgain, Fremlin and Talagrand [BFT78]. The implication (iii) $\Rightarrow$ (i) is Tychonoff's theorem. If some $f \in \overline{\mathcal{R}}$ is not Baire-1, take D_0, D_1, a and b as in Fact 3.1; a sequence in $\mathcal{R}$ converging to f on $D_0 \cup D_1$ has no accumulation point in $B_1(Y)$, because such a point would have no point of continuity on $\overline{D_0}$. $\square$

3.2. The independence property. In Theorem 3.6 below, we show that the obstruction to Baire-1 closures is combinatorial: a sequence of functions whose sublevel and superlevel sets meet in every finite pattern.

Definition 3.3. Let Y be a set. A family $\mathcal{R} \subseteq \mathbb{R}^Y$ has the *independence property* if there are a sequence (r_n) in $\mathcal{R}$ and reals $a < b$ such that for all finite disjoint sets $E, F \subseteq \mathbb{N}$, there is $y \in Y$ with $r_n(y) \leq a$ for $n \in E$ and $r_n(y) \geq b$ for $n \in F$. Otherwise, the family $\mathcal{R}$ is *NIP*.

On the Cantor space, the coordinate functions $p_n(y) = y(n)$ have the independence property, and the closure of $\{p_n : n \in \mathbb{N}\}$ in $\mathbb{R}^{2^{\mathbb{N}}}$ is a copy of $\beta\mathbb{N}$. The next two facts say that this example is the only obstruction.

Fact 3.4 (Rosenthal's dichotomy; see [Ros74, Tod97]). *Let Y be a Polish space and (r_n) a pointwise bounded sequence in $C_p(Y)$. Then the sequence (r_n) has either a pointwise convergent subsequence or a subsequence whose closure in $\mathbb{R}^Y$ is homeomorphic to $\beta\mathbb{N}$.*

Fact 3.5 (Bourgain–Fremlin–Talagrand; see [BFT78, Theorem 4G]). *Let Y be a Polish space and $\mathcal{R} \subseteq C_p(Y)$ a pointwise bounded family. Then we have $\mathcal{R} \subseteq B_1(Y)$ if and only if for every sequence (r_n) in $\mathcal{R}$ and all reals $a < b$, there is $I \subseteq \mathbb{N}$ such that no $y \in Y$ satisfies $r_n(y) \leq a$ for all $n \in I$ and $r_n(y) \geq b$ for all $n \notin I$.*

When the space Y is compact, the condition in Fact 3.5 reduces to a statement about finite patterns, and we obtain the dichotomy announced at the beginning of this section.

Theorem 3.6. *Let Y be a compact metrizable space and $\mathcal{R} \subseteq C_p(Y)$ a pointwise bounded family. Then exactly one of the following holds:*

- (i) *we have $\overline{\mathcal{R}} \subseteq B_1(Y)$;*
- (ii) *the family $\mathcal{R}$ has the independence property.*

The following proof is the classical passage from infinite to finite patterns by compactness (see [DIMW⁺26, Section 1]).

Proof. For a sequence (r_n) in $\mathcal{R}$, reals $a < b$ and $I \subseteq \mathbb{N}$, the sets $\{y : r_n(y) \leq a\}$ for $n \in I$ and $\{y : r_n(y) \geq b\}$ for $n \notin I$ are closed, because each function r_n is continuous. By compactness of Y , these closed sets have empty intersection if and only if finitely many of them do. Hence the condition in Fact 3.5 fails for some (r_n) , a and b exactly when every finite pattern is realized, that is, exactly when the family $\mathcal{R}$ has the independence property. $\square$

We will also need to replace a space of probes by a dense subset, and the next lemma shows that the replacement does not affect the independence property.

Lemma 3.7. *Let Y be a topological space, $\mathcal{R} \subseteq C_p(Y)$ a family and $L \subseteq Y$ a dense set. Then the family $\mathcal{R}$ is NIP if and only if the family $\{r|_L : r \in \mathcal{R}\}$ of restrictions is NIP.*

Sketch. If (r_n) and $a < b$ witness the independence property of $\mathcal{R}$, then, for $a < a' < b' < b$, the open sets on which a finite pattern holds strictly with a' and b' meet L . $\square$

Remark 3.8 (Symmetry). For a formula of a first-order theory, NIP is symmetric in the two variables: the formula $\varphi(x, y)$ is NIP if and only if the formula $\varphi^{\text{opp}}(y, x) = \varphi(x, y)$ is NIP (see [She90, Chapter II, Section 4] and [Sim15a, Lemma 2.5]); if φ has VC dimension d , then the formula φ^{opp} has VC dimension less than 2^{d+1} [Ass83]. The finite combinatorial argument applies, with the reals $a < b$ fixed, to real-valued functions. For the uniform notion of Remark 6.2, the roles of Alice and Bob are therefore interchangeable. Definition 3.3, which asks for an infinite sequence inside a single family, is not symmetric: the relation $a \in b$ between numbers a and finite sets $b \subseteq \mathbb{N}$ has the independence property in the variable a but not in the variable b .

3.3. Rosenthal compacta. A *Rosenthal compactum* is a compact space homeomorphic to a compact subset of $B_1(Y)$, with the topology of pointwise convergence, for some Polish space Y . By Theorem 3.6 and Lemma 3.2, if Y is compact metrizable and $\mathcal{R} \subseteq C_p(Y)$ is pointwise bounded and NIP, then the closure $\overline{\mathcal{R}}$ is a Rosenthal compactum. The classification of Rosenthal compacta (see Section 4) will therefore classify the NIP families that arise below.

3.4. Historical remarks. Rosenthal's dichotomy grew out of his ℓ_1 theorem [Ros74]; Bourgain, Fremlin and Talagrand proved Fact 3.5 in 1978 [BFT78], and Godefroy named Rosenthal compacta [God80] (see also [Ai92, Tka11]). The independence property was isolated by Shelah [She71, She90]; its $\{0, 1\}$ -valued case was found independently by Sauer [Sau72] and Shelah [She72], and by Vapnik and Chervonenkis [VC71, VC74] in learning theory [Val84]. The connection between NIP and Rosenthal compacta is due to Simon [Sim15b], and its use for limits of computations to [DIMW⁺26].

4. SEPARABLE ROSENTHAL COMPACTA FALL INTO FINITELY MANY CANONICAL CLASSES

The goal of this section is to state the trichotomy of Todorćević and the heptachotomy of Argyros, Dodos and Kanellopoulos, and to derive from them a classification of countable NIP families of continuous functions on a compact metrizable space. By Section 3, the pointwise closure of such a family is a separable Rosenthal compactum; hence the classification of these compacta, which describes their failure to be metrizable by finitely many canonical examples, applies to it. We follow the notation of [ADK08].

4.1. Three topological invariants. Every compact metrizable space is hereditarily separable, and by a theorem of Pol every hereditarily separable Rosenthal compactum is first countable (see [Deb13, Section 10]). For a Rosenthal compactum K we therefore have:

$$K \text{ metrizable} \implies K \text{ hereditarily separable} \implies K \text{ first countable.}$$

For separable Rosenthal compacta, the failure of each converse is witnessed by a canonical subspace, and these canonical subspaces are pointwise closures of families indexed by the dyadic tree (see Fact 4.3 below).

4.2. The canonical compacta. Let $2^{<\mathbb{N}}$ be the set of finite binary sequences and $2^{\mathbb{N}}$ the Cantor space. For $t \in 2^{<\mathbb{N}}$, let v_t be the characteristic function of the clopen set $V_t = \{\sigma \in 2^{\mathbb{N}} : t \sqsubset \sigma\}$ of extensions of t , and let $t \smallfrown 0^\infty$ and $t \smallfrown 1^\infty$ be the sequences that extend t by zeros and by ones. Let $\prec$ be the lexicographic order of $2^{\mathbb{N}}$. For $\sigma \in 2^{\mathbb{N}}$, let x_σ^+ and x_σ^- be the characteristic functions of $\{\tau \in 2^{\mathbb{N}} : \sigma \preceq \tau\}$ and $\{\tau \in 2^{\mathbb{N}} : \sigma \prec \tau\}$. For functions f and g on spaces X_1 and X_2 , let (f, g) be the function on the disjoint sum $X_1 \oplus X_2$ that is f on X_1 and g on X_2 ; a point of $2^{\mathbb{N}}$ is regarded as a function on $\mathbb{N}$.

The following compacta appear in the classification:

- the *one-point compactification* $A(2^{\mathbb{N}})$ of a discrete set of size continuum, realized as $\{\delta_\sigma : \sigma \in 2^{\mathbb{N}}\} \cup \{0\}$, where δ_σ is the characteristic function of $\{\sigma\}$, and its separable extension $\hat{A}(2^{\mathbb{N}})$, the pointwise closure of $\{v_t : t \in 2^{<\mathbb{N}}\}$, which equals $\{v_t : t \in 2^{<\mathbb{N}}\} \cup A(2^{\mathbb{N}})$;
- the *split Cantor set* $S(2^{\mathbb{N}}) = \{x_\sigma^+, x_\sigma^- : \sigma \in 2^{\mathbb{N}}\}$, the *split interval* $S(I)$, and the extended split Cantor sets $\hat{S}_+(2^{\mathbb{N}})$ and $\hat{S}_-(2^{\mathbb{N}})$ defined below; the space $\hat{S}_+(2^{\mathbb{N}})$ is a closed subspace of $S(I)$;
- the *Alexandroff duplicate* $D(2^{\mathbb{N}})$ of the Cantor set and its separable extension $\hat{D}(2^{\mathbb{N}})$, and the extended duplicate $\hat{D}(S(2^{\mathbb{N}}))$ of the split Cantor set, both defined below.

Among these, the space $S(2^{\mathbb{N}})$ is hereditarily separable but not metrizable, the space $D(2^{\mathbb{N}})$ is first countable but not separable, and the space $A(2^{\mathbb{N}})$ is not first countable (see [ADK08]).

4.3. The heptachotomy. A *dyadic subtree* of $2^{<\mathbb{N}}$ is a subtree in which every node has exactly two immediate successors; we write it as $(s_t)_{t \in 2^{<\mathbb{N}}}$, using its canonical enumeration, which preserves length and the lexicographic order (see [ADK08, Section 2]). A dyadic subtree is *regular* if each of its levels is contained in a level of $2^{<\mathbb{N}}$. Fix a regular dyadic subtree $R = (s_t)_{t \in 2^{<\mathbb{N}}}$ such that the sequences $s \smallfrown 0^\infty$, for $s \in R$, are pairwise distinct, and so are the sequences $s \smallfrown 1^\infty$.

Definition 4.1. Let X be a Polish space.

- (1) Two pointwise bounded families $\{f_i : i \in I\}$ and $\{g_i : i \in I\}$ of real functions, indexed by the same countable set I , are *equivalent* if the map $f_i \mapsto g_i$ extends to a homeomorphism between their pointwise closures.
- (2) A family $\{f_t : t \in 2^{<\mathbb{N}}\}$ that is relatively compact in $B_1(X)$ is *minimal* if it is equivalent to $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$ for every dyadic subtree (s_t) of $2^{<\mathbb{N}}$.

The seven minimal families are the following, and the pointwise closure of each of them is a separable Rosenthal compactum that contains the family as a discrete set (see [ADK08]):

- (1) $D_1 = \{\frac{1}{|t|+1}v_t : t \in 2^{<\mathbb{N}}\}$, whose closure $A(2^{<\mathbb{N}})$ is the one-point compactification of a countable discrete set;
- (2) $D_2 = \{s \smallfrown 0^\infty : s \in R\}$, a family of points of $2^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$, whose closure is denoted $2^{\leq \mathbb{N}}$;
- (3) $D_3 = \{x_{s \smallfrown 0^\infty}^+ : s \in R\}$, whose closure is $\hat{S}_+(2^{\mathbb{N}})$;
- (4) $D_4 = \{x_{s \smallfrown 1^\infty}^- : s \in R\}$, whose closure is $\hat{S}_-(2^{\mathbb{N}})$;
- (5) $D_5 = \{v_t : t \in 2^{<\mathbb{N}}\}$, whose closure is $\hat{A}(2^{\mathbb{N}})$;
- (6) $D_6 = \{(v_t, t \smallfrown 0^\infty) : t \in 2^{<\mathbb{N}}\}$, whose closure is $\hat{D}(2^{\mathbb{N}})$;
- (7) $D_7 = \{(v_s, x_{s \smallfrown 0^\infty}^+) : s \in R\}$, whose closure is $\hat{D}(S(2^{\mathbb{N}}))$.

The closures in (1) and (2) are metrizable, those in (3) and (4) are hereditarily separable but not metrizable, those in (6) and (7) are first countable but not hereditarily separable, and the closure in (5) is not first countable (see [ADK08]).

Fact 4.2 (Heptachotomy; see [ADK08, Theorem 2]).

- (a) Up to equivalence, there are exactly seven minimal families, namely $D_1, \dots, D_7$, and they are pairwise non-equivalent.
- (b) Let X be a Polish space and $\{f_t : t \in 2^{<\mathbb{N}}\}$ a family that is relatively compact in $B_1(X)$. Then there is a regular dyadic subtree (s_t) of $2^{<\mathbb{N}}$ such that $\{f_{s_t} : t \in 2^{<\mathbb{N}}\}$ is equivalent to one of $D_1, \dots, D_7$.

4.4. The Todorčević trichotomy. Let K be a separable Rosenthal compactum and $\{f_n : n \in \mathbb{N}\}$ a countable dense subset of K . A minimal family D_i canonically embeds into K (with respect to $\{f_n\}$) if there is an increasing injection $\iota : 2^{<\mathbb{N}} \rightarrow \mathbb{N}$, for the enumeration of $2^{<\mathbb{N}}$ fixed in [ADK08], such that $\{f_{\iota(t)} : t \in 2^{<\mathbb{N}}\}$ is equivalent to D_i . In that case the closure of D_i is homeomorphic to a closed subspace of K .

Fact 4.3 (Todorčević's trichotomy in canonical form; see [Tod99] and [ADK08, Theorem 3]). Let K be a separable Rosenthal compactum and $\{f_n : n \in \mathbb{N}\}$ a countable dense subset of K .

- (a) If K consists of bounded functions, is metrizable and is not separable in the supremum norm, then the family D_2 canonically embeds into K .
- (b) If K is hereditarily separable but not metrizable, then the family D_3 or the family D_4 canonically embeds into K .
- (c) If K is first countable but not hereditarily separable, then the family D_6 or the family D_7 canonically embeds into K .
- (d) If K is not first countable, then the family D_5 canonically embeds into K .

In particular, a non-metrizable separable Rosenthal compactum contains a copy of $\hat{S}_+(2^{\mathbb{N}})$ or $\hat{S}_-(2^{\mathbb{N}})$, of $\hat{D}(2^{\mathbb{N}})$ or $\hat{D}(S(2^{\mathbb{N}}))$, or of $\hat{A}(2^{\mathbb{N}})$, according as it is hereditarily separable, first countable but not hereditarily separable, or not first countable.

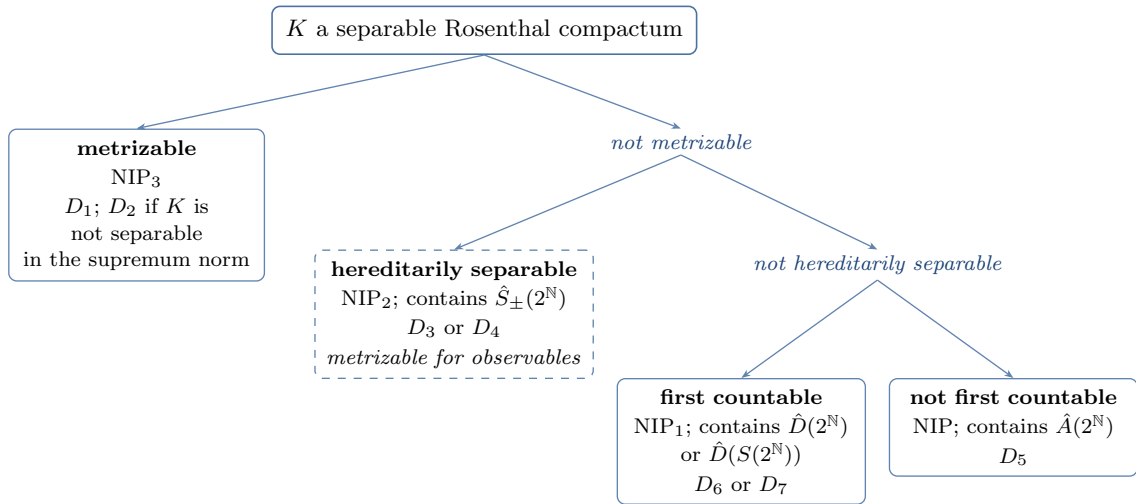


FIGURE 3. The Todorčević trichotomy as a decision tree, with the canonical compacta and the minimal families of the heptachotomy at its leaves (compare [DIMW⁺26, Section 4.2]). For observables the dashed leaf merges with the metrizable one (see Theorem 6.6).

4.5. Levels of NIP. We now transfer the classification to NIP families, following Section 4.2 of [DIMW⁺26].

Definition 4.4. Let Y be a compact metrizable space and $\mathcal{R} \subseteq C_p(Y)$ a pointwise bounded NIP family whose closure $\overline{\mathcal{R}}$ is separable. The family $\mathcal{R}$ is NIP_1 , NIP_2 or NIP_3 if $\overline{\mathcal{R}}$ is, respectively, first countable, hereditarily separable or metrizable.

By the implications among the three invariants, we have $NIP_3 \Rightarrow NIP_2 \Rightarrow NIP_1 \Rightarrow NIP$.

The top level has a concrete meaning. A set $D \subseteq Y$ *separates* a set K of functions on Y if any two distinct members of K differ at some point of D .

Proposition 4.5. *Let Y be a set and $K \subseteq \mathbb{R}^Y$ a set that is compact in the topology of pointwise convergence. Then the space K is metrizable if and only if some countable set $D \subseteq Y$ separates K .*

Proof. Suppose that a countable set D separates K . Restriction to D is a continuous injection of K into $\mathbb{R}^D$, which is metrizable because D is countable. A continuous injection of a compact space into a Hausdorff space is a homeomorphism onto its image, so the space K is metrizable.

Conversely, suppose that K is metrizable. Being compact, it is second countable, and so is $K \times K$. Hence the open set $W = \{(f, g) \in K \times K : f \neq g\}$ is Lindelöf. For $y \in Y$ the set $U_y = \{(f, g) \in K \times K : f(y) \neq g(y)\}$ is open, and these sets cover W . If $\{U_y : y \in D\}$ is a countable subcover, then the set D separates K . $\square$

Thus a NIP family with separable closure is NIP_3 exactly when countably many points of Y distinguish all functions in its closure. When the functions are responses and the points are probes, this says that countably many measurements tell all idealized preparations apart. The other two levels have readings of the same kind.

