

OMITTING TYPES FOR INFINITARY REAL-VALUED LOGIC: A TOPOLOGICAL PERSPECTIVE

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ABSTRACT. We prove an omitting types theorem for countable fragments of infinitary logic for real-valued structures in the sense of Keisler, structures on which no metric is given. The proof is topological: for a countable vocabulary and a countable fragment, the Henkin types form a Čech-complete space, and the required models are obtained from the Baire category theorem.

1. INTRODUCTION

Morley [Mor74] proved the omitting types theorem for countable fragments of $L_{\omega_1\omega}$ by applying the Baire category theorem to a space of countable structures, which he showed to be a Baire space. Following the topological argument of Caicedo and Iovino [CI14] for $[0, 1]$ -valued logic with or without a metric, Eagle [Eag14] extended this proof to infinitary $[0, 1]$ -valued logic for metric structures. In the present paper we extend it to the infinitary logic of Keisler's model theory for real-valued structures [Kei23], in which structures carry no metric and predicates take values in bounded intervals that vary from predicate to predicate; the Baire category theorem is applied to the space of Henkin types, which is Čech-complete when the vocabulary and the fragment are countable.

Theorem A. If the vocabulary is countable, a formal type for a set of sentences of the infinitary language is realized by a structure whose sorts are countable if and only if the type extends to a Henkin type (see Proposition 4, which also treats countable fragments).

Theorem B. If the vocabulary and the fragment are countable, a theory that has a model has one whose sorts are countable and in which no tuple satisfies any of countably many given types, provided that, for each instance of each of these types, the Henkin types that satisfy the theory and the instance form a set with empty interior in the space of Henkin types that satisfy the theory (see Theorem 8).

Organization of the paper. Section 2 is devoted to Čech's form of the Baire category theorem, and Sections 3–8 to the logic. In Section 9 we introduce Henkin types and prove Theorem A; in Section 10 we show that the space of Henkin types is Čech-complete, and in Section 11 we prove Theorem B.

2. TOPOLOGICAL PRELIMINARIES

The topological fact on which the paper rests is Čech’s form of the Baire category theorem, which applies to a class of spaces containing every countable intersection of open subsets of a compact Hausdorff space. The argument of the paper takes place in the space of Henkin types for the sentences of a fragment of the infinitary language (see Section 9), which, as we show in Section 10, is such an intersection when the vocabulary and the fragment are countable. In this section we recall the definitions that Čech’s theorem involves, with Engelking’s *General Topology* [Eng89] as our reference for topology.

A topological space is a **Baire space** if the intersection of every sequence of dense open subsets of the space is dense; Baire spaces were introduced by Bourbaki in 1948 (see [Eng89, notes to Section 3.9]). The spaces to which this property will be applied are Tychonov spaces, that is, completely regular T_1 -spaces; every subspace of a compact Hausdorff space is one (see [Eng89, Theorem 3.2.6]). Let X be a Tychonov space. A sequence $(\mathcal{U}_n)_{n < \omega}$ of open covers of X is **complete** if every family $(F_i)_{i \in I}$ of closed subsets of X that has the finite intersection property, and that has, for every $n < \omega$, a member contained in some element of $\mathcal{U}_n$, has nonempty intersection. The space X is **Čech-complete** if it has a complete sequence of open covers. Čech [Čec37] introduced these spaces in 1937, under the name of topologically complete spaces, as the Tychonov spaces that are G_δ subsets of their Stone–Čech compactifications; their characterization by sequences of open covers, adopted here as the definition, was established independently by Frolík in 1960 and by Arhangel’skiĭ in 1961 (see [Eng89, Theorem 3.9.2 and the notes to Section 3.9]). By Theorems 3.9.1 and 3.9.2 of Engelking [Eng89], a Tychonov space is Čech-complete if and only if it is a G_δ subset of some, or equivalently of every, compactification of it. Consequently, every G_δ subset X of a compact Hausdorff space K is Čech-complete: the closure of X in K is compact (see [Eng89, Theorem 3.1.2]), and is therefore a compactification of X in which X is a G_δ set. The compact Hausdorff spaces in which such subsets will lie are products $\prod_{i \in I} [a_i, b_i]$, where I is a set and $a_i \leq b_i$ are real numbers for each $i \in I$, endowed with the product topology; each interval $[a_i, b_i]$ is compact (see [Eng89, Theorem 3.2.8]), and so is the product, by Tychonov’s theorem (see [Eng89, Theorem 3.2.4]). Čech also proved the following form of the Baire category theorem (see [Eng89, Corollary 3.9.4 and the notes to Section 3.9]).

Theorem 1 (Baire category theorem; Čech [Čec37]). Every Čech-complete space is a Baire space.

3. VOCABULARIES AND STRUCTURES

The models whose existence the omitting types theorem asserts are real-valued structures. Such a structure consists of one or more nonempty sets, called its sorts, of functions from finite products of sorts to sorts, and of bounded real-valued functions on finite products of sorts, called its predicates; classical first-order structures

correspond to the case in which every predicate takes only the values 1 and -1 , read as true and false, respectively.