Proposition 4.6. *Let Y be a compact metrizable space and $\mathcal{R} \subseteq C_p(Y)$ a pointwise bounded NIP family whose closure $K = \overline{\mathcal{R}}$ is separable.*

- (a) *The family is NIP_3 if and only if one countable set $D \subseteq Y$ separates K .*
- (b) *The family is NIP_1 if and only if every $f \in K$ is countably determined: there is a countable set $D_f \subseteq Y$ such that f is the only member of K that agrees with f on D_f .*
- (c) *The family is NIP_2 if and only if there is no set $E \subseteq K$ of cardinality $2^{\aleph_0}$ whose members are finitely separated: for each $f \in E$ there are a finite set $F_f \subseteq Y$ and $\varepsilon_f > 0$ such that every $g \in E \setminus \{f\}$ satisfies $|g(y) - f(y)| \geq \varepsilon_f$ for some $y \in F_f$.*

Proof. (a) is Proposition 4.5. (b) If f is countably determined, then the set $\{f\}$ is the intersection of the countably many open sets $\{g \in K : |g(y) - f(y)| < \frac{1}{m}\}$, for $y \in D_f$ and $m \geq 1$, and in a compact Hausdorff space a point that is a countable intersection of open sets has a countable base of neighborhoods. Conversely, if (U_k) is a countable base at f and each U_k contains a basic neighborhood defined by a finite set $F_k \subseteq Y$, then the set $D_f = \bigcup_k F_k$ works: a member of K that agrees with f on D_f lies in every U_k . (c) A set is finitely separated exactly when it is a discrete subspace, and a separable Rosenthal compactum is hereditarily separable if and only if it has no discrete subspace of cardinality $2^{\aleph_0}$, by a theorem of Todorćević [Tod99] (see also [GM22, Theorem 0.4]). $\square$

The three levels thus measure how many points of Y are needed to determine a limit: one countable set for all of them (NIP_3), a countable set for each of them (NIP_1), or no continuum of limits that finitely many points tell apart one by one (NIP_2). When limits are taken as types and points as parameters, the levels say how many instances of a formula determine a type. This gives a partial answer to the question of [DIMW⁺26] asking for a non-topological characterization of the levels; a finitary characterization, in the spirit of the fat-shattering dimension for NIP, remains open (see Question (5)).

Theorem 4.7. *Let Y be a compact metrizable space and $\{r_n : n \in \mathbb{N}\} \subseteq C_p(Y)$ a pointwise bounded NIP family. Then exactly one of the following holds:*

- (i) *the family is NIP_3 ;*
- (ii) *the family is NIP_2 but not NIP_3 , and there is an increasing injection $\iota : 2^{<\mathbb{N}} \rightarrow \mathbb{N}$ such that $\{r_{\iota(t)}\}$ is equivalent to D_3 or to D_4 ;*
- (iii) *the family is NIP_1 but not NIP_2 , and there is an increasing injection ι such that $\{r_{\iota(t)}\}$ is equivalent to D_6 or to D_7 ;*
- (iv) *the family is not NIP_1 , and there is an increasing injection ι such that $\{r_{\iota(t)}\}$ is equivalent to D_5 .*

The following proof merely combines the facts above.

Proof. By Theorem 3.6 and Lemma 3.2, the closure K of the family is a Rosenthal compactum, and the family is a countable dense subset of it. The four cases are mutually exclusive by the definitions and cover all possibilities. In cases (ii)–(iv), apply Fact 4.3, parts (b)–(d), to K and the dense subset $\{r_n\}$. $\square$

Thus a non-metrizable NIP closure is always explained by a subfamily of the original functions, arranged along the dyadic tree and equivalent to one of five canonical families. Applied to realized types, Theorem 4.7 yields the classification of NIP formulas described in the introduction (see [DIMW⁺26, Section 4.2]). There the realized types are viewed as continuous functions on a compact metrizable space of parameters, and this proviso matters: the space of types of a formula over a countable set of parameters, with no topology on the parameters, is always metrizable, since the countable set of parameters separates the types (see Proposition 4.5).

4.6. Historical remarks. Todorćević proved his trichotomy in 1999 [Tod99], using forcing and absoluteness; Argyros, Dodos and Kanellopoulos extracted the canonical form and the heptachotomy from the methods of his paper in 2008 [ADK08], using Ramsey theory for trees [HL66, Mil81, LSV93]. Glasner and Megrelishvili used the trichotomy to build a hierarchy of tame dynamical systems [GM22], and the levels NIP_1 , NIP_2 and NIP_3 were introduced for limits of computations in [DIMW⁺26].

The hierarchy obtained in this paper differs from these. In the settings of Glasner and Megrelishvili [GM22] and of [DIMW⁺26], the Todorćević trichotomy is applied to families of functions that carry no linear structure, such as enveloping semigroups or sets of computations, and all its classes can occur. The idealized responses of an observable, by contrast, form a convex set (see the proof of Corollary 6.8), and for convex compacta hereditary separability implies metrizability, a special case of a known result (see Remark 6.7): for observables, NIP_2 coincides with NIP_3 (see Theorem 6.6 and Corollary 6.8), and the hierarchy has three levels. At the level of families indexed by the dyadic tree all seven minimal families occur (see Theorem 8.5). The difference is a feature of quantum, or more generally of probabilistic, response spaces.

5. THE IDEALIZED RESPONSES OF AN OBSERVABLE FORM A COMPACT SET OF FUNCTIONS

The goal of this section is to place the responses of an observable in the setting of Section 3: we attach to each bipartite observable a compact metrizable space of probes, on which Alice's responses are continuous, and we identify the pointwise closure of these responses with the responses of Alice's idealized preparations.

5.1. Observables and responses. We recall the setting of [DI26], with the operator theory explained briefly. Alice's and Bob's systems are described by separable complex Hilbert spaces H_1 and H_2 . A *density operator* on a Hilbert space H is a positive trace-class operator of trace 1; density operators are the *actual states* of the system, and we write $\mathfrak{D}(H)$ for the set of them. The trace-class operators, those bounded operators x with $\text{Tr}|x| < \infty$, where $|x| = (x^*x)^{1/2}$, form a Banach space $\mathcal{T}(H)$ under the trace norm $\|x\|_1 = \text{Tr}|x|$, and we write $B(H)$ for the algebra of bounded operators on H . A bounded self-adjoint operator C on $H_1 \otimes H_2$ is an *observable*, and its expectation when the two systems are prepared independently in states ρ and σ is

$$\varphi_C(\rho, \sigma) = \text{Tr}((\rho \otimes \sigma)C).$$

Fixing Alice's state ρ produces an observable on Bob's system alone, the *effective observable* $T_C(\rho) = \text{Tr}_1((\rho \otimes I)C)$, obtained by tracing out Alice's system. It is characterized by $\varphi_C(\rho, \sigma) = \text{Tr}(\sigma T_C(\rho))$ for all $\sigma \in \mathfrak{D}(H_2)$, and it extends linearly to a bounded operator $T_C: \mathcal{T}(H_1) \rightarrow B(H_2)$ with $\|T_C\| \leq \|C\|$, the *slice map*.

Idealized states are described through C*-algebras. A C*-algebra is a norm-closed algebra of bounded operators closed under adjoints; we only use unital ones, which contain the identity. A *state* of a unital C*-algebra $\mathfrak{A}$ is a linear functional ω on $\mathfrak{A}$ that is positive, in the sense that $\omega(a^*a) \geq 0$ for all a , and satisfies $\omega(I) = 1$. The set $\mathcal{S}(\mathfrak{A})$ of states, with the weak* topology (pointwise convergence on $\mathfrak{A}$), is compact; it is metrizable when $\mathfrak{A}$ is norm-separable. Every density operator ρ defines a state $a \mapsto \text{Tr}(\rho a)$; the other states of $B(H)$ are the *idealized states*, limits of actual states that no density operator represents.

5.2. The space of probes. The responses of Alice's preparations only involve Bob's system through the effective observables $T_C(\rho)$. The natural probes are therefore the states of the smallest C*-algebra containing them.

Definition 5.1. Let C be an observable on $H_1 \otimes H_2$. The *probe algebra* $\mathfrak{A}_C \subseteq B(H_2)$ is the unital C*-algebra generated by $T_C(\mathcal{T}(H_1))$, and the *probe space* is $Y_C = \mathcal{S}(\mathfrak{A}_C)$. For $\rho \in \mathfrak{D}(H_1)$, the *response* of ρ is the function $r_\rho: Y_C \rightarrow \mathbb{R}$ given by $r_\rho(\tau) = \tau(T_C(\rho))$, and the *response family* is $\mathcal{R}_C = \{r_\rho: \rho \in \mathfrak{D}(H_1)\}$.

Lemma 5.2. Let C be an observable on $H_1 \otimes H_2$.

- (a) The probe space Y_C is compact and metrizable.
- (b) Every response is continuous on Y_C and bounded by $\|C\|$, so the response family $\mathcal{R}_C$ is a pointwise bounded subset of $C_b(Y_C)$.
- (c) The restrictions to $\mathfrak{A}_C$ of the actual states of Bob's system form a dense subset of Y_C , and for $\sigma \in \mathfrak{D}(H_2)$ we have $r_\rho(\sigma|_{\mathfrak{A}_C}) = \varphi_C(\rho, \sigma)$.

Proof. (a) Since H_1 is separable, the space $\mathcal{T}(H_1)$ is separable, and since the slice map is norm continuous, its image is separable, and hence so is the C*-algebra $\mathfrak{A}_C$ that the image generates. Hence the compact space Y_C is metrizable.

(b) The response r_ρ is evaluation at the element $T_C(\rho)$ of $\mathfrak{A}_C$, which is weak* continuous. Its values are real, because $T_C(\rho)$ is self-adjoint when $\rho \geq 0$ and C is self-adjoint, and $|r_\rho(\tau)| \leq \|T_C(\rho)\| \leq \|C\|$.

(c) Every state of $\mathfrak{A}_C$ extends to a state of $B(H_2)$, by the Hahn–Banach theorem (see [KR83, Theorem 4.3.13]), and the states given by density operators are weak* dense in the state space of $B(H_2)$ (see [DI26, Lemma 2.2]). Restriction to $\mathfrak{A}_C$ is weak* continuous and onto, so it maps a dense set onto a dense set. The formula is the definition of the effective observable. $\square$

By Lemma 5.2(c) and Lemma 3.7, the independence property of $\mathcal{R}_C$ is decided by Bob's actual probes alone. In terms of actual states, the response family has the independence property exactly when there are actual states $\rho_1, \rho_2, \dots$ of Alice's system and reals $a < b$ such that, for any finite disjoint $E, F \subseteq \mathbb{N}$, a single actual probe σ of Bob's system gives $\varphi_C(\rho_n, \sigma) \leq a$ for $n \in E$ and $\varphi_C(\rho_n, \sigma) \geq b$ for $n \in F$. Bob can then read any finite pattern off Alice's sequence with one probe.

Definition 5.3. An observable C on $H_1 \otimes H_2$ is *NIP for Alice's responses* if the response family $\mathcal{R}_C$ is NIP.

5.3. Idealized responses. To describe the closure of $\mathcal{R}_C$, we use the bitranspose of the slice map, which extends it from actual to idealized states. Every bounded operator $T: E \rightarrow F$ between Banach spaces has a *transpose* $T^*: F^* \rightarrow E^*$, given by $T^*(\lambda) = \lambda \circ T$, and a *bitranspose* $T^{**}: E^{**} \rightarrow F^{**}$, which extends T and is continuous for the weak* topologies. Here we regard the slice map as a map $T_C: \mathcal{T}(H_1) \rightarrow \mathfrak{A}_C$. The dual of $\mathcal{T}(H_1)$ is $B(H_1)$, so the bidual of $\mathcal{T}(H_1)$ is $B(H_1)^*$, which contains all states of $B(H_1)$, idealized ones included.

For an observable C on $H_1 \otimes H_2$ and a state ω of $B(H_1)$, let $r_\omega: Y_C \rightarrow \mathbb{R}$ be given by $r_\omega(\tau) = \langle T_C^{**}(\omega), \tau \rangle$.

Lemma 5.4. *Let C be an observable on $H_1 \otimes H_2$. Then we have*

$$\overline{\mathcal{R}_C} = \{r_\omega : \omega \in \mathcal{S}(B(H_1))\}.$$

Moreover, if $\tau \in Y_C$ and $\tilde{\tau}$ is a state of $B(H_2)$ extending τ , then we have $r_\omega(\tau) = (\omega \otimes_{\mathbb{B}} \tilde{\tau})(C)$.

Proof. The map $\omega \mapsto T_C^{**}(\omega)$ is weak* continuous. On bounded sets of $\mathfrak{A}_C^{**}$, the weak* topology is the topology of pointwise convergence on Y_C , because states span $\mathfrak{A}_C^*$. Hence the map $\omega \mapsto r_\omega$ is a continuous map from the compact space $\mathcal{S}(B(H_1))$ to $\mathbb{R}^{Y_C}$, and its image is compact. For an actual state ρ , we have $T_C^{**}(\rho) = T_C(\rho)$; hence the image contains $\mathcal{R}_C$, and, being closed, it contains $\overline{\mathcal{R}_C}$. Conversely, actual states are dense in $\mathcal{S}(B(H_1))$ (see [DI26, Lemma 2.2]), so by continuity the image lies in $\overline{\mathcal{R}_C}$. The last formula is [DI26, Lemma 2.4]; the pairing $\langle T_C^{**}(\omega), \tilde{\tau} \rangle$ depends only on the restriction of $\tilde{\tau}$ to $\mathfrak{A}_C$. $\square$

The functions r_ω are the *idealized responses*: the responses of Alice's idealized preparations to Bob's probes, idealized probes included, with Bob's system idealized first. On actual probes the order does not matter (see [DI26, Corollary 2.5]), and the idealized response satisfies $r_\omega(\sigma|_{\mathfrak{A}_C}) = (\omega \otimes_{\mathbb{B}} \sigma)(C) = (\omega \otimes_{\mathbb{A}} \sigma)(C)$.

6. STABILITY, NIP AND THE INDEPENDENCE PROPERTY ARE CONTINUITY, BAIRE CLASS 1 AND $\beta\mathbb{N}$

The goal of this section is the main result: the three levels of tameness of an observable are three levels of regularity of its idealized responses. In Theorem 6.1 below, we show that stability corresponds to continuity, through weak compactness, and NIP corresponds to the first Baire class, through Theorem 3.6.

For stability we need one more fact from operator theory, Kadison's function representation: on the state space of a unital C^* -algebra $\mathfrak{A}$, the continuous affine real functions are exactly the evaluations $\tau \mapsto \tau(a)$ at self-adjoint elements $a \in \mathfrak{A}$ (see [Kad51]).

Theorem 6.1. *Let C be an observable on $H_1 \otimes H_2$.*

- (a) *The observable C is stable if and only if every idealized response is continuous, that is, $\overline{\mathcal{R}_C} \subseteq C_p(Y_C)$.*
- (b) *The observable C is NIP for Alice's responses if and only if every idealized response is Baire-1, that is, $\overline{\mathcal{R}_C} \subseteq B_1(Y_C)$. In that case, the set $\overline{\mathcal{R}_C}$ of idealized responses is a Rosenthal compactum.*
- (c) *If the observable C is not NIP for Alice's responses, then some sequence of responses of actual preparations of Alice's system has a subsequence whose closure is homeomorphic to $\beta\mathbb{N}$.*

Proof. By Lemma 5.2, Theorem 3.6 applies to the family $\mathcal{R}_C \subseteq C_p(Y_C)$, and part (b) follows from it together with Lemma 3.2. For part (c), let (r_n) be an independent sequence in $\mathcal{R}_C$, with reals $a < b$. Every subsequence of (r_n) is independent, and, by compactness of Y_C (see the proof of Theorem 3.6), for every $I \subseteq \mathbb{N}$ there is a probe at which $r_n \leq a$ for $n \in I$ and $r_n \geq b$ for $n \notin I$. Hence no subsequence of (r_n) converges pointwise, and Fact 3.4 gives a subsequence whose closure is homeomorphic to $\beta\mathbb{N}$.

For part (a), we use the characterization of stability by weak compactness (see [DI26, Theorem 5.1]). Recall that a bounded operator between Banach spaces is *weakly compact* if it maps bounded sets to relatively weakly compact sets, and that, by Gantmacher's theorem, this holds exactly when its bitranspose takes values in the original target space. Suppose first that C is stable. Then the slice map is weakly compact as a map into $B(H_2)$, and hence also as a map into the closed subspace $\mathfrak{A}_C$, whose weak topology is the restriction of that of $B(H_2)$. It follows that the bitranspose T_C^{**} maps $B(H_1)^*$ into $\mathfrak{A}_C$, and every idealized response r_ω is evaluation at an element of $\mathfrak{A}_C$, hence continuous on Y_C .

Conversely, suppose that every idealized response is continuous, and let ω be a state of $B(H_1)$. The element $T_C^{**}(\omega)$ of $\mathfrak{A}_C^{**}$ is self-adjoint, so the idealized response r_ω is a continuous affine real function on Y_C . By Kadison's function representation, there is a self-adjoint $a \in \mathfrak{A}_C$ with $r_\omega(\tau) = \tau(a)$ for all $\tau \in Y_C$. The elements $T_C^{**}(\omega)$ and a of $\mathfrak{A}_C^{**}$ agree on all states, and states span $\mathfrak{A}_C^*$, so we have $T_C^{**}(\omega) = a \in B(H_2)$. Since this holds for every state ω , the observable C is stable (see [DI26, Theorem 5.1]). $\square$

In terms of preparations and probes, Theorem 6.1 describes how the responses of Alice's preparations depend on Bob's probe at each level:

- for a stable observable, the response of each idealized preparation of Alice's system depends continuously on Bob's probe, idealized probes included;
- for an observable that is NIP but unstable, such as the comparison of photon numbers (see Section 7), these responses can be discontinuous, but each is a pointwise limit of a sequence of continuous functions, and together they form a Rosenthal compactum;
- for an observable with the independence property, a sequence of Alice's actual preparations lets Bob read every finite pattern with a single probe, and by Fact 3.4, any such sequence without a convergent subsequence has a subsequence whose closure is a copy of $\beta\mathbb{N}$.

Remark 6.2. In [DI26], stability (no infinite ladder) is distinguished from uniform stability (bounded ladders at every margin, equivalently stability on ultrapowers). The same distinction appears here. Definition 3.3 forbids an infinite independent sequence, whereas *uniform NIP* bounds, at each margin, the size of finite independent sets, that is, the fat-shattering dimension. Uniform NIP implies NIP. Comparisons $[A < B]$ of two quantities are uniformly

NIP, with bounds independent of the Hilbert spaces (see Corollary 7.4), and they are unstable when both quantities have discrete spectra unbounded above, by the ladder argument of Section 7. By Theorem 6.1, their idealized responses form a Rosenthal compactum of Baire-1 functions, not all continuous. As in Part I, the uniform notion is the one that survives ultrapowers: a family is uniformly NIP if and only if its extension to every ultrapower is NIP. Here the extension of a family to an ultrapower is the family of ultralimits of sequences of responses, evaluated at ultralimits of sequences of probes, and a finite set of responses is δ -shattered if it is independent, in the sense of Definition 3.3, with a gap $b - a \geq \delta$. By Łoś's theorem, a δ -shattered set of size n in an ultrapower gives, for every $\delta' < \delta$, a δ' -shattered set of size n in the original family. Hence, if the original family has finite fat-shattering dimension at every margin, so does its extension, which then has no infinite independent sequence; conversely, if the fat-shattering dimension at some margin is infinite, then in an ultrapower by a countably incomplete ultrafilter the finite shattered sets combine, by $\aleph_1$ -saturation, into an infinite independent sequence.

6.1. The finer classification of NIP observables. For NIP observables, the classification of Section 4 applies to the idealized responses. We first check that they form a separable compactum.

Lemma 6.3. *Let C be an observable on $H_1 \otimes H_2$ and (ρ_n) a sequence of density operators that is dense in $\mathfrak{D}(H_1)$ for the trace norm. Then the responses r_{ρ_n} are dense in $\overline{\mathcal{R}_C}$.*

Proof. For density operators ρ and ρ' , we have $\sup_{\tau \in Y_C} |r_\rho(\tau) - r_{\rho'}(\tau)| \leq \|T_C(\rho - \rho')\| \leq \|C\| \|\rho - \rho'\|_1$. Hence every response is a uniform, and so a pointwise, limit of responses r_{ρ_n} , and the closure of $\{r_{\rho_n}\}$ contains $\mathcal{R}_C$ and therefore $\overline{\mathcal{R}_C}$. Such a sequence exists because $\mathcal{T}(H_1)$ is separable. $\square$

Definition 6.4. An observable C that is NIP for Alice's responses is NIP_1 , NIP_2 or NIP_3 if the response family $\mathcal{R}_C$ is (see Definition 4.4).