Formulas speak about structures through a vocabulary, a set of symbols for sorts, functions and predicates; a structure for the vocabulary interprets each symbol as an actual sort, function or predicate. Since symbols must be kept apart from their interpretations, the symbols of a vocabulary are printed in **this color**, the variables, the connectives and the formal expressions built from them in **this color**, and mathematical objects in black.

A **sort descriptor** is a purely syntactic symbol M . Let $\mathbb{M}$ be a nonempty set of sort descriptors. A **function symbol for** $\mathbb{M}$ is a purely syntactic symbol f given together with

- (i) a natural number $n = n_f$, called the **arity of** f , and
- (ii) an $(n+1)$ -tuple $(M_i)_{i \leq n}$ of sort descriptors in $\mathbb{M}$, called the **extended arity of** f .

We indicate the arity and the extended arity of f by writing

$$f : \prod_{i < n} M_i \rightarrow M_n.$$

A function symbol of arity 0 is called a **constant symbol of sort** M_0 . A **predicate symbol for** $\mathbb{M}$ is a purely syntactic symbol p given together with

- (i) a natural number $n = n_p$, called the **arity of** p ,
- (ii) an n -tuple $(M_i)_{i < n}$ of sort descriptors in $\mathbb{M}$, called the **extended arity of** p , and
- (iii) a real number $r = r_p \geq 0$, called the **a priori bound of** p .

We indicate the arity, the extended arity and the a priori bound of p by writing

$$p : \prod_{i < n} M_i \rightarrow [-r, r].$$

A **vocabulary** is a triple $V = (\mathbb{M}, F, P)$ consisting of a nonempty set $\mathbb{M}$ of sort descriptors, a set F of function symbols for $\mathbb{M}$, and a set P of predicate symbols for $\mathbb{M}$ that contains a nullary predicate symbol 1 , called the **logical unity constant**, with a priori bound $r_1 = 1$. The vocabulary V is **countable** if the sets $\mathbb{M}$, F and P are countable. The a priori bounds provide the compactness on which the topology of the paper rests: as we show in Section 4, the value of a formula in a structure lies in an interval $[-r, r]$ determined by the formula alone, whence the types of all structures lie in a product of compact intervals, the compact Hausdorff space of formal types of Section 8.

Let $V = (\mathbb{M}, F, P)$ be a vocabulary. A V -**structure**, or real-valued structure with vocabulary V , is a triple

$$\mathfrak{M} = \langle (M^{\mathfrak{M}})_{M \in \mathbb{M}}, (f^{\mathfrak{M}})_{f \in F}, (p^{\mathfrak{M}})_{p \in P} \rangle$$

in which

- ♠ the interpretation $M^{\mathfrak{M}}$ of each sort descriptor $M \in \mathbb{M}$ is a nonempty set,

- ♠ the interpretation of each function symbol $f : \prod_{i < n} M_i \rightarrow M_n$ in F is a function

$$f^{\mathfrak{M}} : \prod_{i < n} M_i^{\mathfrak{M}} \rightarrow M_n^{\mathfrak{M}},$$

- ♠ the interpretation of each predicate symbol $p : \prod_{i < n} M_i \rightarrow [-r_p, r_p]$ in P is a function

$$p^{\mathfrak{M}} : \prod_{i < n} M_i^{\mathfrak{M}} \rightarrow [-r_p, r_p],$$

and

- ♠ the interpretation of the logical unity constant is $1^{\mathfrak{M}} = 1$.

Here a nullary symbol is interpreted on the one-point set $\prod_{i < 0} M_i^{\mathfrak{M}}$, and its interpretation is identified with its value.

4. FIRST-ORDER AND INFINITARY LANGUAGES

Types, and later Henkin types, are formed from the values that structures assign to formulas; we therefore build, from the symbols of a vocabulary, the terms and the formulas of a first-order language and of its infinitary extension, and interpret them in structures.

For the rest of the paper, we fix a vocabulary $V = (\mathbb{M}, F, P)$.

For each sort descriptor $M \in \mathbb{M}$, we fix a sequence $(x_i^M)_{i < \omega}$ of distinct symbols, not in V , chosen so that the sequences for distinct sort descriptors have no symbol in common. Each symbol x_i^M is called a **variable of sort M** .

Terms of sort M are defined recursively as follows:

- ♠ every variable of sort M is a term of sort M ;
- ♠ if $f : \prod_{i < n} M_i \rightarrow M$ is a function symbol in F and $\bar{\tau} = (\tau_i)_{i < n}$ is a tuple of terms such that τ_i is of sort M_i for each $i < n$, then $f(\bar{\tau})$ is a term of sort M .

In the second clause, we also say that the tuple $\bar{\tau}$ is a term of sort $\prod_{i < n} M_i$. In particular, every constant symbol of sort M is a term of sort M . A term is **closed** if no variable occurs in it.