Proposition 6.5. *Every stable observable is NIP_3 : its idealized responses form a compact metrizable space.*

Proof. By Theorem 6.1(a), every idealized response is continuous on the compact metrizable space Y_C , so a countable dense set of probes separates them; apply Proposition 4.5. $\square$

6.2. Mixtures force metrizability. Idealized responses of an observable form a convex set, because responses depend affinely on Alice's state (see the proof of Corollary 6.8), and in Theorem 6.6 below we show that for convex compacta hereditary separability implies metrizability. This is a special case of a known result on locally connected Rosenthal compacta (see Remark 6.7); we give a direct proof. The mechanism is visible in the split interval: if a convex set of functions contains the two halves $\mathbf{1}_{(a,1]}$ and $\mathbf{1}_{[a,1]}$ of a split pair, it contains their midpoint, which takes the value $\frac{1}{2}$ at a . Let us notice that these midpoints form an uncountable discrete set: the midpoint for a point $b \neq a$ takes the value 0 or 1 at a .

Theorem 6.6 (see [Tod99, Theorems 5 and 7], [Gru88, Lemma 2.4] and Remark 6.7). *Let Y be a set and $K \subseteq \mathbb{R}^Y$ a separable compact convex set, with the topology of pointwise convergence, that is a Rosenthal compactum. If K is hereditarily separable, then the set K is metrizable.*

Proof. Suppose that K is hereditarily separable but not metrizable. By Fact 4.3(b), there is a homeomorphism h from $\hat{S}_+(2^{\mathbb{N}})$ or $\hat{S}_-(2^{\mathbb{N}})$ onto a subspace of K . We treat the case

$\hat{S} = \hat{S}_+(2^{\mathbb{N}})$. The other case is symmetric: the map c that exchanges the digits 0 and 1 reverses $\prec$, and the map $f \mapsto 1 - f \circ c$ carries D_4 to a family of the form of D_3 for the tree $c(R)$. Write $Q = \{s \cap 0^\infty : s \in R\}$ for the countable set of points indexing D_3 , and Σ for the closure of Q in $2^{\mathbb{N}}$. The idea is the one described before the theorem: the midpoints of the images under h of the split pairs of $\hat{S}$ form a subspace of K that is not separable.

We first record the structure of $\hat{S}$. If (σ_i) is a strictly increasing sequence in $2^{\mathbb{N}}$ converging to σ , then we have $x_{\sigma_i}^\pm \rightarrow x_\sigma^\pm$ pointwise; if it is strictly decreasing, then we have $x_{\sigma_i}^\pm \rightarrow x_\sigma^\mp$. Indeed, for $\tau \in 2^{\mathbb{N}}$, the value of $x_{\sigma_i}^\pm$ at τ is eventually 1 if $\tau \succ \sigma$, eventually 0 if $\tau \prec \sigma$, and at $\tau = \sigma$ it is eventually 1 in the increasing case and 0 in the decreasing case. Since $\hat{S}$ is Fréchet–Urysohn (see [BFT78, Theorem 3F]) and every sequence of distinct points of $2^{\mathbb{N}}$ has a strictly monotone subsequence, every point of $\hat{S} \setminus D_3$ has the form x_ρ^+ or x_ρ^- with $\rho \in \Sigma$; and if $x_{\sigma_i}^{e_i} \in \hat{S}$ for a strictly monotone sequence (σ_i) , then the limit computed above lies in $\hat{S}$, because $\hat{S}$ is closed. Note also that $x_\sigma^+ = x_\sigma^-$ exactly when σ is the immediate successor of ρ in $\prec$, and that two of these functions with the same sign are equal only when their indices are.

We use the following fact repeatedly: if $A \subseteq 2^{\mathbb{N}}$ is uncountable, then all but countably many points of A are limits of strictly increasing and of strictly decreasing sequences from A . Indeed, if $\sigma \in A$ is not a limit of a strictly increasing sequence from A , there is $\tau_\sigma \prec \sigma$ with no point of A strictly between them, and the intervals $(\tau_\sigma, \sigma]$ for distinct such σ are disjoint; each of them either contains a basic clopen set or makes τ_σ the immediate predecessor of σ , and both happen for only countably many σ . The decreasing case is the same. Let P be the set of $\sigma \in \Sigma$ that are limits of strictly increasing and of strictly decreasing sequences from Q . The set Σ is uncountable, because distinct branches of the dyadic tree R give distinct limits, and all but countably many points of Σ belong to P , by the same argument applied to Q (an open interval that misses Q also misses Σ); hence the set P is uncountable. For $\sigma \in P$, both x_σ^+ and x_σ^- belong to $\hat{S}$, they differ at σ , and neither equals $x_\rho^\pm$ for $\rho \neq \sigma$, because σ has neither an immediate predecessor nor an immediate successor in $\prec$.

The split pairs indexed by P now supply the midpoints. For $\sigma \in P$, put $F_\sigma^\pm = h(x_\sigma^\pm)$ and $M_\sigma = \frac{1}{2}(F_\sigma^+ + F_\sigma^-)$, which lies in K by convexity. Averaging is continuous for the pointwise topology; hence, if (σ_i) is a strictly increasing (respectively, decreasing) sequence in P converging to σ , then we have $M_{\sigma_i} \rightarrow h(x_\sigma^+)$ (respectively, $h(x_\sigma^-)$). The proof that the midpoints form a non-separable subspace takes three steps.

Step 1: each value is taken by countably many midpoints. Let $m \in K$ and suppose that $A = \{\sigma \in P : M_\sigma = m\}$ is uncountable. Choose $\sigma \in A$ that is a limit of strictly increasing and of strictly decreasing sequences from A . Then $m = h(x_\sigma^+)$ and $m = h(x_\sigma^-)$, so we have $x_\sigma^+ = x_\sigma^-$, a contradiction.

Step 2: few midpoints lie on the copy of $\hat{S}$. Let $E = \{\sigma \in P : M_\sigma \in h(\hat{S})\}$. By Step 1 and the countability of D_3 , all but countably many $\sigma \in E$ satisfy $M_\sigma = h(x_{\rho(\sigma)}^{e(\sigma)})$ for some sign $e(\sigma)$ and some $\rho(\sigma) \in \Sigma$, and $\rho(\sigma) \neq \sigma$, since a midpoint of two distinct points differs from both. Suppose that uncountably many such σ exist. Fix a metric d on $2^{\mathbb{N}}$; then for some n the set E_n of these σ with $d(\sigma, \rho(\sigma)) \geq 1/n$ is uncountable. Choose $\sigma \in E_n$ and a strictly monotone sequence (σ_i) in E_n converging to σ , say increasing. Passing to a subsequence, we may assume that the signs $e(\sigma_i)$ are constant and that the points $\rho(\sigma_i)$ are either constant or strictly monotone, converging to some ρ^* . Then the midpoints M_{σ_i} converge both to $h(x_\sigma^+)$ and to $h(x_{\rho^*}^\pm)$. Since h is injective, we have $x_\sigma^+ = x_{\rho^*}^\pm$, and since $\sigma \in P$ has no immediate

predecessor, we have $\rho^* = \sigma$. Hence we have $d(\sigma_i, \rho(\sigma_i)) \rightarrow 0$, contradicting $\sigma_i \in E_n$. Consequently, the set E is countable.

Step 3: the midpoints form a non-separable subspace. Let $P' = P \setminus E$, which is uncountable, and $L = \{M_\sigma : \sigma \in P'\}$. Let $D \subseteq L$ be countable. By Step 1, there is $\sigma \in P'$ with $M_\sigma \notin D$. If M_σ were in the closure of D , then, since Rosenthal compacta are Fréchet–Urysohn (see [BFT78, Theorem 3F]), there would be a sequence (σ_i) in P' with $M_{\sigma_i} \in D$ and $M_{\sigma_i} \rightarrow M_\sigma$. As $M_\sigma \notin D$, no point occurs infinitely often in (σ_i) , so a subsequence is strictly monotone and converges to some $\rho \in \Sigma$, and then the midpoints M_{σ_i} converge to $h(x_\rho^+)$ or $h(x_\rho^-)$. Thus we have $M_\sigma \in h(\hat{S})$, contradicting $\sigma \notin E$. Hence the subspace L is not separable, and the space K is not hereditarily separable, a contradiction. $\square$

Remark 6.7. The proof uses convexity only to know that midpoints of points of K lie in K . The same midpoints settle a question about the split interval $S = [0, 1] \times \{0, 1\}$, ordered lexicographically, with the order topology. In a remark following Proposition 10 of [God80], the space $P(S)$ of probability measures on S is stated to be hereditarily separable. Let us notice that this is not the case: for $0 < a < 1$, let $a^- = (a, 0)$ and $a^+ = (a, 1)$, and let m_a be the midpoint $\frac{1}{2}(\delta_{a^-} + \delta_{a^+})$. The set $\{s \in S : s \leq a^-\}$ is clopen, so its indicator f_a is continuous, and $m_a(f_a) = \frac{1}{2}$, while $m_b(f_a) \in \{0, 1\}$ for $b \neq a$. Hence the set $\{m_a : 0 < a < 1\}$ is an uncountable discrete subspace of $P(S)$, exactly as in the proof of Theorem 6.6. Todorćević has pointed out to us that Theorem 6.6 is known, and that it holds in the larger class of locally connected Rosenthal compacta, which contains the convex ones: it follows from Theorems 5 and 7 of [Tod99] and an argument of Gruenhage, Lemma 2.4 of [Gru88]. The proof above is direct and uses only the midpoints, the same midpoints that refute the remark of [God80]. The theorem answers, for convex compacta, the question of which classes of the Todorćević trichotomy occur, in the spirit of the hierarchy of tame systems of Glasner and Megrelishvili [GM22].

Corollary 6.8. *Every observable that is NIP_2 is NIP_3 .*

Proof. The set $\overline{\mathcal{R}_C}$ of idealized responses is convex. Indeed, we have $r_{t\rho+(1-t)\rho'} = tr_\rho + (1-t)r_{\rho'}$ for $0 \leq t \leq 1$, because the slice map and evaluation at a probe are linear, and the set $\mathfrak{D}(H_1)$ is convex; hence the family $\mathcal{R}_C$ is convex. The map $(f, g) \mapsto tf + (1-t)g$ is continuous on $\mathbb{R}^{Y_C} \times \mathbb{R}^{Y_C}$ for the pointwise topology and sends $\mathcal{R}_C \times \mathcal{R}_C$ into $\mathcal{R}_C$; hence it sends the closure $\overline{\mathcal{R}_C} \times \overline{\mathcal{R}_C}$ into $\overline{\mathcal{R}_C}$. The set $\overline{\mathcal{R}_C}$ is also separable, by Lemma 6.3, and a Rosenthal compactum, by Theorem 6.1(b). It follows from Theorem 6.6 that, if the set $\overline{\mathcal{R}_C}$ is hereditarily separable, then it is metrizable. $\square$

For observables, then, the Todorćević trichotomy has three classes. For families of functions it has four: the family D_3 consists of continuous functions on the Cantor set, since each set $\{\tau : \tau \succeq s \cap 0^\infty\}$ is clopen, and its closure $\hat{S}_+(2^\mathbb{N})$ is hereditarily separable but not metrizable, so the family D_3 is NIP_2 but not NIP_3 . This example is the order of the real line with real parameters, whose realized types take only the values 0 and 1. Expectations introduce mixtures, and mixtures of the two halves of each split pair turn the split interval into a space with an uncountable discrete subset; for the order of positions, the resulting space is the Helly space (see Section 7). Expectations average over Alice’s mixed preparations, and for these averages hereditary separability forces metrizability (see Theorem 6.6).

Theorem 6.9. *Let C be an observable on $H_1 \otimes H_2$ that is NIP for Alice’s responses, and let (ρ_n) be a trace-norm dense sequence in $\mathfrak{D}(H_1)$. Then exactly one of the following holds:*

- (i) *the observable C is NIP_3 ;*
- (ii) *the observable C is NIP_1 but not NIP_2 , and there is an increasing injection $\iota: 2^{<\mathbb{N}} \rightarrow \mathbb{N}$ such that the responses $\{r_{\rho_{\iota(t)}} : t \in 2^{<\mathbb{N}}\}$ are equivalent to D_6 or to D_7 ;*
- (iii) *the observable C is not NIP_1 , and there is an increasing injection ι such that these responses are equivalent to D_5 .*

Proof. By Lemma 5.2, the family $\{r_{\rho_n}\}$ is a pointwise bounded family of continuous functions on the compact metrizable space Y_C , and it is NIP because it is contained in $\mathcal{R}_C$. By Lemma 6.3, its closure is $\overline{\mathcal{R}_C}$. Apply Theorem 4.7; its case (ii) does not occur, by Corollary 6.8. $\square$

Physically, when the idealized responses of a NIP observable do not form a metrizable space, the reason is always visible in Alice’s actual preparations: a family of them, arranged along the dyadic tree, has idealized limits that reproduce one of three canonical compacta. The next section locates the comparisons $[A < B]$ in this classification.

6.3. Historical remarks. Kadison proved the function representation used in Theorem 6.1 in 1951 [Kad51]; the NIP version of Grothendieck’s picture [Gro52, DI26], with Baire-1 functions in place of continuous ones, follows [DIMW⁺26].

7. THREE PHYSICAL ORDERS REALIZE THE THREE TODORČEVIĆ CLASSES

The goal of this section is to show that each of the three Todorčević classes of Theorem 6.9 contains a physically natural observable. All three are orders, and we show that they are unstable and NIP. In Theorem 7.8 we show that the comparison of two discrete quantities, such as photon numbers, has a metrizable compactum of idealized responses, so it is NIP_3 . In Theorem 7.8 and Proposition 7.9 we show that the comparison of two continuous quantities, such as the positions of two particles in a box or two quadratures of light, has the Helly space as its compactum of idealized responses, so it is NIP_1 but not NIP_2 . In Theorems 7.10 and 7.12 we show that the causal order of spacetime, “Bob’s detection lies in the causal future of Alice’s event”, is NIP but not NIP_1 , and so is the comparison of two phases, “Bob’s angle is ahead of Alice’s by less than half a turn”: on a line, monotone responses cannot hide a jump, but on a circle the jumps at the two ends of a half-turn can cancel.

7.1. Relations between classical quantities. All three orders relate a classical quantity measured by Alice to one measured by Bob, and their responses can be computed from classical probability distributions. Let (X_1, m_1) and (X_2, m_2) be σ -finite measure spaces such that the Hilbert spaces $H_j = L^2(X_j, m_j)$ are separable, let $\alpha: X_1 \rightarrow \mathbb{R}^p$ and $\beta: X_2 \rightarrow \mathbb{R}$ be measurable, and let $R \subseteq \mathbb{R}^p \times \mathbb{R}$ be a Borel set. The *relation observable* C_R is multiplication on $L^2(X_1 \times X_2, m_1 \otimes m_2) = H_1 \otimes H_2$ by the indicator of $\{(s, t) : (\alpha(s), \beta(t)) \in R\}$; it represents the event “Alice’s value and Bob’s value are related by R ”. For a state ρ of Alice’s system, the *distribution* of Alice’s value in ρ is the Borel probability measure $\mu_\rho(U) = \text{Tr}(\rho M_{1_{\alpha^{-1}(U)}})$ on $\mathbb{R}^p$, where M_f denotes multiplication by f , and the distribution μ_σ of Bob’s value in a state σ of Bob’s system is defined in the same way. For $y \in \mathbb{R}$, let $R^y = \{x \in \mathbb{R}^p : (x, y) \in R\}$.

By the spectral theorem, every self-adjoint operator on a separable Hilbert space is unitarily equivalent to multiplication by a real measurable function on a space $L^2(X, m)$ of this kind. Thus, for self-adjoint operators A on H_1 and B on H_2 , realized as multiplication by α and by β , the *comparison* $[A < B]$ is the relation observable C_R for $R = \{(x, y) : x < y\}$; it is the projection $(E_A \otimes E_B)(R)$, where E_A and E_B are the spectral measures, so it does not depend on the realization. When A and B are bounded, the operator $I \otimes B - A \otimes I$ is multiplication by $\beta(t) - \alpha(s)$, and $[A < B]$ is its spectral projection for the interval $(0, \infty)$, or equivalently the spectral projection of $A \otimes I - I \otimes B$ for the interval $(-\infty, 0)$.

Lemma 7.1. *Let C_R be a relation observable. For states ρ of Alice's system and σ of Bob's, we have*

$$\varphi_{C_R}(\rho, \sigma) = (\mu_\rho \otimes \mu_\sigma)(R) = \int g_\rho d\mu_\sigma, \quad g_\rho(y) = \mu_\rho(R^y),$$

and the effective observable $T_{C_R}(\rho)$ is multiplication by $g_\rho \circ \beta$.

Proof. For unit vectors $\psi \in H_1$ and $\chi \in H_2$, with distributions μ_ψ and μ_χ in the corresponding pure states, Tonelli's theorem and a change of variables give

$$\langle \psi \otimes \chi, C_R(\psi \otimes \chi) \rangle = \iint \mathbf{1}_R(\alpha(s), \beta(t)) |\psi(s)|^2 |\chi(t)|^2 dm_1(s) dm_2(t) = (\mu_\psi \otimes \mu_\chi)(R).$$

A density operator is a countable convex combination of pure states, $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, converging in trace norm, and then we have $\mu_\rho = \sum_j p_j \mu_{\psi_j}$. Both sides of the first identity are affine in each variable and preserve such combinations, so the first identity holds for all states; the second is Tonelli's theorem, and g_ρ is measurable by the same theorem. Finally, for every unit vector $\chi \in H_2$,

$$\langle \chi, T_{C_R}(\rho) \chi \rangle = \varphi_{C_R}(\rho, |\chi\rangle\langle\chi|) = \int g_\rho(\beta(t)) |\chi(t)|^2 dm_2(t),$$

and a bounded operator on a complex Hilbert space is determined by these values. $\square$

In Lemma 7.2 and Proposition 7.3 below, we show that when the functions g_ρ are monotone, or differences of monotone functions, the responses cannot shatter large sets. Recall from Remark 6.2 that a finite set of responses is δ -shattered if every pattern on it is realized with reals $a < b$ such that $b - a \geq \delta$.

Lemma 7.2. *Let $\delta > 0$ and $m = \lceil 9/\delta \rceil$. Let $h_1, \dots, h_n: \mathbb{R} \rightarrow \mathbb{R}$ be functions of the form $h_i = u_i - v_i$, where $u_i, v_i: \mathbb{R} \rightarrow [0, 1]$ are nondecreasing, and let $a < b$ be reals with $b - a \geq \delta$. Suppose that for every $S \subseteq \{1, \dots, n\}$ there is a Borel probability measure ν_S on $\mathbb{R}$ with $\int h_i d\nu_S \geq b$ for $i \in S$ and $\int h_i d\nu_S \leq a$ for $i \notin S$. Then we have $2^n \leq (2nm + 1)^m$. In particular, we have $n \leq N(\delta)$, where $N(\delta)$ is the largest integer n satisfying this inequality; it is finite and depends only on δ .*

Proof. We first approximate a measure by m equally weighted points. Let ν be a Borel probability measure on $\mathbb{R}$, with distribution function $F(t) = \nu((-\infty, t])$. For $k = 1, \dots, m$, let $y_k = \min\{t : F(t) \geq k/m\}$; the minimum exists because F is nondecreasing and right continuous, with limits 0 and 1 at $-\infty$ and $+\infty$. Let $\nu' = \frac{1}{m} \sum_{k=1}^m \delta_{y_k}$. Since $y_k \leq t$ exactly when $F(t) \geq k/m$, we have $\nu'((-\infty, t]) = \lfloor mF(t) \rfloor / m$, so $|\nu((-\infty, t]) - \nu'((-\infty, t])| \leq 1/m$ for every t , and letting t increase to a given point, the same bound holds for the half-lines $(-\infty, t)$. Hence $|\nu(J) - \nu'(J)| \leq 1/m$ for every set J of the form $\emptyset, \mathbb{R}, [t, \infty)$ or (t, ∞) . For a nondecreasing $u: \mathbb{R} \rightarrow [0, 1]$, every set $\{u > s\}$ has this form, and $\int u d\nu = \int_0^1 \nu(\{u > s\}) ds$;

therefore

$$(1) \quad \left| \int u \, d\nu - \int u \, d\nu' \right| \leq \frac{1}{m}.$$

Next we discretize the functions. Let $\tilde{u}_i = \lfloor mu_i \rfloor / m$ and $\tilde{v}_i = \lfloor mv_i \rfloor / m$. These functions are nondecreasing, take values in $\{0, \frac{1}{m}, \dots, 1\}$, and differ from u_i and v_i by at most $\frac{1}{m}$. Each of the $2n$ functions $\tilde{u}_i, \tilde{v}_i$ changes value at most m times along $\mathbb{R}$, so the map $Q(y) = (\tilde{u}_1(y), \dots, \tilde{u}_n(y), \tilde{v}_1(y), \dots, \tilde{v}_n(y))$ takes at most $2nm + 1$ values.

For each S , let $y_1^S, \dots, y_m^S$ be the points constructed from ν_S as above, and let $w(S) = (Q(y_1^S), \dots, Q(y_m^S))$. By (1), applied to u_i and to v_i , and by the bound on the discretization,

$$\left| \int h_i \, d\nu_S - \frac{1}{m} \sum_{k=1}^m (\tilde{u}_i(y_k^S) - \tilde{v}_i(y_k^S)) \right| \leq \frac{4}{m} \leq \frac{4\delta}{9},$$

and the average on the left is determined by $w(S)$. If $S \neq S'$, choose i in one of the two sets but not the other, say $i \in S \setminus S'$. Then $\int h_i \, d\nu_S - \int h_i \, d\nu_{S'} \geq \delta > \frac{8\delta}{9}$, so the two averages differ, and $w(S) \neq w(S')$. Thus the map $S \mapsto w(S)$ is injective, and $2^n \leq (2nm + 1)^m$. $\square$

Proposition 7.3. *Let C_R be a relation observable and $c \geq 1$ a constant such that, for every state ρ of Alice's system, $g_\rho = c(u_\rho - v_\rho)$ for two nondecreasing functions $u_\rho, v_\rho: \mathbb{R} \rightarrow [0, 1]$. Then, for every $\delta > 0$, no set of more than $N(\delta/2c)$ responses of Alice's states is δ -shattered by points of the probe space. Hence the observable C_R is uniformly NIP, with bounds that do not depend on the Hilbert spaces, and in particular it is NIP.*

Proof. Suppose that the responses $r_{\rho_1}, \dots, r_{\rho_n}$ are δ -shattered with reals $a < b$, the pattern S being realized by $\tau_S \in Y_C$. Let $a' = a + \delta/4$ and $b' = b - \delta/4$. The set of probes τ with $r_{\rho_i}(\tau) > b'$ for $i \in S$ and $r_{\rho_i}(\tau) < a'$ for $i \notin S$ is open, by Lemma 5.2(b), and contains τ_S , so, by Lemma 5.2(c), it contains the restriction of an actual state σ_S of Bob's system. By Lemma 7.1, the value of r_{ρ_i} there is $\int g_{\rho_i} \, d\mu_{\sigma_S}$, and Lemma 7.2, applied with $h_i = g_{\rho_i}/c = u_{\rho_i} - v_{\rho_i}$, $\nu_S = \mu_{\sigma_S}$ and the reals $a'/c < b'/c$, gives $n \leq N(\delta/2c)$. An infinite independent sequence with reals $a < b$ would contain $(b - a)$ -shattered sets of every finite size, so the observable C_R is NIP. $\square$

Corollary 7.4. *For all self-adjoint operators A on H_1 and B on H_2 , the comparison $[A < B]$ is uniformly NIP, with bounds that depend only on the margin, and in particular it is NIP.*

Proof. Here $R^y = (-\infty, y)$, so the function $g_\rho(y) = \mu_\rho((-\infty, y))$ is nondecreasing, with values in $[0, 1]$, and $g_\rho = g_\rho - 0$ is the difference of the nondecreasing function g_ρ and the constant function 0. Apply Proposition 7.3 with $c = 1$. $\square$

Remark 7.5 (Commuting observables and the general framework). In the orders of this section, as in the label-controlled observables of Section 8, the probe algebra is commutative, and the responses are integrals against classical probability distributions. Let us notice that this does not reduce the paper to classical probability, for four reasons: First, the definitions of Section 5 and Theorems 6.1, 6.9 and 6.6 are proved for every observable on $H_1 \otimes H_2$, with no commutativity assumption, and they apply equally to observables such as the swap test, whose probe algebra contains all compact operators. The examples show that every class of the classification is inhabited, and that commuting observables already suffice is information in itself, just as in model theory the dividing lines are first exhibited by the simplest structures, such as the random graph and the dense linear order. Second, the framework imposes restrictions that hold for all observables: the idealized responses always form a compact convex set, because they are the affine image of the state space of $B(H_1)$

(see Lemma 5.4), and by Corollary 6.8 convexity alone removes a class of the Todorčević trichotomy. Third, noncommutativity enters where the physics requires it: in Section 9 the responses are magnetizations of the Ising model in a transverse field, a Hamiltonian whose terms do not commute, and Theorem 9.6 shows that the classification detects its phase transition. Finally, whenever an observable is a relation between commuting quantities measured by Alice and by Bob, Lemma 7.1 reduces its classification to classical probability, and Proposition 7.3 shows how that reduction yields bounds that do not depend on the Hilbert spaces. Whether noncommutative probe algebras produce phenomena that commuting observables cannot is Question (3).

7.2. Order observables. Let $\Xi \subseteq [0, 1]$ be a compact set containing 0 and 1. We consider two cases:

- *discrete*: the set Ξ is $\{1 - \frac{1}{l+1} : l \in \mathbb{N}\} \cup \{1\}$, the Hilbert spaces are $H_1 = H_2 = \ell^2(\mathbb{N})$, and the operator $A = B$ is the diagonal operator with eigenvalue $1 - \frac{1}{l+1}$ on the l -th basis vector; this operator is a bounded relabeling of the number operator, and the order of its eigenvalues is that of the photon numbers;
- *continuous*: the set Ξ is $[0, 1]$, the Hilbert spaces are $H_1 = H_2 = L^2[0, 1]$, and the operator $A = B$ is multiplication by x , the position of a particle confined to the interval $[0, 1]$.

In both cases, let $C = [A < B]$ be the spectral projection of $I \otimes B - A \otimes I$ for the interval $(0, \infty)$, which represents the event “Bob’s value exceeds Alice’s”. For a state ρ , let μ_ρ be the distribution of A in ρ , a Borel probability measure on Ξ . In the discrete case every probability measure on $\Xi \setminus \{1\}$ occurs, and in the continuous case the measures that occur are exactly the absolutely continuous ones (for a probability density p , take the pure state given by the unit vector $\psi = \sqrt{p}$). By Lemma 7.1,

$$\varphi_C(\rho, \sigma) = (\mu_\rho \otimes \mu_\sigma)(\{(x, y) : x < y\}) = \int g_\rho d\mu_\sigma, \quad g_\rho(y) = \mu_\rho([0, y)).$$

The function $g_\rho : \Xi \rightarrow [0, 1]$ is nondecreasing, with $g_\rho(0) = 0$ and $g_\rho(1) = 1$ in both cases, since μ_ρ gives no mass to the point 1; in the continuous case it is also continuous. Accordingly, the effective observable is $T_C(\rho) = g_\rho(B)$. By Corollary 7.4, the observable C is uniformly NIP, and in particular NIP, and it is unstable by the interleaving argument of [DI26, Proposition 4.6(3)], which uses only the order of the eigenvalues, in the discrete case, and by the same ladder argument in the continuous case, with Alice’s positions a_k and Bob’s positions b_l interleaved, $a_1 < b_1 < a_2 < b_2 < \dots$.

7.3. The probe space and the idealized responses. The functions g_ρ are nondecreasing, and in Proposition 7.7 below, we show that their pointwise limits describe all idealized responses. Let M_Ξ be the set of nondecreasing functions $g : \Xi \rightarrow [0, 1]$ with $g(0) = 0$ and $g(1) = 1$, with the topology of pointwise convergence. It is a closed subset of $[0, 1]^\Xi$, hence compact. In Lemma 7.6 below, we show that, although M_Ξ contains discontinuous functions, integration against a fixed measure is continuous on it.

Lemma 7.6. *Let ν be a Borel probability measure on Ξ and (g_i) a net in M_Ξ converging pointwise to g . Then we have $\int g_i d\nu \rightarrow \int g d\nu$.*

Proof. Let $\varepsilon > 0$ be given. Since ν has only finitely many atoms of mass at least ε , there are points $0 = p_0 < p_1 < \dots < p_m = 1$ of Ξ such that each open cell $c_k = \Xi \cap (p_{k-1}, p_k)$ has $\nu(c_k) < \varepsilon$. Put $\eta_i = \max_k |g_i(p_k) - g(p_k)|$, which tends to 0. On c_k both g_i and g take values

between their values at p_{k-1} and at p_k , because they are nondecreasing. Hence

$$\left| \int_{c_k} g_i d\nu - \int_{c_k} g d\nu \right| \leq \nu(c_k)(g(p_k) - g(p_{k-1}) + 2\eta_i),$$

and summing over k , together with the contribution of the points p_k , gives $|\int g_i d\nu - \int g d\nu| \leq \varepsilon + 3\eta_i$, since the differences $g(p_k) - g(p_{k-1})$ add up to at most 1. $\square$

Proposition 7.7. *In both cases, the probe algebra $\mathfrak{A}_C$ is $\{f(B) : f \in C(\Xi)\}$, so the probe space Y_C is the space $P(\Xi)$ of Borel probability measures on Ξ with the weak* topology. The map $\Phi : r \mapsto (r(\delta_y))_{y \in \Xi}$ is a homeomorphism from $\overline{\mathcal{R}_C}$ onto M_Ξ , and every idealized response is given by $r(\nu) = \int \Phi(r) d\nu$.*

Proof. The functions g_ρ for states ρ whose distribution has full support in Ξ separate the points of Ξ , and the map $f \mapsto f(B)$ is an isometric *-isomorphism from $C(\Xi)$ onto its image, because the spectrum of B is Ξ . By the Stone–Weierstrass theorem, the unital C*-algebra generated by the operators $g_\rho(B)$ is $\{f(B) : f \in C(\Xi)\}$, and the slice map takes values in it. By the Riesz representation theorem, its state space is $P(\Xi)$, and the response of ρ is $r_\rho(\nu) = \int g_\rho d\nu$.

The map Φ is continuous, because its coordinates are evaluations. Let $r \in \overline{\mathcal{R}_C}$ be the pointwise limit of a net (r_{ρ_i}) . Then the functions $g_{\rho_i} = \Phi(r_{\rho_i})$ converge pointwise to $g = \Phi(r) \in M_\Xi$, and Lemma 7.6 gives $r(\nu) = \lim_i \int g_{\rho_i} d\nu = \int g d\nu$. Thus the response r is determined by $\Phi(r)$, the map Φ is injective, and, as a continuous injection from a compact space, it is a homeomorphism onto its image. The image is compact and contains the functions g_ρ .

It remains to be shown that the functions g_ρ are dense in M_Ξ . Let $g \in M_\Xi$ and a finite set $F \subseteq \Xi$ containing 0 and 1 be given. In the continuous case, the piecewise linear interpolation $\tilde{g}$ of g on F is continuous and nondecreasing, with values 0 at 0 and 1 at 1; hence the function $\tilde{g}$ is g_ρ for a state whose distribution has the absolutely continuous distribution function $\tilde{g}$, and it agrees with g on F . In the discrete case, write $F = \{f_0 < f_1 < \dots < f_m\}$, with $f_0 = 0$ and $f_m = 1$. The probability measure with mass $g(f_{j+1}) - g(f_j)$ at f_j for $0 \leq j < m$ is the distribution of a state ρ with finitely many photons, and $g_\rho(f_j) = g(f_j) - g(f_0) = g(f_j)$ for every j , so the function g_ρ agrees with g on F . Hence every neighborhood of g contains some g_ρ . $\square$

7.4. Location in the classification. We can now place both comparisons in the classification of Theorem 6.9.

Theorem 7.8. *Let $C = [A < B]$ be as above.*

- (a) *In the discrete case, the observable C is unstable and NIP_3 .*
- (b) *In the continuous case, the observable C is unstable and NIP_1 , but not NIP_2 . Hence, for every trace-norm dense sequence (ρ_n) of states of Alice's particle, there is an increasing injection $\iota : 2^{<\mathbb{N}} \rightarrow \mathbb{N}$ such that the responses $\{r_{\rho_{\iota(t)}} : t \in 2^{<\mathbb{N}}\}$ are equivalent to D_6 or to D_7 .*

Proof. (a) By Proposition 7.7, the compactum of idealized responses is homeomorphic to a subspace of $[0, 1]^\Xi$ with Ξ countable, which is metrizable.

(b) By Proposition 7.7, the compactum of idealized responses is homeomorphic to $M_{[0,1]}$, a copy of the Helly space. It is first countable: given $g \in M_{[0,1]}$, the countable set D_g of rational points, jump points of g , and the endpoints determines a countable neighborhood base at

g . Indeed, let $y \in [0, 1]$ and $\varepsilon > 0$ be given. If $y \notin D_g$, then the function g is continuous at y , and there are rationals $p < y < q$ with $g(q) - g(p) < \varepsilon/2$. For every $h \in M_{[0,1]}$ with $|h - g| < \varepsilon/4$ at p and q , we have $h(p) \leq h(y) \leq h(q)$, because h is nondecreasing, and hence we have $|h(y) - g(y)| < \varepsilon$.

The compactum is not hereditarily separable. For $0 < a < 1$, let h_a be 0 on $[0, a)$, $\frac{1}{2}$ at a and 1 on $(a, 1]$. The set $\{h \in M_{[0,1]} : \frac{1}{4} < h(a) < \frac{3}{4}\}$ is a neighborhood of h_a that contains no other h_b , so the set $\{h_a : 0 < a < 1\}$ is an uncountable discrete subspace.

Thus the observable is NIP_1 but not NIP_2 . The last statement is Theorem 6.9(ii). $\square$

The idealized responses of the continuous comparison have a direct physical reading, which is also a probabilistic one. Alice's idealized preparations are described by arbitrary nondecreasing functions g , where $g(y)$ is the probability that Alice's particle lies to the left of y , and the value of g at a jump records how the particle approaches the jump point: from the left, from the right, or both. The function h_a belongs to a half-and-half mixture of a particle concentrating at a from the left and one concentrating at a from the right, and Bob's idealized probe concentrated at a reads the value $\frac{1}{2}$. In the discrete case no such splitting is possible, because the eigenvalues are isolated, and the only idealization is the escape to infinity.

7.5. Unconfined quantities. Let us notice that confinement to a box is not what produces the Helly space: a comparison depends only on the order of the two outcomes, so it is unchanged when both quantities are replaced by the same increasing function of themselves, and a suitable function turns unconfined quantities into positions in a box (see Proposition 7.9 below).

Proposition 7.9. *Let A and B be self-adjoint operators, each unitarily equivalent to multiplication by x on $L^2(\mathbb{R})$; for instance, the positions of two particles on a line, or quadratures of two modes of light. Then the comparison $[A < B]$ is unstable, and its idealized responses form a copy of the Helly space of nondecreasing functions $[0, 1] \rightarrow [0, 1]$ taking the value 0 at 0 and 1 at 1. Hence the comparison is NIP_1 but not NIP_2 .*

Proof. We may assume that A and B are multiplication by x on $L^2(\mathbb{R})$. Let $\theta: \mathbb{R} \rightarrow (0, 1)$ be the increasing diffeomorphism $\theta(x) = \frac{1}{2}(1 + \tanh x)$. The operator U given by $(U\psi)(u) = \psi(\theta^{-1}(u)) |(\theta^{-1})'(u)|^{1/2}$ is a unitary from $L^2(\mathbb{R})$ onto $L^2[0, 1]$ that carries multiplication by $\theta(x)$ to multiplication by u . Since θ is strictly increasing, we have $x < y$ exactly when $\theta(x) < \theta(y)$, so the spectral projection $[A < B]$ equals $[\theta(A) < \theta(B)]$, and $U \otimes U$ carries it to the comparison of the continuous case. A unitary of each factor preserves expectation values, carries actual and idealized states to actual and idealized states, and therefore carries idealized responses to idealized responses by a homeomorphism. The statement now follows from Proposition 7.7 and Theorem 7.8(b). $\square$

What matters, then, is that the quantities are continuous, not that they are confined. For quantities whose expectation values matter, such as energies, the rescaling by θ is not available, because it is not linear (see the discussion of the logical setting in the introduction).

7.6. The causal order is not first countable. A comparison is first countable because its idealized responses are monotone, and a monotone function has at most countably many jumps, each with two sides (see the proof of Theorem 7.8(b)). A relation between Bob's value and two values of Alice's, one on each side, removes this restriction (see Theorem 7.10 below), and the natural example is the causal order of spacetime. In units in which the speed of light is 1, an event (t, ξ) of a $(1 + 1)$ -dimensional spacetime can influence a detection by Bob at the point y at time 0 exactly when $|y - \xi| \leq -t$, that is, when y lies between the

points $a = \xi + t$ and $b = \xi - t$ at which the two light rays through the event cross Bob's time slice. We describe Alice's event by these null coordinates and confine everything to the causal diamond over Bob's interval: Alice's Hilbert space is $H_1 = L^2([0, 1]^2)$, with variables (a, b) , Bob's is $H_2 = L^2[0, 1]$, with variable y , and the *causal-order observable* C is multiplication by the indicator of

$$R = \{(a, b, y) : a \leq y \leq b\}.$$

Events with $a > b$ lie in the future of Bob's slice and influence none of his detections.

For a state ρ of Alice's system, let μ_ρ be the distribution of (a, b) in ρ , an absolutely continuous probability measure on $[0, 1]^2$, and similarly for Bob. The observable C is the relation observable C_R for this relation, so, by Lemma 7.1, $\varphi_C(\rho, \sigma) = (\mu_\rho \otimes \mu_\sigma)(R) = \int g_\rho d\mu_\sigma$, where $g_\rho(y) = \mu_\rho(\{(a, b) : a \leq y \leq b\})$ is the probability that Alice's event can influence a detection at y . The function g_ρ is continuous, because the sections of R at y and at y' differ by a set contained in two strips of area at most $|y - y'|$ each, whose μ_ρ -measure tends to 0 because μ_ρ is absolutely continuous, and the effective observable is $T_C(\rho) = g_\rho(B)$, multiplication by g_ρ .

Theorem 7.10. *The causal-order observable is NIP but not NIP₁. Hence, for every trace-norm dense sequence (ρ_n) of states of Alice's system, there is an increasing injection $\iota : 2^{<\mathbb{N}} \rightarrow \mathbb{N}$ such that the responses $\{r_{\rho_{\iota(t)}} : t \in 2^{<\mathbb{N}}\}$ are equivalent to D_5 .*

Proof. For every state ρ , the event $\{a \leq y \leq b\}$ is the event $\{a \leq y\}$ with the event $\{a \leq y, b < y\}$ removed, so

$$g_\rho(y) = \mu_\rho(\{(a, b) : a \leq y\}) - \mu_\rho(\{(a, b) : a \leq y, b < y\}),$$

a difference of two nondecreasing functions with values in $[0, 1]$. By Proposition 7.3, with $c = 1$, the observable C is uniformly NIP, and in particular NIP. By Lemma 6.3 the compactum of idealized responses is separable, so its level is defined.