If $p : \prod_{i < n} M_i \rightarrow [-r_p, r_p]$ is a predicate symbol in P and $\bar{\tau}$ is a term of sort $\prod_{i < n} M_i$, then $p(\bar{\tau})$ is called an **atomic V -formula** with **a priori bound** $r_{p(\bar{\tau})} := r_p$. The other V -formulas are formed from the atomic ones by means of symbols called **logical connectives**:

- ♠ for each rational number s , the symbol $s \cdot$ is a logical connective;
- ♠ the symbol $\vee$ is a logical connective;
- ♠ for each variable x , the symbol $\sup_x$ is a logical connective.

The V -formulas and their a priori bounds are defined recursively: every atomic V -formula is a V -formula, and if φ and ψ are V -formulas with a priori bounds r_φ and r_ψ , respectively, then, for every rational number s and every variable x , the expressions $s \cdot \varphi$, $\varphi \vee \psi$ and $\sup_x \varphi$ are V -formulas with a priori bounds

$$r_{s \cdot \varphi} := |s| r_\varphi, \quad r_{\varphi \vee \psi} := \max\{r_\varphi, r_\psi\}, \quad r_{\sup_x \varphi} := r_\varphi,$$

respectively. The set of all V -formulas is called the **first-order language for V** and is denoted by L^V .

The set $\text{free}(\xi)$ of **variables free in a V -formula ξ** is defined recursively as follows:

- ♠ if ξ is atomic, of the form $p(\bar{\tau})$, then $\text{free}(\xi)$ is the set of variables that occur in the terms of the tuple $\bar{\tau}$;
- ♠ if ξ is of the form $s \cdot \varphi$, then $\text{free}(\xi) := \text{free}(\varphi)$;
- ♠ if ξ is of the form $\varphi \vee \psi$, then $\text{free}(\xi) := \text{free}(\varphi) \cup \text{free}(\psi)$;
- ♠ if ξ is of the form $\sup_x \varphi$, then $\text{free}(\xi) := \text{free}(\varphi) \setminus \{x\}$.

A **V -sentence** is a V -formula with no free variables, that is, a V -formula φ such that $\text{free}(\varphi) = \emptyset$.

For a term τ and a tuple $\bar{x}$ of distinct variables, we write $\tau(\bar{x})$ to indicate that every variable occurring in τ is among $\bar{x}$; similarly, for a formula φ , we write $\varphi(\bar{x})$ to indicate that every variable free in φ is among $\bar{x}$. If $\bar{x} = (x_i)_{i < n}$, where x_i is a variable of sort M_i for each $i < n$, we write

$$\bar{x}M := \prod_{i < n} M_i$$

and say that $\bar{x}$ is a variable of sort $\bar{x}M$; for a V -structure $\mathfrak{M}$, we write

$$\bar{x}M^{\mathfrak{M}} := \prod_{i < n} M_i^{\mathfrak{M}}.$$

The infinitary language $L_{\omega_1\omega}^V$ is obtained by adding to the formation rules of L^V a new logical connective $\bigvee$, called the **infinitary connective of countable join**: the language $L_{\omega_1\omega}^V$ is the smallest set of expressions that contains the atomic V -formulas, is closed under the connectives $s \cdot$, $\vee$ and $\sup_x$, for all rational numbers s and all variables x , and contains the expression $\bigvee \Phi$ for every sequence $\Phi = (\varphi_n)_{n < \omega}$ of its members such that, for some finite tuple $\bar{x}$ of variables, every φ_n is of the form $\varphi_n(\bar{x})$, and such that $\sup_{n < \omega} r_{\varphi_n}$ is finite. The a priori bounds and the free variables of the formulas $s \cdot \varphi$, $\varphi \vee \psi$ and $\sup_x \varphi$ are given by the clauses stated for L^V ; the formula $\bigvee \Phi$ has the a priori bound $r_{\bigvee \Phi} := \sup_{n < \omega} r_{\varphi_n}$, and its free variables are those of the formulas φ_n , that is, $\text{free}(\bigvee \Phi) := \bigcup_{n < \omega} \text{free}(\varphi_n)$, a subset of the finite set of variables of $\bar{x}$.

We also write $\bigvee_n \varphi_n$ for $\bigvee \Phi$, and we denote the set of sentences of $L_{\omega_1\omega}^V$ by $S_{\omega_1\omega}^V$. A formula of $L_{\omega_1\omega}^V$ may have infinite length, but only finitely many free variables. The two languages differ in size: if V is countable, the first-order language L^V is countable, whereas the infinitary language $L_{\omega_1\omega}^V$ has the cardinality of the continuum; for this reason, the Baire category argument of this paper takes place in countable fragments (see Section 6).