The probe algebra consists of operators M_f of multiplication by continuous functions f , with $\|M_f\| = \sup |f|$. Hence every state of $\mathfrak{A}_C$ extends to a state of $\{M_f : f \in C[0, 1]\}$, which is isomorphic to $C[0, 1]$ (see [KR83, Theorem 4.3.13]), and by the Riesz representation theorem every state τ of $\mathfrak{A}_C$ is represented by a probability measure ν on $[0, 1]$, in the sense that $\tau(M_f) = \int f d\nu$ for $M_f \in \mathfrak{A}_C$; then we have $r_\rho(\tau) = \int g_\rho d\nu$. For each point $c \in (0, 1)$, the evaluation $\tau_c(M_f) = f(c)$ is a state of $\mathfrak{A}_C$, Bob's probe localized at c .

Let ρ_0 be a state whose distribution is uniform on $[\frac{1}{2}, 1] \times [0, \frac{1}{2}]$, an event after Bob's slice. Then we have $g_{\rho_0} = 0$, so the zero function is a response. For $c \in (0, 1)$ and large n , let $\rho_{c,n}$ be a state whose distribution is uniform on $[c - \frac{2}{n}, c - \frac{1}{n}] \times [c + \frac{1}{n}, c + \frac{2}{n}]$, an event at position near c shortly before Bob's slice. Then we have $g_{\rho_{c,n}}(c) = 1$ and $g_{\rho_{c,n}}(y) = 0$ when $|y - c| > \frac{2}{n}$. Hence the function $g_{\rho_{c,n}}$ converges pointwise to the indicator of $\{c\}$, and by dominated convergence we have $r_{\rho_{c,n}}(\tau) \rightarrow \nu(\{c\})$ for every probe τ and every measure ν representing it. The limit r_c is an idealized response, with $r_c(\tau_c) = 1$.

Suppose that the zero function had a countable base of neighborhoods (U_k) in the compactum of idealized responses. Each U_k contains a set $\{\tau : |r(\tau)| < \varepsilon_k \text{ for } \tau \in F_k\}$ with F_k finite. Choose a representing measure for each probe in $F = \bigcup_k F_k$; together these measures have countably many atoms, so some $c \in (0, 1)$ is an atom of none of them. Then $r_c(\tau) = 0$ for $\tau \in F$, so the response r_c lies in every U_k . Since the space is Hausdorff, the intersection of a base of neighborhoods of a point is that point, so we have $r_c = 0$, contradicting $r_c(\tau_c) = 1$. Hence the compactum is not first countable. The last statement is Theorem 6.9(iii). $\square$

The proof has a direct physical reading. The idealized response r_c is that of an event on Bob's time slice at the point c : its light cone meets the slice in the single point c , so only probes that give positive weight to the point c itself detect it. There is one such event for each point c , and countably many of Bob's probes cannot tell all of them apart from an event that influences nothing; this is the failure of first countability. The relation R is the intersection of two comparisons, $a \leq y$ and $y \leq b$, one for each light ray. A single comparison has monotone idealized responses; the two rays together allow idealized responses supported on a single point. Here the Todorčević trichotomy is used in full: it turns the failure of first countability into a dyadic tree of actual events whose responses are equivalent to the canonical family D_5 . The proof suggests a tree of this kind, although we do not claim that the trichotomy produces this one: events whose light cones meet Bob's slice in the intervals of a Cantor construction, each emitted later than its predecessor.

7.7. The comparison of phases is not first countable. Orders on a line give first countable compacta of idealized responses (see Theorem 7.8 and Proposition 7.9). In Theorem 7.12 below, we show that orders on a circle need not. Identify the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with $[0, 1)$. Alice and Bob each hold a particle on a ring, a planar rotor, with Hilbert spaces $H_1 = H_2 = L^2(\mathbb{T})$, the angle acting by multiplication. The *phase comparison* is the relation observable $C = C_R$ for

$$R = \{(x, y) \in [0, 1)^2 : y - x \in (0, \frac{1}{2}) \bmod 1\},$$

the event “Bob's angle is ahead of Alice's by less than half a turn”. For a state ρ of Alice's rotor, the distribution μ_ρ of her angle is an absolutely continuous probability measure on $\mathbb{T}$, and every such measure occurs. By Lemma 7.1, $\varphi_C(\rho, \sigma) = \int g_\rho d\mu_\sigma$, where $g_\rho(y) = \mu_\rho((y - \frac{1}{2}, y))$ is the probability that Alice's angle lies in the open half-circle ending at y ; the function g_ρ is continuous on $\mathbb{T}$, because μ_ρ has no atoms.

Lemma 7.11. *For every probability measure μ on $\mathbb{T} = [0, 1)$, the function $g(y) = \mu((y - \frac{1}{2}, y))$ on $[0, 1)$ is $2(u - v)$ for two nondecreasing functions $u, v: [0, 1) \rightarrow [0, 1]$.*

Proof. Let $G(y) = \mu([0, y))$. For $y \in [\frac{1}{2}, 1)$ the half-circle is the interval $(y - \frac{1}{2}, y)$, so we have $g(y) = G(y) - \mu([0, y - \frac{1}{2}])$; for $y \in [0, \frac{1}{2})$ it is $(y + \frac{1}{2}, 1) \cup [0, y)$, so we have $g(y) = G(y) + 1 - \mu([0, y + \frac{1}{2}])$. Let $V(y) = \mu([0, y + \frac{1}{2}]) - 1$ for $0 \leq y < \frac{1}{2}$ and $V(y) = \mu([0, y - \frac{1}{2}])$ for $\frac{1}{2} \leq y < 1$. Then we have $g = G - V$, the function G is nondecreasing with values in $[0, 1]$, and V is nondecreasing with values in $[-1, 1]$: it is nondecreasing on each half, and at $\frac{1}{2}$ it passes from $\lim_{y \uparrow 1/2} V(y) = 0$ to $V(\frac{1}{2}) = \mu(\{0\}) \geq 0$. Put $u = \frac{1}{2}(G + 1)$ and $v = \frac{1}{2}(V + 1)$. $\square$

Theorem 7.12. *The phase comparison is uniformly NIP, with bounds that depend only on the margin, and it is not NIP_1 : its compactum of idealized responses is not first countable. Hence, for every trace-norm dense sequence (ρ_n) of states of Alice's rotor, there is an increasing injection $\iota: 2^{<\mathbb{N}} \rightarrow \mathbb{N}$ such that the responses $\{r_{\rho_{\iota(t)}} : t \in 2^{<\mathbb{N}}\}$ are equivalent to D_5 .*

Proof. By Lemma 7.11, extended constantly outside $[0, 1)$, Proposition 7.3 applies with $c = 2$, and the observable is uniformly NIP.

As in the proof of Theorem 7.10, every probe τ is represented by a probability measure ν on $\mathbb{T}$, with $r_\rho(\tau) = \int g_\rho d\nu$, and for $c \in \mathbb{T}$ the evaluation $\tau_c(M_f) = f(c)$ is a probe, Bob's rotor localized at c . Let ρ_0 be the uniform state. Every open half-circle has measure $\frac{1}{2}$, so $r_{\rho_0}(\tau) = \frac{1}{2}$ for every probe τ .

For $a \in \mathbb{T}$ and large n , let $\rho_{a,n}$ be a state whose distribution is uniform, with weight $\frac{1}{2}$ each, on the arcs $B_1 = (a - \frac{1}{n}, a)$ and $B_2 = (a + \frac{1}{2}, a + \frac{1}{2} + \frac{1}{n})$: half of the angle just behind a , half just ahead of the antipode. Write $A_y = (y - \frac{1}{2}, y)$. If $y = a$, then the half-circle A_y contains B_1 and B_2 , and we have $g_{\rho_{a,n}}(a) = 1$. If $y = a + \frac{1}{2}$, then the half-circle $A_y = (a, a + \frac{1}{2})$ meets neither, and we have $g_{\rho_{a,n}}(a + \frac{1}{2}) = 0$. Otherwise the endpoints of A_y are at some positive distance d from a and from $a + \frac{1}{2}$; an open half-circle whose endpoints differ from these two antipodal points contains exactly one of them, and once $\frac{1}{n} < d$ it contains the arc B_1 just behind a if it contains a , and the arc B_2 just ahead of $a + \frac{1}{2}$ if it contains $a + \frac{1}{2}$, but not both; so we have $g_{\rho_{a,n}}(y) = \frac{1}{2}$. Hence the function $g_{\rho_{a,n}}$ converges pointwise to $g_a = \frac{1}{2} + \frac{1}{2}\mathbf{1}_{\{a\}} - \frac{1}{2}\mathbf{1}_{\{a+1/2\}}$, and by dominated convergence we have $r_{\rho_{a,n}}(\tau) \rightarrow \frac{1}{2} + \frac{1}{2}\nu(\{a\}) - \frac{1}{2}\nu(\{a + \frac{1}{2}\})$ for every probe τ and every measure ν representing it. The limit r_a is an idealized response, with $r_a(\tau_a) = 1$.

Suppose that r_{ρ_0} had a countable base of neighborhoods (U_k) in the compactum of idealized responses. Each U_k contains a set $\{r : |r(\tau) - \frac{1}{2}| < \varepsilon_k \text{ for } \tau \in F_k\}$ with F_k finite. The measures representing the probes in $F = \bigcup_k F_k$ have countably many atoms, so there is $a \in \mathbb{T}$ such that neither a nor $a + \frac{1}{2}$ is one of them. Then we have $r_a(\tau) = \frac{1}{2}$ for every $\tau \in F$, so the response r_a lies in every U_k , and we have $r_a = r_{\rho_0}$, contradicting $r_a(\tau_a) = 1$. The last statement is Theorem 6.9(iii). $\square$

The proof has a direct physical reading. The uniform state responds $\frac{1}{2}$ to every probe. An antipodal pair of angles, one approached from behind and the other from ahead, also responds $\frac{1}{2}$, except that a probe at a reads 1 and a probe at the antipode reads 0: the jumps at the two ends of the half-turn point in opposite directions and cancel everywhere else. There is one such pair for each point of the circle, and countably many probes cannot tell them all from the uniform state. On a line this cannot happen, because a monotone response cannot hide a jump (see the proof of Theorem 7.8(b)); on a circle there is no first or last point. The same contrast appears in topological dynamics: Glasner and Megrelishvili showed that compact metric linearly ordered systems have first countable enveloping semigroups, while the circular analogue of Helly's space is not first countable [GM22, Theorem 8.8, Proposition 8.10 and Remark 8.11]. Their spaces consist of order-preserving maps of the circle, while the responses here are real valued, so neither result implies the other.

7.8. Interference fringes are not NIP. Orders are tame. Oscillation at ever finer scales is not, as we show in Proposition 7.13 below, and the simplest physical instance is an interference pattern. Alice prepares a mode of light in one of the states e_n , $n \in \mathbb{N}$, whose interference pattern on Bob's screen has wavenumber $k_n = 8^n$, and Bob's detector, at the position $x \in [0, 2\pi]$, clicks with probability $f_n(x) = \frac{1}{2}(1 + \cos k_n x)$: it asks whether the photon lands on a bright fringe. The *fringe observable* is the label-controlled observable $C = \sum_n |e_n\rangle\langle e_n| \otimes M_{f_n}$ on $\ell^2(\mathbb{N}) \otimes L^2[0, 2\pi]$.

Proposition 7.13. *The fringe observable is not NIP: Alice's states e_n form an independent sequence, and the closure of their responses contains a copy of $\beta\mathbb{N}$.*

Proof. Let $E, F \subseteq \mathbb{N}$ be finite and disjoint. Choose $y \in [0, 1)$ whose base-8 digit in place $n+1$ is 0 for $n \in F$ and 4 for $n \in E$, and put $x = 2\pi y$. Then we have $\cos k_n x = \cos(2\pi\{8^n y\})$, where $\{8^n y\}$ is the fractional part, which lies in $[0, \frac{1}{8})$ for $n \in F$ and in $[\frac{1}{2}, \frac{5}{8})$ for $n \in E$; hence $f_n(x) \geq \frac{3}{4}$ for $n \in F$ and $f_n(x) \leq \frac{1}{4}$ for $n \in E$. A state σ of Bob's detector concentrated near x gives, by continuity of the finitely many functions f_n involved, expectation values at least $\frac{5}{8}$ on F and at most $\frac{3}{8}$ on E . By Lemma 5.2(c), these states are probes, so the sequence (e_n) is independent. Every subsequence is independent as well, so no subsequence of the

responses converges pointwise on the compact metrizable probe space, and by Fact 3.4 some subsequence has closure homeomorphic to $\beta\mathbb{N}$. $\square$

The contrast with Section 7 is the physical content of NIP. A comparison asks on which side of a threshold a value lies, and finitely many probes can only cut an ordered line into finitely many pieces; a fringe of wavenumber 8^n cuts Bob's screen into 8^n pieces, and the fringes of different wavenumbers are independent of one another, like the digits of a number.

8. THE SEVEN MINIMAL FAMILIES IN PHYSICS

The goal of this section is to show that each of the seven minimal families $D_1, \dots, D_7$ of the heptachotomy is realized by a tree of preparations of a physically natural observable, and that each realization displays what is characteristic of its family. For whole compacta of idealized responses, mixtures merge the hereditarily separable class with the metrizable one (see Corollary 6.8); in Theorem 8.5 below, we show that for trees of preparations all seven families occur, because a tree may have a closure of any of the seven kinds inside a larger convex compactum, as the split interval sits inside the Helly space.

8.1. Label-controlled observables. Several of our observables are of the following kind. Alice's system is a register that holds a label i , and the label selects the test applied to Bob's system: a detector that accepts Bob's system at the point x with probability $f_i(x)$.

Definition 8.1. Let L be a compact metrizable space, λ a Borel probability measure on L with full support, I a countable set and $(f_i)_{i \in I}$ a family of continuous functions $L \rightarrow [0, 1]$. The *label-controlled observable* is

$$C = \sum_{i \in I} |e_i\rangle\langle e_i| \otimes M_{f_i} \quad \text{on } \ell^2(I) \otimes L^2(L, \lambda),$$

where (e_i) is the standard basis of $\ell^2(I)$ and M_f is multiplication by f .

The sum converges strongly, because its terms act on orthogonal subspaces, and $0 \leq C \leq I$. When L contains a countable dense set D of isolated points, we take $\lambda = \sum_{x \in D} w_x \delta_x$ with positive weights; then the space $L^2(L, \lambda)$ is a weighted copy of $\ell^2(D)$, and Bob's system is located at one of the points of D .

Lemma 8.2. *Let C be as in Definition 8.1, and suppose that the functions f_i separate the points of L . Then the probe algebra is $\mathfrak{A}_C = \{M_f : f \in C(L)\}$, the probe space Y_C is the space $P(L)$ of Borel probability measures on L with the weak* topology, and the response of the label state e_i is $r_{e_i}(\nu) = \int f_i d\nu$.*

Proof. For a density operator ρ , tracing out Alice's system gives $T_C(\rho) = \sum_i \langle e_i, \rho e_i \rangle M_{f_i}$, a norm-convergent sum, so the slice map takes values in $\{M_f : f \in C(L)\}$. Since λ has full support and f is continuous, we have $\|M_f\| = \sup |f|$; hence the map $f \mapsto M_f$ is an isometric *-isomorphism from $C(L)$ onto its image. The unital C*-subalgebra of $C(L)$ generated by the functions f_i is $C(L)$, by the Stone–Weierstrass theorem, because they separate points. Hence we have $\mathfrak{A}_C = \{M_f : f \in C(L)\}$, and by the Riesz representation theorem its states are the measures $\nu \in P(L)$, acting by $M_f \mapsto \int f d\nu$. $\square$

8.2. Transfer from points to measures. The responses of our observables are integrals $\nu \mapsto \int g d\nu$ of functions g on a compact space against Bob's probes. The next lemma shows that a family of such responses has the same closure as the family of integrands, by a homeomorphism that matches the families term by term.

Lemma 8.3. *Let L be a compact metrizable space and $(f_t)_{t \in T}$ a countable family of continuous functions $L \rightarrow [0, 1]$ that is relatively compact in $B_1(L)$. For $t \in T$ and $\nu \in P(L)$, put $F_t(\nu) = \int f_t d\nu$. Then the families $\{F_t\}$, of functions on $P(L)$, and $\{f_t\}$, of functions on L , are equivalent, and $\{F_t\}$ is NIP.*

Proof. Every sequence (f_{t_n}) has a subsequence that converges pointwise on L . Indeed, by Fact 3.4, otherwise a subsequence would have closure homeomorphic to $\beta\mathbb{N}$, which is not Fréchet–Urysohn, while the closure of $\{f_t\}$ is a Rosenthal compactum and therefore Fréchet–Urysohn (see [BFT78, Theorem 3F]). By the dominated convergence theorem, the corresponding subsequence of (F_{t_n}) converges pointwise on $P(L)$.

Consequently the family $\{F_t\}$ is NIP. If a sequence (F_{t_n}) and reals $a < b$ witnessed the independence property, then, since $P(L)$ is compact and the functions F_t are continuous, for each $E \subseteq \mathbb{N}$ there would be ν with $F_{t_n}(\nu) \leq a$ for $n \in E$ and $F_{t_n}(\nu) \geq b$ for $n \notin E$ (see the proof of Theorem 3.6); the same holds for every subsequence, and taking E to contain every other index of a subsequence shows that no subsequence converges at the corresponding ν . By Theorem 3.6, the closure of $\{F_t\}$ is contained in $B_1(P(L))$; it is a Rosenthal compactum, hence Fréchet–Urysohn.

Let Φ send a function G on $P(L)$ to the function $x \mapsto G(\delta_x)$ on L . The map Φ is continuous for the pointwise topologies, and $\Phi(F_t) = f_t$. The map Φ is injective on the closure of $\{F_t\}$: if G lies in that closure, then the function G is the pointwise limit of a sequence (F_{t_n}) , so the sequence (f_{t_n}) converges pointwise to $\Phi(G)$, which is therefore a bounded Borel function, and by dominated convergence $G(\nu) = \lim_n \int f_{t_n} d\nu = \int \Phi(G) d\nu$ for every ν . Thus the function G is determined by $\Phi(G)$. As a continuous injection on a compact space, the map Φ is a homeomorphism from the closure of $\{F_t\}$ onto its image, which is compact, contains $\{f_t\}$ and is contained in the closure of $\{f_t\}$; hence the image is that closure. $\square$

We also need to discard points that carry no information about limits.

Lemma 8.4. *Let L be a set, $L_0 \subseteq L$, and $(g_t)_{t \in T}$ a countable pointwise bounded family of real functions on L whose pointwise closure K , a compact space, is Fréchet–Urysohn. Suppose that*

- (i) *the restrictions $g_t|_{L_0}$ are pairwise distinct, and none of them lies in the closure of the others;*
- (ii) *any two limits of sequences (g_{t_k}) with pairwise distinct indices that agree on L_0 are equal.*

Then the families (g_t) and $(g_t|_{L_0})$ are equivalent.

Proof. Restriction $\pi: K \rightarrow \mathbb{R}^{L_0}$ is continuous, and its image is the closure of $(g_t|_{L_0})$. Let $f \in K$ be different from every g_t . Since K is Fréchet–Urysohn, f is the limit of a sequence (g_{t_k}) ; since K is Hausdorff and f is not a g_t , no index occurs infinitely often, so after passing to a subsequence the indices are pairwise distinct. By (i), π is injective on the family, and $\pi(g_t)$ is not $\pi(f)$ for such a limit f , because $\pi(f)$ lies in the closure of $\{g_s|_{L_0} : s \neq t\}$. By (ii), π is injective on the limits. Hence the map π is a continuous injection of the compact space K , and so a homeomorphism onto its image. $\square$

8.3. Three settings. The realizations use three physical settings.

Photon numbers. This is the discrete comparison of Section 7, with $\Xi = \{x_l : l \in \mathbb{N}\} \cup \{1\}$, $x_l = 1 - \frac{1}{l+1}$. Alice’s number state e_l has $g_l = g_{e_l}$, the indicator of $\{x_j : j > l\} \cup \{1\}$; the probe space is $P(\Xi)$ and $r_{e_l}(\nu) = \int g_l d\nu$ (see Proposition 7.7).