Let $\mathfrak{M}$ be a V -structure. For a term $\tau(\bar{x})$ of sort M , where $\bar{x} = (x_j)_{j < m}$, the **interpretation of $\tau(\bar{x})$ in $\mathfrak{M}$** is the function $\tau^{\mathfrak{M}} : \bar{x}M^{\mathfrak{M}} \rightarrow M^{\mathfrak{M}}$ defined recursively, for $\bar{a} = (a_j)_{j < m}$ in $\bar{x}M^{\mathfrak{M}}$, as follows:

- ♠ the interpretation of the variable x_j is the projection $x_j^{\mathfrak{M}}(\bar{a}) := a_j$;

- ♠ if $f : \prod_{i < n} M_i \rightarrow M$ is a function symbol and $\bar{\tau} = (\tau_i)_{i < n}$ is a term of sort $\prod_{i < n} M_i$, then the interpretation of the term $f(\bar{\tau})$ is given by $(f(\bar{\tau}))^{\mathfrak{M}}(\bar{a}) := f^{\mathfrak{M}}((\tau_i^{\mathfrak{M}}(\bar{a}))_{i < n})$.

The interpretation of a closed term τ is a function on the one-point set ${}_{\bar{x}}M^{\mathfrak{M}}$ with $\bar{x}$ empty, and is identified with its value, an element of $M^{\mathfrak{M}}$. The **interpretation of a formula** $\xi(\bar{x})$ of $L_{\omega_1\omega}^V$ in $\mathfrak{M}$ is the function $\xi^{\mathfrak{M}}$ on ${}_{\bar{x}}M^{\mathfrak{M}}$ defined recursively, for $\bar{a}$ in ${}_{\bar{x}}M^{\mathfrak{M}}$, as follows:

- ♠ if ξ is atomic, of the form $p(\bar{\tau})$ with $\bar{\tau} = (\tau_i)_{i < n}$, then

$$\xi^{\mathfrak{M}}(\bar{a}) := p^{\mathfrak{M}}((\tau_i^{\mathfrak{M}}(\bar{a}))_{i < n});$$

- ♠ if ξ is of the form $s \cdot \varphi$, where s is a rational number and $\varphi \in L_{\omega_1\omega}^V$, then

$$\xi^{\mathfrak{M}}(\bar{a}) := s \cdot \varphi^{\mathfrak{M}}(\bar{a});$$

- ♠ if ξ is of the form $\varphi \vee \psi$, where $\varphi, \psi \in L_{\omega_1\omega}^V$, then

$$\xi^{\mathfrak{M}}(\bar{a}) := \max\{\varphi^{\mathfrak{M}}(\bar{a}), \psi^{\mathfrak{M}}(\bar{a})\};$$

- ♠ if ξ is of the form $\sup_{\bar{y}} \varphi(\bar{x}\bar{y})$, where $\varphi(\bar{x}\bar{y}) \in L_{\omega_1\omega}^V$ and $\bar{y}$ is not among $\bar{x}$, then

$$\xi^{\mathfrak{M}}(\bar{a}) := \sup_{b \in {}_{\bar{y}}M^{\mathfrak{M}}} \varphi^{\mathfrak{M}}(\bar{a}b);$$

- ♠ if ξ is of the form $\bigvee \Phi$, where $\Phi = (\varphi_n(\bar{x}))_{n < \omega}$ is a sequence of formulas of $L_{\omega_1\omega}^V$, then

$$\xi^{\mathfrak{M}}(\bar{a}) := \sup_{n < \omega} \varphi_n^{\mathfrak{M}}(\bar{a}).$$

An induction on formulas shows that the suprema in this definition are finite and that $\xi^{\mathfrak{M}}$ takes its values in $[-r_\xi, r_\xi]$. When ξ is a sentence, the tuple $\bar{x}$ may be taken empty, and $\xi^{\mathfrak{M}}$ is identified with its value, a real number in $[-r_\xi, r_\xi]$.

5. EXPANSION BY CONSTANTS

In the method introduced by Henkin in 1949, the elements of a structure are named by new constant symbols; the first step is to add these symbols to the fixed vocabulary. Let $C = \{c_n^M : n < \omega, M \in \mathbb{M}\}$ be a set of distinct constant symbols, not in V , such that c_n^M is of sort M for every $n < \omega$ and every $M \in \mathbb{M}$. The **expansion of V by C** is the vocabulary $V_C := (\mathbb{M}, F \cup C, P)$, which is countable if V is; the infinitary language $L_{\omega_1\omega}^{V_C}$ is denoted by $L_{\omega_1\omega}^C$, and its set of sentences by $S_{\omega_1\omega}^C$.

Let $\mathfrak{M} = \langle (M^{\mathfrak{M}})_{M \in \mathbb{M}}, (f^{\mathfrak{M}})_{f \in F}, (p^{\mathfrak{M}})_{p \in P} \rangle$ be a V -structure. An **expansion of $\mathfrak{M}$ to V_C** is a V_C -structure

$$\mathfrak{N} = \langle (M^{\mathfrak{N}})_{M \in \mathbb{M}}, (f^{\mathfrak{N}})_{f \in F \cup C}, (p^{\mathfrak{N}})_{p \in P} \rangle$$

that agrees with $\mathfrak{M}$ on the symbols of V , that is, such that $M^{\mathfrak{N}} = M^{\mathfrak{M}}$ for every $M \in \mathbb{M}$, $f^{\mathfrak{N}} = f^{\mathfrak{M}}$ for every $f \in F$, and $p^{\mathfrak{N}} = p^{\mathfrak{M}}$ for every $p \in P$; such an expansion is determined by the elements $(c_n^M)^{\mathfrak{N}}$ of $M^{\mathfrak{N}}$ that interpret the new

constant symbols. Conversely, every V_C -structure $\mathfrak{N}$ is an expansion of exactly one V -structure, its V -**reduct**, obtained by forgetting the interpretations of the symbols in C .

In the structures built by the Henkin construction, every element has a name. A V_C -structure $\mathfrak{N}$ is **canonical** if, for every sort descriptor $M \in \mathbb{M}$, every element of $M^{\mathfrak{N}}$ is the interpretation in $\mathfrak{N}$ of a closed V_C -term of sort M .

6. FRAGMENTS

The Baire category argument requires the Henkin types of Section 9 to form a G_δ subset of a compact space; in Section 10 we write them as an intersection of open sets attached to sentences, an intersection that is countable only when the sentences are countable in number, whereas the infinitary language has the cardinality of the continuum even when the vocabulary is countable (see Section 4). The argument is therefore carried out in fragments, sets of formulas closed under the first-order connectives, under subformulas and under substitution of terms, which are countable when they are generated by countably many formulas over a countable vocabulary.

Let ξ be a formula of $L_{\omega_1\omega}^V$. The set $\text{sub}(\xi)$ of **subformulas of ξ** is defined recursively as follows:

- ♠ if ξ is atomic, we set $\text{sub}(\xi) := \{\xi\}$;
- ♠ if ξ is of the form $s \cdot \varphi$, we set $\text{sub}(\xi) := \{\xi\} \cup \text{sub}(\varphi)$;
- ♠ if ξ is of the form $\varphi \vee \psi$, we set $\text{sub}(\xi) := \{\xi\} \cup \text{sub}(\varphi) \cup \text{sub}(\psi)$;
- ♠ if ξ is of the form $\sup_x \varphi$, we set $\text{sub}(\xi) := \{\xi\} \cup \text{sub}(\varphi)$;
- ♠ if ξ is of the form $\bigvee_n \varphi_n$, we set $\text{sub}(\xi) := \{\xi\} \cup \bigcup_{n < \omega} \text{sub}(\varphi_n)$.

A **fragment** of $L_{\omega_1\omega}^V$ is a set $\mathcal{F} \subseteq L_{\omega_1\omega}^V$ with the following properties:

- (F1) it contains L^V ;
- (F2) it is closed under subformulas, that is, $\text{sub}(\varphi) \subseteq \mathcal{F}$ for every $\varphi \in \mathcal{F}$;
- (F3) it is closed under the first-order connectives, that is, if $\varphi, \psi \in \mathcal{F}$, then $s \cdot \varphi \in \mathcal{F}$ for every rational number s , $\varphi \vee \psi \in \mathcal{F}$, and $\sup_x \varphi \in \mathcal{F}$ for every variable x ;
- (F4) it is closed under substitution of terms, that is, if $\varphi(\bar{x}y) \in \mathcal{F}$ and τ is a term of sort ${}_y M$ that is free for y in φ (no free occurrence of y in φ lies within a subformula $\sup_z \psi$ of φ such that z occurs in τ), then the formula $\varphi(\bar{x}\tau)$, obtained from φ by substituting τ for the free occurrences of y , belongs to $\mathcal{F}$.

The intersection of a nonempty family of fragments of $L_{\omega_1\omega}^V$ is a fragment, and $L_{\omega_1\omega}^V$ is itself a fragment; hence, for every set $\Phi \subseteq L_{\omega_1\omega}^V$, the intersection of all fragments containing Φ is the smallest fragment containing Φ , which is called the **fragment generated by Φ** . If V and Φ are countable, then the fragment generated by Φ is countable. Indeed, let $\mathcal{F}_0 := L^V \cup \bigcup_{\varphi \in \Phi} \text{sub}(\varphi)$, and, for each $k < \omega$, let $\mathcal{F}_{k+1}$ be obtained by adding to $\mathcal{F}_k$ the formulas that (F2)–(F4) require for members of $\mathcal{F}_k$. Since every formula has countably many subformulas, and since there are countably many rational numbers, variables and terms, an induction on k shows

that every set $\mathcal{F}_k$ is countable. The union of the increasing sequence $(\mathcal{F}_k)_{k < \omega}$ is a fragment that contains Φ and is contained in every fragment containing Φ ; it is therefore the fragment generated by Φ , and it is countable.