Positions in a box. This is the continuous comparison of Section 7: the probe space is $P([0, 1])$, and a state ρ of Alice's particle with distribution μ_ρ has response $\nu \mapsto \int g_\rho d\nu$, where $g_\rho(y) = \mu_\rho([0, y])$. We place a Cantor set in the box. Let $\psi(\sigma) = \frac{3}{4} - \sum_n \sigma(n)3^{-n-1}$ for $\sigma \in 2^{\mathbb{N}}$; this is an order-reversing homeomorphism of $(2^{\mathbb{N}}, <)$ onto a Cantor set $C_0 \subseteq [\frac{1}{4}, \frac{3}{4}]$. For $s \in 2^{<\mathbb{N}}$, let I_s be the smallest interval containing $\psi(V_s)$; it has length $\ell_s = \frac{1}{2}3^{-|s|}$, left endpoint $\psi(s \smallfrown 1^\infty)$ and right endpoint $\psi(s \smallfrown 0^\infty)$. If τ is the immediate successor of σ in $<$, then the interval $(\psi(\tau), \psi(\sigma))$ is a component of $[0, 1] \setminus C_0$; the other components are $[0, \frac{1}{4})$ and $(\frac{3}{4}, 1]$. Let us notice that the components adjacent to I_s have length at least $\ell_s/2$: each of them is either $[0, \frac{1}{4})$ or $(\frac{3}{4}, 1]$, of length $\frac{1}{4} \geq \ell_s/2$, or the gap between $I_{u \smallfrown 0}$ and $I_{u \smallfrown 1}$ for some proper initial segment u of s , of length $\ell_u/3 \geq \ell_s$.

Registers. Alice's system is a chain of spins of variable finite length, with Hilbert space $\ell^2(2^{<\mathbb{N}})$; the basis vector e_t is the chain of length $|t|$ whose spin at site n is up exactly when $t(n) = 1$. Bob probes it in two ways. In *scanning*, Bob's probe sits at a site $n \in \mathbb{N}$ and asks whether Alice's spin at that site is up: the test is $a_t(n) = t(n)$ for $n < |t|$ and $a_t(n) = 0$ otherwise. In *template matching*, Bob holds a reference configuration u , finite or infinite, and asks whether Alice's chain coincides with the beginning of his reference: the test is $b_t(u) = 1$ if $t \sqsubseteq u$ and $b_t(u) = 0$ otherwise. We extend a_t to the one-point compactification $L_a = \mathbb{N} \cup \{\infty\}$ by $a_t(\infty) = 0$, and we let L_b be the space of finite and infinite binary sequences, with the ultrametric $d(u, u') = 2^{-|u \wedge u'|}$ for $u \neq u'$, where $u \wedge u'$ is the longest common initial segment; it is compact, the finite sequences are isolated and dense, and the functions b_t are continuous, since $\{u : t \sqsubseteq u\}$ is clopen. The *scanning*, *template* and *combined* observables are the label-controlled observables with labels $2^{<\mathbb{N}}$, on L_a , L_b and $L_a \oplus L_b$, with the functions a_t , b_t and (a_t, b_t) , and with measures concentrated on $\mathbb{N}$, on the finite sequences, and on both. These functions separate the points of L_a , of L_b and of $L_a \oplus L_b$; for instance, the empty label gives $a_\emptyset = 0$ on L_a and $b_\emptyset = 1$ on L_b . Hence Lemma 8.2 applies to all three observables.

8.4. The realization. In Theorem 8.5 below, we show that the three settings realize all seven minimal families.

Theorem 8.5. *Each minimal family is realized by a tree of actual preparations of a physically natural observable: there are pure states $(\rho_t)_{t \in 2^{<\mathbb{N}}}$ of Alice's system, given by pairwise orthogonal vectors, such that the responses $\{r_{\rho_t}\}$ are equivalent to D_i , as follows.*

- D_1 : the comparison of photon numbers, with Alice's number states in any order;
- D_2 : scanning, with Alice's chains e_{s_t} , whose scanning functions are $s_t \smallfrown 0^\infty$, for $s_t \in R$;
- D_3 : the comparison of positions, with Alice's particle spread over the gap of C_0 immediately to the right of $\psi(s_t \smallfrown 0^\infty)$;
- D_4 : the comparison of positions, with Alice's particle spread over the gap of C_0 immediately to the left of $\psi(s_t \smallfrown 1^\infty)$;
- D_5 : template matching, with Alice's chains e_t ;
- D_6 : the combined observable, with Alice's chains e_t ;
- D_7 : the comparison of positions, with Alice's particle spread half over a short interval just to the left of I_{s_t} and half over a short interval just to the right of it.

Moreover, the causal order of spacetime realizes D_5 in the sense of Theorem 7.10.

Proof. In each case the probe space is $P(L)$ for a compact metrizable space L , the responses are $r_{\rho_t}(\nu) = \int g_t d\nu$ with g_t continuous, and the closure of $\{g_t\}$ consists of Baire-1 functions;

by Lemma 8.3, it suffices to show that $\{g_t\}$ is equivalent to D_i . Here (s_t) is the canonical enumeration of the subtree R of Section 4.

D_1 . Let $\beta: 2^{<\mathbb{N}} \rightarrow \mathbb{N}$ be a bijection and $g_t = g_{\beta(t)}$. Every sequence of pairwise distinct members converges to the indicator of $\{1\}$, which is not a member, and each member is isolated. Hence the closure is a countable compact space with a single non-isolated point, and so is the closure $D_1 \cup \{0\}$ of D_1 ; the bijection $g_t \mapsto \frac{1}{|t|+1}v_t$, together with the limit points, is a homeomorphism. The space Ξ is countable, so every function on it is Baire-1, every subset of a countable space being an F_σ .

D_2 . Here $g_t = a_{s_t}$ is $s_t \frown 0^\infty$ on $\mathbb{N}$ and 0 at ∞ . This is the family D_2 , extended continuously to $\mathbb{N} \cup \{\infty\}$; every function in its closure vanishes at ∞ , so the extension does not change the closure, and every function on the countable space L_a is Baire-1.

D_3 and D_4 . Let $\sigma_t = s_t \frown 0^\infty$. If $\sigma_t \neq 0^\infty$, it has an immediate predecessor σ_t^- , and $G_t = (\psi(\sigma_t), \psi(\sigma_t^-))$ is a gap; if $\sigma_t = 0^\infty$, let $G_t = (\frac{3}{4}, 1]$. Let ρ_t be uniform on G_t . Then $g_t = 0$ on $[0, \psi(\sigma_t)]$, $g_t = 1$ to the right of G_t , and g_t is linear on G_t . On C_0 , $g_t(\psi(\tau)) = 0$ exactly when $\tau \succeq \sigma_t$; that is, $g_t \circ \psi = 1 - x_{\sigma_t}^+$. The sequences σ_t are pairwise distinct, so the gaps G_t are pairwise distinct. We apply Lemma 8.4 with $L_0 = C_0$. The closure of $\{g_t\}$ consists of nondecreasing functions, so it is a Rosenthal compactum and Fréchet–Urysohn. Condition (i) holds because D_3 is discrete in its closure. For (ii), let J be a component of $[0, 1] \setminus C_0$. For pairwise distinct t_k , eventually $G_{t_k} \neq J$, and then the function g_{t_k} is constant on $\bar{J}$; hence every such limit f is constant on $\bar{J}$, and so is determined on J by its value at an endpoint of J in C_0 . Thus the family $\{g_t\}$ is equivalent to $\{g_t|_{C_0}\}$, which is carried to D_3 by the homeomorphism ψ followed by the map $f \mapsto 1 - f$, a homeomorphism for the pointwise topology. The case of D_4 is symmetric: with $\sigma_t = s_t \frown 1^\infty$ and its immediate successor σ_t^+ , the gap is $(\psi(\sigma_t^+), \psi(\sigma_t))$, or $[0, \frac{1}{4}]$ if $\sigma_t = 1^\infty$, and $g_t \circ \psi = 1 - x_{\sigma_t}^-$.

D_5 . Here $g_t = b_t$ on L_b ; its restriction to the infinite sequences is v_t . For pairwise distinct t_k , eventually t_k is longer than any given finite u , so every such limit vanishes on the finite sequences. Hence Lemma 8.4, with $L_0 = 2^\mathbb{N}$, applies, condition (i) holding because D_5 is discrete in its closure $\hat{A}(2^\mathbb{N})$. The closure consists of Baire-1 functions, because its restrictions to the closed set $2^\mathbb{N}$ lie in $\hat{A}(2^\mathbb{N}) \subseteq B_1(2^\mathbb{N})$ and the remaining points are isolated.

D_6 . Here $g_t = (a_t, b_t)$; its restriction to $\mathbb{N} \oplus 2^\mathbb{N}$ is $(t \frown 0^\infty, v_t)$, which is the member $(v_t, t \frown 0^\infty)$ of D_6 with the two summands written in the other order. Every limit vanishes at ∞ and on the finite sequences, as above, the closure consists of Baire-1 functions, by the arguments for D_2 and D_5 , and Lemma 8.4 applies with $L_0 = \mathbb{N} \oplus 2^\mathbb{N}$.

D_7 . Let e_t^L and e_t^R be the left and right endpoints of I_{s_t} , and $\ell = \ell_{s_t}$. Let ρ_t be the state whose distribution is half uniform on $A_t = [e_t^L - \frac{\ell}{2}, e_t^L - \frac{\ell}{4}]$ and half uniform on $B_t = [e_t^R + \frac{\ell}{4}, e_t^R + \frac{\ell}{2}]$; these intervals lie in the components adjacent to I_{s_t} . On C_0 , g_t equals 0 to the left of I_{s_t} , $\frac{1}{2}$ on I_{s_t} and 1 to the right; that is, $g_t \circ \psi = 1 - x_{\sigma_t}^+ + \frac{1}{2}v_{s_t}$ with $\sigma_t = s_t \frown 0^\infty$, since $\psi(\sigma_t) = e_t^R$. On the closure of D_7 we have $v \leq x$ and both take the values 0 and 1, so the map $(v, x) \mapsto (1 - x + \frac{1}{2}v) \circ \psi^{-1}$ is a continuous injection, and it carries D_7 to $\{g_t|_{C_0}\}$; since D_7 is discrete in its closure, condition (i) of Lemma 8.4 holds. For (ii), note that A_t lies in the component whose right endpoint is $\psi(s_t \frown 1^\infty)$, and B_t in the component whose left endpoint is $\psi(s_t \frown 0^\infty)$. These sequences are pairwise distinct for $s_t \in R$, so each component of $[0, 1] \setminus C_0$ contains at most one interval A_t and at most one interval B_t . For pairwise distinct t_k , eventually neither A_{t_k} nor B_{t_k} lies in a given component J , and then the function g_{t_k} is constant on $\bar{J}$. Hence every such limit is constant on $\bar{J}$, and is determined by its values on C_0 .

Finally, in the photon and register cases the states are distinct basis vectors, and in the box we take the vectors $\sqrt{d\mu_{\rho_t}/dy}$, whose supports meet in at most finitely many points, so they are orthogonal. $\square$

The realizations display what distinguishes the families.

- D_1 , *escape to infinity*: an oscillator excited ever higher has a single idealized limit, and each actual preparation is isolated from it.
- D_2 , *infinite configurations*: the idealized limits of the chains are the infinite spin configurations, a Cantor set of them, and countably many site probes tell them all apart, so the closure is metrizable; but any two configurations differ by 1 at some site, so the closure is not separable in the supremum norm.
- D_3 and D_4 , *one-sided localization*: a particle that approaches a point from one side only produces one half of a split pair, and the split Cantor set appears; the closure is hereditarily separable but not metrizable.
- D_5 , *point-supported limits*: light cones that shrink to points on Bob's slice, or templates that become infinitely long, give idealized responses that see a single point, and the closure is not first countable.
- D_6 , *twins over a metrizable base*: each infinite configuration has two idealized versions, one that every template misses and one that the template of the configuration itself detects; the scanning probes alone see a Cantor set.
- D_7 , *twins over a split base*: a particle that approaches a point from both sides at once produces the midpoint of a split pair, as the symmetric state does in the Ising model (see Section 9); its twin structure sits over the split Cantor set.

In Proposition 8.6 and Corollary 8.7 below, we show that these observables are NIP, and hence they can be placed in the classification of Theorem 6.9. To prove that they are NIP, we use a theorem of Stegall. Recall that a subset of a Banach space is *weakly precompact* if it is bounded and every sequence in it has a weakly Cauchy subsequence, and that the *absolutely convex hull* of a set consists of the finite combinations $\sum_k a_k x_k$ of its elements with $\sum_k |a_k| \leq 1$. Stegall's theorem states that the absolutely convex hull of a weakly precompact set is weakly precompact (see [Rod21], where the result is attributed to Stegall). **Proposition 8.6.** *Let C be a label-controlled observable whose functions f_i separate the points of L and have the property that every sequence of them has a subsequence converging pointwise on L . Then the observable C is NIP for Alice's responses.*

Proof. For a density operator ρ , put $p_i = \langle e_i, \rho e_i \rangle$, a probability distribution on I , and $g_\rho = \sum_i p_i f_i$, a norm-convergent sum in $C(L)$; then we have $r_\rho(\nu) = \int g_\rho d\nu$. By dominated convergence, a sequence of functions f_i that converges pointwise on L is weakly Cauchy in $C(L)$; hence the set $\{f_i\}$ is weakly precompact. By Stegall's theorem, so is its absolutely convex hull, and so is the norm closure of that hull: given y_n in the closure, choose x_n in the hull with $\|x_n - y_n\| < 1/n$ and a weakly Cauchy subsequence of (x_n) ; the corresponding subsequence of (y_n) is weakly Cauchy. The partial sums of g_ρ lie in the hull; hence every function g_ρ lies in this closure. Consequently, every sequence (r_{ρ_n}) has a subsequence that converges pointwise on $P(L) = Y_C$, and, as in the proof of Lemma 8.3, an independent sequence has no such subsequence. $\square$

Corollary 8.7.

- (a) *The scanning observable is unstable and NIP₃, but not uniformly NIP.*
- (b) *The template observable is NIP but not NIP₁.*

- (c) *The scanning observable is not NIP for Bob's responses, that is, the observable obtained by exchanging the two tensor factors is not NIP for Alice's responses. Hence NIP for Alice's responses does not imply NIP for Bob's.*

Proof. (a) Every sequence of functions a_t on the countable set L_a has a pointwise convergent subsequence, by a diagonal argument, so the observable is NIP by Proposition 8.6. For a trace-norm dense sequence (ρ_n) , the closure of the responses is that of $\{r_{\rho_n}\}$ (see Lemma 6.3), which by Lemma 8.3 is homeomorphic to the closure of $\{g_{\rho_n}\}$ in $\mathbb{R}^{L_a}$; since L_a is countable, this space is metrizable. For each m , the chains $t_1, \dots, t_m$ of length 2^m with $t_i(n)$ the i -th binary digit of n are shattered, with values 0 and 1, by the site probes $n < 2^m$; hence the fat-shattering dimension is infinite at every margin $\delta \leq 1$. Finally, the observable is unstable: the functions a_{1^k} converge pointwise to the indicator of $\mathbb{N}$, so the idealized response $\nu \mapsto \nu(\mathbb{N})$, which is not continuous on $P(L_a)$, is a limit of responses (see Theorem 6.1(a)).

(b) Let (t_k) be a sequence of labels. By compactness of L_b , a subsequence converges in L_b ; if the limit is finite, the subsequence is eventually constant. If the limit is an infinite sequence σ , then $b_{t_k}(u) \rightarrow 0$ for finite u , and $b_{t_k}(\tau) \rightarrow 0$ for infinite $\tau \neq \sigma$, since eventually t_k agrees with σ beyond the first place where σ and τ differ; passing to a further subsequence, $b_{t_k}(\sigma)$ is constant. Hence the observable is NIP by Proposition 8.6. By Theorem 8.5, the closure of its responses contains a closed copy of $\hat{A}(2^{\mathbb{N}})$, which is not first countable.

(c) Bob's site states e_n , $n \in \mathbb{N}$, form an independent sequence: for disjoint finite sets $E, F \subseteq \mathbb{N}$, Alice's chain t of length $1 + \max(E \cup F)$ with $t(n) = 1$ exactly for $n \in F$ gives the value 1 on F and 0 on E . Alice's actual states are points of the probe space of the exchanged observable (see Lemma 5.2(c)), so the family of Bob's responses has the independence property. This is the relation $a \in b$ of Remark 3.8, realized by a physical observable. $\square$

The scanning observable shows the difference between NIP and uniform NIP (see Remark 6.2): its idealized responses form a metrizable compactum, yet its responses are not learnable with a number of probes bounded independently of the distribution of the probes [ABDCBH97], because Bob's site probes can read off every finite pattern of sufficiently long chains.

9. SPONTANEOUS SYMMETRY BREAKING IS A SPLIT PAIR

The goal of this section and the next is to show that the levels of Section 4 have a physical meaning beyond bipartite observables. Many central notions of physics are defined by an order of limits of a family of responses: a preparation, such as a finite sample, has a response, such as its magnetization as a function of an external field, and the notion depends on whether the sample becomes infinite before or after the field is switched off. Sections 3 and 4 apply verbatim to a pointwise bounded family $\mathcal{R}$ of continuous functions on a compact metrizable space Y of probes. The family is *stable* if, for all sequences (r_k) in $\mathcal{R}$ and (y_l) in Y , the iterated limits $\lim_l \lim_k r_k(y_l)$ and $\lim_k \lim_l r_k(y_l)$ are equal whenever both exist; by Grothendieck's double limit criterion [Gro52], this holds exactly when every idealized response is continuous, and then the family is NIP₃, by the argument of Proposition 6.5. The family is NIP exactly when every idealized response is of the first Baire class (see Theorem 3.6), and its level is given by Definition 4.4. The families below need not be convex, so Theorem 6.6 does not apply to them; their closures are computed by hand. The physical hypotheses are stated explicitly, and the physics enters only through standard facts, which we cite.

The standard example of an order of limits is spontaneous symmetry breaking: for a ferromagnet below its critical temperature, letting the external field vanish after the volume becomes infinite gives a nonzero magnetization, and the other order gives zero [FV17]. In Theorem 9.2 below, we show that this is an instability in our sense, absent above the critical temperature, and that, with an offset field that moves the transition point, the idealized responses below the critical temperature are NIP₁ but not NIP₂: the two pure phases form a split pair, and the symmetric state is their midpoint, as in the proof of Theorem 6.6.

9.1. The Ising model. We recall the facts we need; all of them can be found in [FV17]. Fix a dimension $d \geq 2$ and an inverse temperature $\beta > 0$. For $N \geq 1$, let $\Lambda_N = \{-N, \dots, N\}^d$ with periodic boundary conditions, and for a real field h let

$$Z_N(h) = \sum_{\sigma \in \{-1,1\}^{\Lambda_N}} \exp\left(\beta \sum_{x \sim y} \sigma_x \sigma_y + h \sum_x \sigma_x\right), \quad p_N(h) = \frac{1}{|\Lambda_N|} \log Z_N(h),$$

where $x \sim y$ ranges over the nearest-neighbor pairs of the torus; in this convention the field h already includes the factor β . The magnetization per site in the Gibbs state is $m_N(h) = p'_N(h)$.