Let $\mathcal{F}$ be a fragment of $L_{\omega_1\omega}^V$. Since $L_{\omega_1\omega}^V$ is contained in the infinitary language $L_{\omega_1\omega}^C$ of the expanded vocabulary V_C of Section 5, we may consider the fragment of $L_{\omega_1\omega}^C$ generated by $\mathcal{F}$, which we denote by $\mathcal{F}_C$; if V and $\mathcal{F}$ are countable, then so is $\mathcal{F}_C$, by the preceding paragraph applied to the countable vocabulary V_C . We denote the set of sentences of $\mathcal{F}$ by $S^{\mathcal{F}}$ and the set of sentences of $\mathcal{F}_C$ by $S_C^{\mathcal{F}}$.

7. CONDITIONS AND THEORIES

In real-valued logic, the role of the statements of classical logic is played by conditions, formal strict inequalities between a sentence and a rational number. Sets of conditions, called theories, specify the models to be constructed, whereas the types to be omitted are expressed by closed conditions; strict conditions are taken as the basic ones because each of them defines an open set in the topology of Section 10, as the Baire category argument requires.

A **lower-bound condition** is a formal expression $\varphi > r$, and an **upper-bound condition** is a formal expression $\varphi < r$, where φ is a sentence of the infinitary language $L_{\omega_1\omega}^C$ of the expanded vocabulary V_C (see Section 5) and r is a rational number. A **condition** is a lower-bound condition or an upper-bound condition, and a **theory** is a set of conditions. For a set Σ of sentences, a Σ -**theory** is a theory whose conditions are on sentences in Σ .

For a rational number r , we write

- ♠ $\varphi \geq r$ for the countable theory $\{\varphi > r - \frac{1}{n} : n \in \mathbb{Z}_+\}$, and
- ♠ $\varphi \leq r$ for the countable theory $\{\varphi < r + \frac{1}{n} : n \in \mathbb{Z}_+\}$.

The expressions $\varphi \geq r$ and $\varphi \leq r$ are called **closed conditions**.

Let $\mathfrak{N}$ be a V_C -structure. We say that $\mathfrak{N}$ **satisfies** the condition $\varphi > r$, and write $\mathfrak{N} \models \varphi > r$, if $\varphi^{\mathfrak{N}} > r$; it satisfies the condition $\varphi < r$ if $\varphi^{\mathfrak{N}} < r$. A V_C -structure that satisfies every condition of a theory is called a **model** of the theory; the same term is used for V -structures and theories whose conditions are on V -sentences. Thus a structure satisfies the closed condition $\varphi \geq r$ exactly when the value of φ in it is at least r .

Under the reading of 1 as true and -1 as false, the connectives generalize those of classical first-order logic, as the following equivalences show; they hold for every V_C -structure $\mathfrak{N}$ and every rational number r . Multiplication by -1 plays the role of negation: for a sentence φ , if $-\varphi$ denotes the sentence obtained by applying the connective $s \cdot$ with $s = -1$, then

$$\begin{aligned} \mathfrak{N} \models -\varphi > r & \text{ iff } (-\varphi)^{\mathfrak{N}} > r \\ & \text{ iff } -(\varphi^{\mathfrak{N}}) > r \\ & \text{ iff } \varphi^{\mathfrak{N}} < -r. \end{aligned}$$

In particular, for $r = 0$, a V_C -structure satisfies $-\varphi > 0$ exactly when it satisfies $\varphi < 0$. The binary connective $\vee$ is the counterpart of disjunction: for sentences φ and ψ , we have

$$\begin{aligned} \mathfrak{N} \models \varphi \vee \psi > r & \text{ iff } \max\{\varphi^{\mathfrak{N}}, \psi^{\mathfrak{N}}\} > r \\ & \text{ iff } \mathfrak{N} \models \varphi > r \text{ or } \mathfrak{N} \models \psi > r. \end{aligned}$$

The supremum $\sup_x$ takes the place of the existential quantifier: for a formula $\varphi(x)$ of $L_{\omega_1\omega}^C$, the condition $\sup_x \varphi > r$ is characterized as follows:

$$\begin{aligned} \mathfrak{N} \models \sup_x \varphi > r & \text{ iff } (\sup_x \varphi)^{\mathfrak{N}} > r \\ & \text{ iff } \sup \{\varphi^{\mathfrak{N}}(a) : a \in {}_x M^{\mathfrak{N}}\} > r \\ & \text{ iff } \varphi^{\mathfrak{N}}(a) > r \text{ for some } a \in {}_x M^{\mathfrak{N}}. \end{aligned}$$

Finally, the join $\bigvee$ generalizes countable disjunction: for a sequence Φ of sentences of $L_{\omega_1\omega}^C$ whose join is a sentence of $L_{\omega_1\omega}^C$, the condition $\bigvee \Phi > r$ is characterized in the same way:

$$\begin{aligned} \mathfrak{N} \models \bigvee \Phi > r & \text{ iff } \left(\bigvee \Phi\right)^{\mathfrak{N}} > r \\ & \text{ iff } \sup \{\varphi^{\mathfrak{N}} : \varphi \in \Phi\} > r \\ & \text{ iff } \varphi^{\mathfrak{N}} > r \text{ for some } \varphi \in \Phi. \end{aligned}$$

8. TYPES OF STRUCTURES AND FORMAL TYPES

The type of a structure records the values that the structure assigns to a set of sentences; formal types are families of such values that need not come from any structure. Among the latter are the Henkin types, which, as we show in Section 9, do come from structures; the Baire category argument takes place in the space they form, a subspace of the space of formal types with the product topology.

Let Φ be a set of sentences of the infinitary language $L_{\omega_1\omega}^C$ of the expanded vocabulary V_C . We write $\prod \Phi$ for the product $\prod_{\varphi \in \Phi} [-r_\varphi, r_\varphi]$, where r_φ is the a priori bound of φ , endowed with the product topology, in which it is a compact Hausdorff space (see Section 2); its elements are called **formal Φ -types**. For a formal Φ -type $\mathfrak{m}$ and a sentence $\varphi \in \Phi$, we write $\varphi^{\mathfrak{m}}$ for the φ -coordinate of $\mathfrak{m}$, and we say that $\mathfrak{m}$ satisfies a condition $\varphi > r$ or $\varphi < r$ if $\varphi^{\mathfrak{m}} > r$ or $\varphi^{\mathfrak{m}} < r$, respectively. If $\mathfrak{N}$ is a V_C -structure, the **Φ -type of $\mathfrak{N}$** is the formal Φ -type

$$\Phi\text{-tp}(\mathfrak{N}) := (\varphi^{\mathfrak{N}})_{\varphi \in \Phi}.$$

When Φ is contained in the set $S_{\omega_1\omega}^V$ of sentences of $L_{\omega_1\omega}^V$, the same definition applies to a V -structure $\mathfrak{M}$; the $S_{\omega_1\omega}^V$ -type $S_{\omega_1\omega}^V\text{-tp}(\mathfrak{M})$ is called the **full type** of $\mathfrak{M}$, and, similarly, the full type of a V_C -structure is its type for the set $S_{\omega_1\omega}^C$ of sentences of $L_{\omega_1\omega}^C$.

9. HENKIN TYPES

In his proof of the completeness theorem, Henkin [Hen49] built a model of a consistent theory out of new individual constants, chosen so that every existential sentence of the theory has a witness among them. The Henkin conditions below are the real-valued form of this requirement: they single out the formal types in which the value of a sentence formed with a connective from other sentences is computed from their values, and in which the value of a supremum over a variable is the supremum of the values obtained by substituting closed terms for that variable. In this section we show that these formal types are exactly the types of canonical structures, that is, of the structures in which every element is the value of a closed term, and we deduce, with the help of a downward Löwenheim–Skolem theorem for countable fragments, which formal types are realized by structures.

Let $\mathcal{G}$ be a fragment of the infinitary language $L_{\omega_1\omega}^C$ of the expanded vocabulary V_C , and let $S^{\mathcal{G}}$ be the set of its sentences. For a variable x , a formula φ whose free variables are among x and a closed V_C -term τ of sort ${}_xM$, we write $\varphi(\tau)$ for the sentence obtained by substituting τ for the free occurrences of x in φ , as in condition (F4) of the definition of a fragment (see Section 6). A formal type $\mathfrak{m} \in \prod S^{\mathcal{G}}$ is an $S^{\mathcal{G}}$ -**Henkin type** if it satisfies the following **Henkin conditions**:

- (H1) $1^{\mathfrak{m}} = 1$;
- (H2) $(s \cdot \varphi)^{\mathfrak{m}} = s \varphi^{\mathfrak{m}}$ for every $\varphi \in S^{\mathcal{G}}$ and every $s \in \mathbb{Q}$;
- (H3) $(\bigvee \Phi)^{\mathfrak{m}} = \sup\{\varphi^{\mathfrak{m}} : \varphi \in \Phi\}$ for every sentence $\bigvee \Phi \in S^{\mathcal{G}}$;
- (H4) $(\sup_x \varphi)^{\mathfrak{m}} = \sup\{\varphi(\tau)^{\mathfrak{m}} : \tau \text{ is a closed } V_C\text{-term of sort } {}_xM\}$ for every variable x and every formula $\varphi \in \mathcal{G}$ whose free variables are among x ;
- (H5) $(\varphi \vee \psi)^{\mathfrak{m}} = \max\{\varphi^{\mathfrak{m}}, \psi^{\mathfrak{m}}\}$ for all $\varphi, \psi \in S^{\mathcal{G}}$.

Every sentence that occurs in these conditions belongs to $S^{\mathcal{G}}$: the logical unity constant 1 by (F1); the sentences $s \cdot \varphi$, $\varphi \vee \psi$ and $\sup_x \varphi$ by (F3); the members of Φ by (F2), since the free variables of $\bigvee \Phi$ are those of its members; and the sentences $\varphi(\tau)$ by (F4). The language $L_{\omega_1\omega}^C$ is a fragment of itself, whose set of sentences is $S_{\omega_1\omega}^C$; its $S_{\omega_1\omega}^C$ -Henkin types, which are elements of $\prod S_{\omega_1\omega}^C$, are simply called **Henkin types**.