- (F1) The function p_N is smooth, convex and even. Hence the function m_N is continuous, nondecreasing and odd, with $|m_N| \leq 1$ and $m_N(0) = 0$.
- (F2) The limit $p(h) = \lim_N p_N(h)$ exists for every h ; it is convex and even, and it is differentiable at every $h \neq 0$, by the Lee–Yang theorem [LY52]. Since p_N and p are convex, $m_N(h) \rightarrow p'(h)$ at every $h \neq 0$, by Griffiths' lemma on derivatives of convex functions.
- (F3) The spontaneous magnetization $m^*(\beta) = p'(0^+)$ is the right derivative of p at 0, and $p'(0^-) = -m^*(\beta)$. The critical inverse temperature $\beta_c = \inf\{\beta : m^*(\beta) > 0\}$ is positive, and it is finite by the argument of Peierls [Pei36, Gri64]; since $m^*(\beta)$ is the magnetization of the plus state, it is nondecreasing in β , by the inequalities of Griffiths, Kelly and Sherman, and $m^*(\beta) > 0$ for $\beta > \beta_c$ (see [FV17, Chapter 3]).

Since p' is nondecreasing and odd, (F3) gives $p'(t) \geq m^*$ for $t > 0$ and $p'(t) \leq -m^*$ for $t < 0$.

9.2. Preparations and probes. A preparation consists of a finite system Λ_N and an offset field $a \in [-1, 1]$, an idealized uniform field built into the sample, and a probe is an applied field $h \in Y = [-2, 2]$. The offset is a device that moves the transition point; without it, the family $\{m_N\}$ has a single transition point, at $h = 0$. The system then feels the field $h - a$, and the response of the preparation (N, a) is the function

$$r_{N,a}(h) = m_N(h - a) \quad (h \in Y).$$

Let $\mathcal{R} = \{r_{N,a} : N \geq 1, a \in [-1, 1]\}$. Each $r_{N,a}$ is continuous and nondecreasing, and $r_{N,a'} \rightarrow r_{N,a}$ uniformly as $a' \rightarrow a$, because m_N is uniformly continuous on compact sets; hence the responses with rational a are dense in $\mathcal{R}$, and the closure $\overline{\mathcal{R}}$ is separable. Every function in $\overline{\mathcal{R}}$ is nondecreasing, hence of the first Baire class, so the closure $\overline{\mathcal{R}}$ is a Rosenthal compactum, and by Theorem 3.6 the family $\mathcal{R}$ is NIP. The idealized responses are described by the next lemma.

We write $m^* = m^*(\beta)$, and for $a \in [-1, 1]$ and $c \in [-m^*, m^*]$ we let $g_{a,c} : Y \rightarrow \mathbb{R}$ be given by $g_{a,c}(h) = p'(h - a)$ for $h \neq a$ and $g_{a,c}(a) = c$.

Lemma 9.1. *The following hold:*

- (i) every $g_{a,c}$ with $-1 < a < 1$ belongs to $\overline{\mathcal{R}}$;
- (ii) every function in $\overline{\mathcal{R}} \setminus \mathcal{R}$ equals $g_{a,c}$ for some $a \in [-1, 1]$ and $c \in [-m^*, m^*]$.

Proof. (i) Let $-1 < a < 1$. Since $\overline{\mathcal{R}}$ is closed and $g_{a,c} \rightarrow g_{a,\pm m^*}$ pointwise as $c \rightarrow \pm m^*$, it suffices to take $|c| < m^*$, and by symmetry $0 \leq c < m^*$. If $c = 0$, then we have $r_{N,a} \rightarrow g_{a,0}$, by (F1) at $h = a$ and by (F2) elsewhere. If $0 < c < m^*$, fix $\eta > 0$; by (F2), $m_N(\eta) \rightarrow p'(\eta) \geq m^* > c$, so for all large N the intermediate value theorem gives $\varepsilon_N \in (0, \eta)$ with $m_N(\varepsilon_N) = c$, because $m_N(0) = 0$. Since η is arbitrary, we may choose $\varepsilon_N \rightarrow 0$. Put $a_N = a - \varepsilon_N$. Then we have $r_{N,a_N}(a) = m_N(\varepsilon_N) = c$. For $h \neq a$, let $t = h - a$ and fix η' with $0 < \eta' < |t|$; for large N we have $t - \eta' \leq h - a_N \leq t + \eta'$, so by monotonicity $m_N(t - \eta') \leq r_{N,a_N}(h) \leq m_N(t + \eta')$, and by (F2) the limits of the outer terms are $p'(t - \eta')$ and $p'(t + \eta')$, which tend to $p'(t)$ as $\eta' \rightarrow 0$ because p' is continuous at $t \neq 0$. Since $a > -1$, the offsets a_N lie in $[-1, 1]$ for large N , and $r_{N,a_N} \rightarrow g_{a,c}$.

(ii) Let $g \in \overline{\mathcal{R}} \setminus \mathcal{R}$. Since $\overline{\mathcal{R}}$ is a Rosenthal compactum, it is Fréchet–Urysohn [BFT78, Theorem 3F], so the function g is the limit of a sequence r_{N_k, a_k} . Passing to a subsequence, we may assume that $a_k \rightarrow a$ and that either N_k is constant or $N_k \rightarrow \infty$. In the first case $g = r_{N,a} \in \mathcal{R}$, by uniform continuity, which is excluded. In the second case, the argument of the previous paragraph gives $g(h) = p'(h - a)$ for $h \neq a$. Finally, for every $\eta > 0$ and all large k we have $|a - a_k| < \eta$, hence we have $|m_{N_k}(a - a_k)| \leq m_{N_k}(\eta)$, whose limit is $p'(\eta)$; letting $\eta \rightarrow 0$ gives $|g(a)| \leq m^*$. Thus we have $g = g_{a,g(a)}$. $\square$

9.3. The classification detects the phase transition. The lemma reduces the classification of the family to the behavior of p' at 0.

Theorem 9.2. *Let $d \geq 2$ and $\beta > 0$ be given.*

- (a) *If $m^*(\beta) = 0$, in particular if $\beta < \beta_c$, then every idealized response is continuous: the family $\mathcal{R}$ is stable, and $\overline{\mathcal{R}}$ is metrizable.*
- (b) *If $m^*(\beta) > 0$, in particular if $\beta > \beta_c$, then the family $\mathcal{R}$ is unstable, and $\overline{\mathcal{R}}$ is first countable but not hereditarily separable: the family is NIP_1 but not NIP_2 .*

Already the family $\{m_N\}$ without offsets is stable exactly when $m^(\beta) = 0$; it is NIP_3 for every β .*

Proof. (a) If $m^* = 0$, then the pressure p is differentiable everywhere, so its derivative p' is continuous, and by Lemma 9.1(ii) every idealized response has the form $g_{a,0}(h) = p'(h - a)$, hence is continuous. By Grothendieck’s double limit criterion [Gro52], a family of continuous functions on a compact space whose pointwise closure consists of continuous functions has the double limit property, so the family $\mathcal{R}$ is stable. The closure is metrizable by the argument of Proposition 6.5: evaluation at the rational points of Y is a continuous injection on continuous functions.

(b) Let $m^* > 0$. For the family without offsets, take $h_l = 1/l$ and $N_k \rightarrow \infty$. By (F2) and (F3), $\lim_l \lim_k m_{N_k}(h_l) = \lim_l p'(1/l) = m^*$, whereas, by continuity of each m_N and (F1), $\lim_k \lim_l m_{N_k}(h_l) = \lim_k m_{N_k}(0) = 0$. Hence the family $\{m_N\}$, and with it the family $\mathcal{R}$, is unstable.

Every function in $\overline{\mathcal{R}}$ is nondecreasing, and the argument in the proof of Theorem 7.8(b) shows that the space of nondecreasing functions $Y \rightarrow [-1, 1]$ is first countable; hence so is $\overline{\mathcal{R}}$. For $a \in (-1, 1)$, the neighborhood $\{g : |g(a)| < m^*/2\}$ of $g_{a,0}$ contains no $g_{b,0}$ with $b \neq a$, because $|g_{b,0}(a)| = |p'(a - b)| \geq m^*$. Hence the set $\{g_{a,0} : -1 < a < 1\}$ is an uncountable discrete subspace, and $\overline{\mathcal{R}}$ is not hereditarily separable.

The first part of the last statement follows from (a), applied with $a = 0$, and from the computation at the beginning of (b). For the second part, the sequence (m_N) converges

pointwise, by (F1) and (F2), so its closure is countable, and a countable compact Hausdorff space is metrizable. $\square$

The proof has a transparent physical reading. At the transition point $h = a$, the idealized responses take every value c between $-m^*$ and m^* . The extreme values $\pm m^*$ are the magnetizations of the two pure phases, obtained by letting the field reach the transition from one side after the volume has become infinite; they form a split pair. The value $c = (2\lambda - 1)m^*$ is the magnetization of the mixture of the two phases with weights λ and $1 - \lambda$, and it is reached through finite-size rounding of the transition [BK90], by tuning the offset as the volume grows. The uncountable discrete set that destroys hereditary separability consists of the symmetric responses $g_{a,0}$, the midpoints of the split pairs, exactly as in the proof of Theorem 6.6. Above the critical temperature there is no split pair, and the idealized responses are continuous. In this sense the classification sees the phase transition: crossing β_c moves the family with offsets from the stable, metrizable class to the class of NIP_1 families that are not NIP_2 , the class of the Helly space. Without offsets, the family stays NIP_3 , and the transition is seen by stability alone.

9.4. The quantum Ising model. In Theorem 9.6 below, we show that Theorem 9.2 holds verbatim for a Hamiltonian whose terms do not commute: the Ising model in a transverse field. The only ingredient of the classical proof that requires an argument for this model is the differentiability of the pressure away from zero field. It can be derived from the inequality of Griffiths, Hurst and Sherman, which holds for this model through its space-time representation [BG09, Lemma 4.6]; we derive it instead from a Lee–Yang theorem, which gives more, as we show in Propositions 9.4 and 9.5 below: the partition function has no zeros off the imaginary axis of the complex longitudinal field, and the pressure is real analytic away from zero field.

Fix $d \geq 1$, a coupling $J > 0$, a transverse field $\Gamma \geq 0$ and an inverse temperature $\beta > 0$. On the torus Λ_N of Section 9, with Hilbert space $(\mathbb{C}^2)^{\otimes \Lambda_N}$, Pauli matrices σ_i^x, σ_i^z at each site i and $S_N = \sum_i \sigma_i^z$, let

$$H_N = -J \sum_{i \sim j} \sigma_i^z \sigma_j^z - \Gamma \sum_i \sigma_i^x, \quad Z_N(h) = \text{Tr} \exp(-\beta H_N + h S_N),$$

for complex h ; as in Section 9, the longitudinal field h includes the factor β . For real h put $p_N(h) = |\Lambda_N|^{-1} \log Z_N(h)$ and $m_N = p'_N$. For $\Gamma = 0$ this is the classical model with βJ in place of β .

- (Q1) The function p_N is smooth, convex and even, and $m_N(h)$ is the expectation of $S_N/|\Lambda_N|$ in the Gibbs state; hence the function m_N is continuous, nondecreasing and odd, with $|m_N| \leq 1$, because $\|S_N\| = |\Lambda_N|$, and $m_N(0) = 0$. Indeed, by cyclicity of the trace, $\frac{d}{dh} \text{Tr} e^{X+hS} = \text{Tr}(S e^{X+hS})$ for self-adjoint matrices X and S , and $h \mapsto \log \text{Tr} e^{X+hS} = \sup_{\omega} \{\text{Tr} \omega(X+hS) - \text{Tr} \omega \log \omega\}$, the supremum over density matrices ω , is convex as a supremum of affine functions; the unitary $U = \prod_i \sigma_i^x$ commutes with H_N and satisfies $U S_N U^* = -S_N$, so $Z_N(-h) = Z_N(h)$ for all complex h .
- (Q2) The limit $p(h) = \lim_N p_N(h)$ exists for every real h ; it is convex and even. For free boundary conditions the existence is standard [Sim93], and the periodic Hamiltonian differs from the free one by at most J times the number of bonds that wrap around the torus, which is at most $d(2N+1)^{d-1}$, so the two pressures differ by $O(1/N)$, because $|\log \text{Tr} e^A - \log \text{Tr} e^B| \leq \|A - B\|$.

In the proof of Proposition 9.4 below, we show that the partition function of the quantum model is a limit of partition functions of classical Ising models on a larger graph, in which each site of the torus is repeated M times, one copy for each slice of “imaginary time”, by the correspondence between quantum spin systems and classical Ising systems in one more dimension due to Suzuki [Suz76]. The copies of a site are coupled ferromagnetically, so the classical Lee–Yang theorem applies to each of these classical models: none of them vanishes off the imaginary axis. A limit of holomorphic functions without zeros has no zeros either, unless it vanishes identically (see Fact 9.3(b)), and so the quantum partition function inherits the zero-free half-planes. In Proposition 9.5 below, we derive the analyticity of the pressure from a uniform bound and two classical theorems of complex analysis. The proof uses only the following standard facts.

Fact 9.3.

- (a) Lee–Yang theorem [LY52] (see [FV17, Chapter 3]). *Let G be a finite graph with vertex set V and edge set E , let $J_e \geq 0$ for $e \in E$, and let h_v be complex numbers with $\operatorname{Re} h_v > 0$ for $v \in V$. Then $\sum_{s \in \{-1,1\}^V} \exp(\sum_{e=\{u,w\} \in E} J_e s_u s_w + \sum_{v \in V} h_v s_v) \neq 0$.*
- (b) Hurwitz’s theorem [Tit39]. *If holomorphic functions without zeros on a connected open set $U \subseteq \mathbb{C}$ converge uniformly on compact subsets of U to f , then either f has no zeros in U or f vanishes identically.*
- (c) Borel–Carathéodory inequality [Tit39]. *If f is holomorphic on a neighborhood of the closed disc $|z - z_0| \leq R$ and $\operatorname{Re} f \leq A$ there, then $|f(z) - f(z_0)| \leq \frac{2r}{R-r}(A - \operatorname{Re} f(z_0))$ for $|z - z_0| \leq r < R$.*
- (d) Vitali’s theorem [Tit39]. *A sequence of holomorphic functions on a connected open set U that is uniformly bounded on each compact subset of U and converges at every point of a set with an accumulation point in U converges uniformly on compact subsets of U ; by Weierstrass’s theorem, so do the derivatives.*

Proposition 9.4 (Lee–Yang theorem for the quantum Ising model). *For every N , the function Z_N is entire, $Z_N(h) \neq 0$ whenever $\operatorname{Re} h \neq 0$, and $|Z_N(h)| \leq Z_N(\operatorname{Re} h)$.*

Proof. Write $X = \beta J \sum_{i \sim j} \sigma_i^z \sigma_j^z + h S_N$, which is diagonal in the basis $\{|s\rangle : s \in \{-1, 1\}^{\Lambda_N}\}$ of eigenvectors of the operators σ_i^z , and $W = \beta \Gamma \sum_i \sigma_i^x$. The function $Z_N(h) = \operatorname{Tr} e^{X+W}$ is entire, because the exponential series converges locally uniformly in h .

Step 1: the Lie–Trotter formula. For matrices A and B we have $e^{A/M} e^{B/M} = I + (A + B)/M + O(M^{-2})$, uniformly for A and B in a bounded set; taking the principal logarithm, $e^{A/M} e^{B/M} = \exp((A + B)/M + R_M)$ with $\|R_M\| = O(M^{-2})$, and hence we have $(e^{A/M} e^{B/M})^M = \exp(A + B + M R_M) \rightarrow e^{A+B}$, uniformly for A and B in a bounded set. Applied to X and W , this gives $Z_{N,M}(h) = \operatorname{Tr}(e^{X/M} e^{W/M})^M \rightarrow Z_N(h)$ as $M \rightarrow \infty$, uniformly for h in compact subsets of $\mathbb{C}$.

Step 2: a classical ferromagnet. Suppose first that $\Gamma > 0$, and put $\varepsilon = \beta \Gamma / M$. Since $e^{\varepsilon \sigma^x} = \cosh \varepsilon I + \sinh \varepsilon \sigma^x$, the matrix element $\langle s | e^{\varepsilon \sigma^x} | s' \rangle$ of a single spin is $\cosh \varepsilon$ if $s = s'$ and $\sinh \varepsilon$ if $s \neq s'$, that is, $\langle s | e^{\varepsilon \sigma^x} | s' \rangle = c_\varepsilon e^{J_\varepsilon s s'}$ with $c_\varepsilon = (\sinh \varepsilon \cosh \varepsilon)^{1/2} > 0$ and $J_\varepsilon = \frac{1}{2} \log \coth \varepsilon > 0$. Inserting the basis between the factors of $(e^{X/M} e^{W/M})^M$ gives, for $M \geq 3$,

$$Z_{N,M}(h) = c_\varepsilon^{M|\Lambda_N|} \sum_{s^1, \dots, s^M} \exp \sum_{k=1}^M \left(\frac{\beta J}{M} \sum_{i \sim j} s_i^k s_j^k + J_\varepsilon \sum_i s_i^k s_i^{k+1} + \frac{h}{M} \sum_i s_i^k \right),$$

where $s^1, \dots, s^M$ range over $\{-1, 1\}^{\Lambda_N}$ and $s^{M+1} = s^1$. This is $c_\varepsilon^{M|\Lambda_N|}$ times the partition function of a classical Ising model on the finite graph $\Lambda_N \times \{1, \dots, M\}$, with nonnegative couplings and the same field h/M at every site. By Fact 9.3(a), it does not vanish when $\operatorname{Re} h > 0$. Moreover, by the triangle inequality applied to the sum, $|Z_{N,M}(h)| \leq Z_{N,M}(\operatorname{Re} h)$. If $\Gamma = 0$, then the function Z_N itself is a classical Ising partition function, and the same two statements hold directly.

Step 3: Hurwitz's theorem. The functions $Z_{N,M}$ have no zeros in the connected open set $\{\operatorname{Re} h > 0\}$ and converge to Z_N uniformly on its compact subsets, and Z_N is not identically zero there, since $Z_N(1) > 0$. By Fact 9.3(b), Z_N has no zeros in $\{\operatorname{Re} h > 0\}$, and by the evenness in (Q1) none in $\{\operatorname{Re} h < 0\}$. The inequality $|Z_N(h)| \leq Z_N(\operatorname{Re} h)$ follows from Step 2 in the limit. $\square$

Proposition 9.5. *The pressure p is real analytic on $\mathbb{R} \setminus \{0\}$, and $m_N(h) \rightarrow p'(h)$ for every $h \neq 0$.*

Proof. By evenness it suffices to treat $h > 0$. The half-plane $\{\operatorname{Re} h > 0\}$ is simply connected and Z_N does not vanish there, so the function Z_N has a holomorphic logarithm on it, and we let f_N be $|\Lambda_N|^{-1}$ times the logarithm that is real on $(0, \infty)$; thus $f_N = p_N$ on $(0, \infty)$ and $\operatorname{Re} f_N(h) = |\Lambda_N|^{-1} \log |Z_N(h)| \leq p_N(\operatorname{Re} h)$, by Proposition 9.4.

We bound p_N on the real line uniformly in N . The torus has $d|\Lambda_N|$ bonds, so we have $\|\beta H_N\| \leq \beta(dJ + \Gamma)|\Lambda_N|$; since the trace of the exponential of a self-adjoint matrix of size $2^{|\Lambda_N|}$ and norm at most c lies between $2^{|\Lambda_N|}e^{-c}$ and $2^{|\Lambda_N|}e^c$, we get $|p_N(0) - \log 2| \leq \beta(dJ + \Gamma) =: b$, and since $|p'_N| \leq 1$, also $|p_N(x) - \log 2| \leq b + |x|$ for real x .

Fix $x_0 > 0$, and put $R = x_0/2$ and $r = x_0/4$. On the disc $|h - x_0| \leq R$ we have $\operatorname{Re} f_N \leq \log 2 + b + \frac{3}{2}x_0$, and $f_N(x_0) = p_N(x_0) \geq \log 2 - b - x_0$. By the Borel–Carathéodory inequality (see Fact 9.3(c)),

$$|f_N(h) - f_N(x_0)| \leq \frac{2r}{R-r} \left(\sup_{|z-x_0| \leq R} \operatorname{Re} f_N(z) - \operatorname{Re} f_N(x_0) \right) \leq 2(2b + \tfrac{5}{2}x_0)$$

for $|h - x_0| \leq r$. Since also $|f_N(x_0)| \leq \log 2 + b + x_0$, the functions f_N are uniformly bounded on each disc $\{|h - x_0| < x_0/4\}$, and therefore locally uniformly bounded on the connected open set U that is the union of these discs, which contains $(0, \infty)$. They converge on $(0, \infty)$, by (Q2). By Vitali's theorem (see Fact 9.3(d)), they converge uniformly on compact subsets of U to a holomorphic function F , which equals p on $(0, \infty)$; hence the pressure p is real analytic there, and $m_N = f'_N \rightarrow F' = p'$ on $(0, \infty)$. $\square$

Put $m^*(\beta, \Gamma) = p'(0^+)$, the spontaneous magnetization; by evenness, $p'(0^-) = -m^*(\beta, \Gamma)$. With offsets as in Section 9, let $r_{N,a}(h) = m_N(h - a)$ for $N \geq 1$, $a \in [-1, 1]$ and $h \in Y = [-2, 2]$, and let $\mathcal{R}_\Gamma$ be the family of these functions.

Theorem 9.6. *Let $d \geq 1$, $J > 0$, $\Gamma \geq 0$ and $\beta > 0$ be given.*

- (a) *If $m^*(\beta, \Gamma) = 0$, then every idealized response is continuous: the family $\mathcal{R}_\Gamma$ is stable, and its closure is metrizable.*
- (b) *If $m^*(\beta, \Gamma) > 0$, then the family $\mathcal{R}_\Gamma$ is unstable, and its closure is first countable but not hereditarily separable: the family is NIP_1 but not NIP_2 . For each $a \in (-1, 1)$, the functions g_{a, m^*} and $g_{a, -m^*}$ of Lemma 9.1 lie in the closure and differ only at a , and their midpoints $g_{a, 0}$ form an uncountable discrete set.*

The family $\{m_N\}$ without offsets is stable exactly when $m^(\beta, \Gamma) = 0$, and it is NIP_3 in all cases.*

Proof. The discussion preceding Lemma 9.1, the lemma and the proof of Theorem 9.2 use only the following properties: m_N is continuous, nondecreasing and odd, with $|m_N| \leq 1$ and $m_N(0) = 0$; the limit p exists, is convex and even, and is differentiable at every $h \neq 0$, with $m_N(h) \rightarrow p'(h)$; and $p'(0^\pm) = \pm m^*$. These hold by (Q1), (Q2) and Proposition 9.5, and the proofs apply verbatim with $m^*(\beta, \Gamma)$ in place of $m^*(\beta)$. $\square$

In physical terms, $g_{a, \pm m^*}$ are the magnetizations of the two pure phases at the transition point $h = a$, and $g_{a, 0}$ is that of the symmetric state; the classification sees the quantum phase diagram. Let us notice that both cases of the theorem occur: Björnberg and Grimmett [BG09] write the Hamiltonian as $-\frac{1}{2}\lambda \sum \sigma^z \sigma^z - \delta \sum \sigma^x - \gamma \sum \sigma^z$, so that $\lambda = 2J$, $\delta = \Gamma$ and $h = \beta\gamma$, and they define the spontaneous magnetization as the limit, as $\gamma \downarrow 0$, of the infinite-volume magnetization at field γ on the same periodic boxes Λ_N , through the space-time representation. By translation invariance and Proposition 9.5, that magnetization is $p'(\beta\gamma)$, and since the right derivative of a convex function is the right limit of its derivative, their spontaneous magnetization is $p'(0^+) = m^*(\beta, \Gamma)$. For $d = 1$ and $\beta < \infty$ it vanishes, and for $d \geq 2$ and $\Gamma > 0$ there is a critical value of $2J/\Gamma$, depending on $\beta\Gamma$, below which $m^* = 0$ and above which $m^* > 0$ (see [BG09, Section 1 and Theorems 1.1–1.2], where the existence of the critical value is attributed to standard methods). Order at low temperature and small Γ/J goes back to Ginibre [Gin69].

The Lee–Yang theorem [LY52] was extended to general classical ferromagnets by Newman [New74] and by Lieb and Sokal [LS81], and Lee–Yang theorems for quantum spin systems go back to Asano [Asa70] and to Suzuki and Fisher [SF71]. Applied to H_N , these results, in the form given by Fröhlich and Rodriguez [FR12, Theorem 5], exclude zeros only for $\operatorname{Re} h > \beta\Gamma$; the point of Proposition 9.4 is the whole half-plane. We have not found it stated in the literature, but its proof, an approximation of the quantum model by classical ferromagnets in the spirit of Simon and Griffiths [SG73] and of Suzuki [Suz76], combines standard tools, and it may be known to experts.

10. CHIRAL SYMMETRY BREAKING AND METALLIC CONDUCTION ARE INSTABILITIES

The goal of this section is to show that two orders of limits of quantum theory, the Banks–Casher relation for the chiral condensate and the criterion of Scalapino, White and Zhang for superconductivity, are statements about stability. In both, the response is an integral against a family of spectral measures, and in Theorems 10.3 and 10.4 below we show that stability fails exactly when spectral weight accumulates at zero energy. In both, the transition occurs at a single value of the probe, and the next lemma shows that the family is then NIP_3 , whether or not it is stable.

Lemma 10.1. *Let Q be a compact metrizable space, q_0 a point of Q that is not isolated, and $q \mapsto g_q$ a map from Q into $\mathbb{R}^{[0,1]}$ that is continuous for the topology of pointwise convergence.*

Suppose that g_q is continuous for $q \neq q_0$, and that $G = g_{q_0}$ is continuous on $(0, 1]$. Then the family $\mathcal{R} = \{g_q : q \neq q_0\}$ is NIP₃, and it is stable if and only if G is continuous at 0. If $G(0^+) = \lim_{s \rightarrow 0^+} G(s)$ exists, then for $q_k \rightarrow q_0$ with $q_k \neq q_0$ and $s_l \rightarrow 0$ with $s_l > 0$,

$$\lim_l \lim_k g_{q_k}(s_l) = G(0^+), \quad \lim_k \lim_l g_{q_k}(s_l) = G(0).$$

Proof. The image of Q is compact, contains $\mathcal{R}$ and is contained in $\overline{\mathcal{R}}$, because q_0 is not isolated; hence it is $\overline{\mathcal{R}}$. Its members are continuous on $(0, 1]$, so two distinct members differ at a rational point of $[0, 1]$, and $\overline{\mathcal{R}}$ is metrizable by Proposition 4.5. The function G is of the first Baire class, as the pointwise limit of the continuous functions that equal G on $[1/n, 1]$ and are linear on $[0, 1/n]$ with value $G(0)$ at 0; hence the family is NIP, by Theorem 3.6, and it is NIP₃. By Grothendieck's double limit criterion it is stable exactly when every member of $\overline{\mathcal{R}}$ is continuous, that is, exactly when G is continuous at 0. The iterated limits follow from the continuity of $q \mapsto g_q$ and of each g_{q_k} . $\square$

10.1. The chiral condensate. We work with a lattice regularization in a box of Euclidean volume V , with an anti-self-adjoint Dirac operator D_V , as for staggered fermions. Its eigenvalues come in pairs $\pm i\lambda_n$ with $\lambda_n > 0$, and we restrict to the sector of trivial topology, in which there are no exact zero modes; in a sector of topological charge $\nu \neq 0$ the zero modes add $|\nu|/(mV)$ to the condensate, which is then unbounded as $m \rightarrow 0$. The spectral density is averaged over gauge fields with a measure that does not depend on the mass m : either the quenched theory, or the valence condensate at a fixed sea quark mass, which is the setting of the Banks–Casher relation. The chiral condensate at quark mass $m > 0$ is

$$\Sigma_V(m) = \frac{1}{V} \text{Tr}(D_V + m)^{-1} = \int \frac{2m}{m^2 + \lambda^2} d\kappa_V(\lambda), \quad \kappa_V = \frac{1}{V} \sum_n \delta_{\lambda_n},$$

and we put $\Sigma_V(0) = 0$, the value given by continuity. We assume:

- (C1) each κ_V is a finitely supported positive measure on $(0, \Lambda]$, of total mass at most M_0 , where the cutoff Λ and the bound M_0 do not depend on V ;
- (C2) as $V \rightarrow \infty$ along a given sequence of volumes, κ_V converges weakly on $[0, \Lambda]$ to a measure κ that, on some interval $[0, \delta]$, has a density f continuous at 0.

Hypothesis (C1) holds on the lattice. Hypothesis (C2) is the physical assumption at issue, the existence of a limiting density of near-zero eigenvalues per unit volume; its value at 0 is $f(0)$. A preparation is a volume V , a probe is a mass $m \in [0, 1]$, and the response is Σ_V . Each Σ_V is continuous on $[0, 1]$, because κ_V has finitely many atoms, all at positive λ .

Lemma 10.2. *Assume (C1) and (C2). Then $\Sigma_V(m) \rightarrow \Sigma(m) = \int \frac{2m}{m^2 + \lambda^2} d\kappa(\lambda)$ for every $m > 0$, the function Σ is continuous on $(0, 1]$, and $\Sigma(m) \rightarrow \pi f(0)$ as $m \rightarrow 0^+$.*

Proof. For fixed $m > 0$, the function $\lambda \mapsto 2m/(m^2 + \lambda^2)$ is continuous and bounded on $[0, \Lambda]$, so the first statement is weak convergence, and the second follows from dominated convergence. For the third, choose δ so small that f is bounded on $[0, \delta]$. On $(\delta, \Lambda]$ the integrand is at most $2m/\delta^2$, so that part tends to 0. On $[0, \delta]$, the substitution $\lambda = mu$ gives

$$\int_0^\delta \frac{2m}{m^2 + \lambda^2} f(\lambda) d\lambda = \int_0^{\delta/m} \frac{2}{1 + u^2} f(mu) du,$$

which tends to $f(0) \int_0^\infty \frac{2 du}{1 + u^2} = \pi f(0)$ by dominated convergence. The measure κ has no atom at 0, because it has a density near 0. $\square$

Theorem 10.3. *Assume (C1) and (C2).*

(a) *The iterated limits of the condensate are*

$$\lim_{m \rightarrow 0} \lim_{V \rightarrow \infty} \Sigma_V(m) = \pi f(0), \quad \lim_{V \rightarrow \infty} \lim_{m \rightarrow 0} \Sigma_V(m) = 0.$$

(b) *The family $\{\Sigma_V\}$ is stable if and only if $f(0) = 0$, that is, if and only if chiral symmetry is not spontaneously broken [BC80].*

(c) *In all cases the family is NIP_3 .*

Proof. Let Q consist of the numbers $1/V$ along the sequence of volumes, together with $q_0 = 0$, put $g_{1/V} = \Sigma_V$, and let $G(m) = \Sigma(m)$ for $m > 0$ and $G(0) = 0$. By Lemma 10.2 and $\Sigma_V(0) = 0$, the map $q \mapsto g_q$ is continuous, G is continuous on $(0, 1]$, and $G(0^+) = \pi f(0)$. Apply Lemma 10.1. $\square$

Letting the volume grow before the mass vanishes keeps a density of near-zero modes, and the condensate is $\pi f(0)$; the other order removes every mode from the neighborhood of 0 before the mass vanishes, and the condensate is 0. When the mass is tied to the volume, as in the microscopic regime of Leutwyler and Smilga, where $\mu = mV\Sigma$ is held fixed [LS92, VW00], the condensate takes values strictly between 0 and $\pi f(0)$. As for the Ising model, these values become idealized responses only when there is a continuum of transition points (see Section 11).

10.2. The current response. Let H be the Hamiltonian of a clean many-body system at zero temperature, with ground state Ω , and let j_q be the transverse current operator at wave vector q , normalized per unit volume. For a finite system these are operators and a vector; for an infinite system, the objects below are the infinite-volume limits of the corresponding finite-volume spectral measures, and the theorem is a statement about these measures. The paramagnetic current response at imaginary frequency $i\omega$ has the spectral (Lehmann) representation

$$\Pi(q, i\omega) = \int \frac{2\nu}{\nu^2 + \omega^2} d\mu_q(\nu),$$

where μ_q is the spectral measure of $H - E_0$ in the vector $j_q\Omega$, restricted to $(0, \infty)$; for a finite system, $\mu_q = \sum_n |\langle n, j_q\Omega \rangle|^2 \delta_{E_n - E_0}$. The total response is $\mathcal{K}(q, i\omega) = D_0 - \Pi(q, i\omega)$, where D_0 is the diamagnetic term, the expectation of minus the kinetic energy along the field on a lattice, and n/m in the continuum. Following Scalapino, White and Zhang, the *superfluid weight* and the *Drude weight* are, up to the factor πe^2 of [SWZ93],

$$D_s = \lim_{q \rightarrow 0} \mathcal{K}(q, 0), \quad D = \lim_{\omega \rightarrow 0} \lim_{q \rightarrow 0} \mathcal{K}(q, i\omega).$$

The superfluid weight is the transverse static limit of Scalapino, White and Zhang. Their Drude weight, and that of Kohn [Koh64], is taken at $q = 0$ and continued from imaginary to real frequency; it agrees with D when the measures below depend continuously on q at $q = 0$ away from zero energy, and the imaginary-frequency limit agrees with the real-frequency one. In an insulator $D_s = D = 0$; in a clean normal metal $D > 0 = D_s$; in a clean superconductor $D_s = D > 0$ [SWZ93]. In a disordered metal the Drude weight vanishes, and at positive temperature the normal fluid gives $D > D_s$ even in a superconductor; both cases lie outside the hypotheses below.

It is convenient to absorb a factor into the measure: put $d\zeta_q(\nu) = \frac{2}{\nu} d\mu_q(\nu)$, so that

$$\Pi(q, i\omega) = \int \frac{\nu^2}{\nu^2 + \omega^2} d\zeta_q(\nu), \quad \Pi(q, 0) = \zeta_q((0, \Lambda]).$$

We assume:

- (S1) each ζ_q , for $0 < q \leq 1$, is a positive measure on $(0, \Lambda]$ of total mass at most D_0 , where the bandwidth Λ does not depend on q ;
- (S2) regarding each ζ_q as a measure on $[0, \Lambda]$, the map $q \mapsto \int h d\zeta_q$ is continuous on $(0, 1]$ and converges to $\int h d\zeta_0$ as $q \rightarrow 0$, for a measure ζ_0 on $[0, \Lambda]$ and every $h \in C[0, \Lambda]$.

The bound in (S1) is the statement $\mathcal{K}(q, 0) \geq 0$, that the ground state is a local minimum of the energy as a function of a static vector potential. This is an assumption: for lattice fermions it can fail, as Lieb's flux-phase theorem shows [Lie94]. The compact support in (S1) is a convenience: since $\nu^2/(\nu^2 + \omega^2)$ extends continuously to $[0, \infty]$, the proof below works for tight families of measures on $(0, \infty)$. The limit ζ_0 may have an atom at 0: weight of current-carrying excitations whose energy tends to 0 with q . A preparation is a wave vector $q \in (0, 1]$, a probe is a frequency $\omega \in [0, 1]$, and the response is $g_q(\omega) = \Pi(q, i\omega)$.

Theorem 10.4. *Assume (S1) and (S2).*

- (a) *We have $D_s = D_0 - \zeta_0([0, \Lambda])$ and $D = D_0 - \zeta_0((0, \Lambda])$, so that $D - D_s = \zeta_0(\{0\})$.*
- (b) *The family of responses $\{g_q : 0 < q \leq 1\}$ is stable if and only if $D = D_s$. In particular, the responses of a clean superconductor and of an insulator are stable, and those of a clean normal metal are unstable.*
- (c) *In all cases the family is NIP₃.*

Proof. Write $\kappa_\omega(\nu) = \nu^2/(\nu^2 + \omega^2)$. For $\omega > 0$ the function κ_ω is continuous on $[0, \Lambda]$ with $\kappa_\omega(0) = 0$, while $\kappa_0 = 1$ on $(0, \Lambda]$. Each g_q is continuous on $[0, 1]$, by dominated convergence. By (S2), applied to the continuous function 1, we have $g_q(0) = \zeta_q((0, \Lambda]) \rightarrow \zeta_0([0, \Lambda])$, which gives D_s ; applied to κ_ω with $\omega > 0$, it gives $g_q(\omega) \rightarrow G(\omega) = \int_{(0, \Lambda]} \kappa_\omega d\zeta_0$, and $G(\omega) \rightarrow \zeta_0((0, \Lambda])$ as $\omega \rightarrow 0$, which gives D . This proves (a). Put $g_0 = G$ with $G(0) = \zeta_0([0, \Lambda])$. The map $q \mapsto g_q$ is continuous on $[0, 1]$, by (S2) and the computation just made, and G is continuous on $(0, 1]$ with $G(0) - G(0^+) = \zeta_0(\{0\}) = D - D_s$. Lemma 10.1 gives (b) and (c). $\square$

The difference $D - D_s$ is the weight of current-carrying excitations whose energy tends to zero with the wave vector, the gapless quasiparticles of a metal, and the presence of such weight is exactly the failure of stability. In a superconductor the gap keeps this weight away from zero energy, and in an insulator there is no current-carrying weight at low energy.

11. QUESTIONS

- (1) *Light cones.* In Theorem 7.10 Bob detects at a fixed time. Locate the causal order in the classification when Bob's detection time is also uncertain, and in $1 + d$ dimensions.
- (2) *The minimal families of the box.* By Theorem 8.5, the comparison of positions in a box realizes D_3 , D_4 and D_7 . Does it realize D_6 ? Which of D_6 and D_7 does Theorem 7.8(b) produce from a given dense sequence?
- (3) *Noncommutative probes.* Find a natural observable that is NIP but unstable and whose probe algebra is not commutative, and locate it in the classification. Can noncommutativity produce a phenomenon in the classification that observables with commutative

probe algebras cannot, for instance in which minimal families of the heptachotomy arise from a given dense sequence of preparations?

- (4) *Orientations.* By Theorem 7.12, the comparison of phases on a circle is NIP but not NIP_1 . Locate in the classification the comparison of directions on a sphere, “Bob’s direction lies in the open hemisphere centered at Alice’s”, and relative orientations in higher dimensions.
- (5) *Finitary levels.* Is there a finitary, combinatorial characterization of NIP_1 , NIP_2 and NIP_3 , in the spirit of the fat-shattering dimension for NIP, sharpening Proposition 4.6?
- (6) *Quantum phase transitions.* Extend Theorem 9.6 to the ground state of the Ising model in a transverse field, where the transition is driven by Γ at zero temperature.
- (7) *A continuum of transitions.* Is there a natural family of chiral condensates with a continuum of transition points, obtained for instance from twisted boundary conditions or a chemical potential, that is NIP_1 but not NIP_2 ?
- (8) *Positive temperature.* Extend Theorem 10.4 to positive temperature, where the Matsubara frequencies are discrete and the Drude weight is defined by extrapolation.
- (9) *Unbounded formulas.* Classify the energy functional $\rho \mapsto \text{Tr}(\rho H)$, with the energy constraints $\text{Tr}(\rho H) \leq E$ as shards, in the setting of Keisler.
- (10) *Forcing.* Can the Todorćević trichotomy, or its consequences for observables, be proved without forcing?

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