To pass from formal types back to structures, we follow Henkin and take as elements the closed terms themselves; the structure so obtained depends only on the values of the atomic sentences. Let $\mathcal{A}$ be the set of atomic V_C -sentences; by (F1), it is contained in $S^{\mathcal{G}}$ for every fragment $\mathcal{G}$ of $L_{\omega_1\omega}^C$. An **atomic type** is a formal type $\mathfrak{a} \in \prod \mathcal{A}$ such that $1^{\mathfrak{a}} = 1$. Every atomic type $\mathfrak{a}$ determines a V_C -structure $\mathfrak{N}_{\mathfrak{a}}$, the **term structure of $\mathfrak{a}$** , as follows. For each $M \in \mathbb{M}$, the set $M^{\mathfrak{N}_{\mathfrak{a}}}$ consists of the closed V_C -terms of sort M ; it is nonempty, since it contains the constants c_n^M . Each function symbol f of V_C is interpreted syntactically, by $f^{\mathfrak{N}_{\mathfrak{a}}}(\bar{\tau}) := f(\bar{\tau})$, and each predicate symbol p by $p^{\mathfrak{N}_{\mathfrak{a}}}(\bar{\tau}) := p(\bar{\tau})^{\mathfrak{a}}$. The latter is a function into $[-r_p, r_p]$, because the a priori bound of the atomic sentence $p(\bar{\tau})$ is r_p ; moreover, $1^{\mathfrak{N}_{\mathfrak{a}}} = 1^{\mathfrak{a}} = 1$, and $\mathfrak{N}_{\mathfrak{a}}$ is therefore a V_C -structure. No quotient of the set of terms is needed, since the vocabulary has no equality symbol. An induction on

terms shows that $\tau^{\mathfrak{N}_a} = \tau$ for every closed V_C -term τ ; hence $\mathfrak{N}_a$ is canonical, and $\mathcal{A}\text{-tp}(\mathfrak{N}_a) = \alpha$.

Lemma 2. Let $\mathcal{G}$ be a fragment of $L_{\omega_1\omega}^C$.

- (a) If $\mathfrak{N}$ is a canonical V_C -structure, then $S^{\mathcal{G}}\text{-tp}(\mathfrak{N})$ is an $S^{\mathcal{G}}$ -Henkin type.
- (b) Two $S^{\mathcal{G}}$ -Henkin types that agree on $\mathcal{A}$ are equal.
- (c) A formal type $\mathfrak{m} \in \prod S^{\mathcal{G}}$ is the $S^{\mathcal{G}}$ -type of a canonical V_C -structure if and only if it is an $S^{\mathcal{G}}$ -Henkin type; in that case, it is the $S^{\mathcal{G}}$ -type of the term structure of its restriction to $\mathcal{A}$.

Proof. The idea is that the Henkin conditions repeat the recursive definition of the values of sentences, except that a supremum over a sort becomes a supremum over closed terms; they therefore determine the value of every sentence from the values of the atomic ones, which is part (b), and they hold in a canonical structure, in which every element is the value of a closed term and the two suprema coincide, which is part (a). Part (b) is proved by induction on the rank of formulas: atomic formulas have rank 0; the rank of $s \cdot \varphi$ and of $\sup_x \varphi$ is the rank of φ plus one; the rank of $\varphi \vee \psi$ is the larger of the ranks of φ and ψ plus one; and the rank of $\bigvee \Phi$ is the supremum of the ranks of the members of Φ plus one. Substituting closed terms for variables changes only the atomic subformulas, and therefore preserves the rank.

(a) Conditions (H1), (H2), (H3) and (H5) hold for the values $\sigma^{\mathfrak{N}}$ of the sentences $\sigma \in S^{\mathcal{G}}$, by the definition of a V_C -structure and the recursive definition of the interpretation of formulas. For (H4), let $\varphi \in \mathcal{G}$ be a formula whose free variables are among x . The substitution lemma, $\varphi(\tau)^{\mathfrak{N}} = \varphi^{\mathfrak{N}}(\tau^{\mathfrak{N}})$ for every closed V_C -term τ of sort ${}_x M$, is proved by a routine induction on φ ; since $\mathfrak{N}$ is canonical, every element of ${}_x M^{\mathfrak{N}}$ is of the form $\tau^{\mathfrak{N}}$ for such a term τ , whence

$$(\sup_x \varphi)^{\mathfrak{N}} = \sup_{b \in {}_x M^{\mathfrak{N}}} \varphi^{\mathfrak{N}}(b) = \sup_{\tau} \varphi^{\mathfrak{N}}(\tau^{\mathfrak{N}}) = \sup_{\tau} \varphi(\tau)^{\mathfrak{N}}.$$

(b) Let $\mathfrak{m}$ and $\mathfrak{m}'$ be $S^{\mathcal{G}}$ -Henkin types that agree on $\mathcal{A}$. We show by induction on the rank of σ that $\sigma^{\mathfrak{m}} = \sigma^{\mathfrak{m}'}$ for every $\sigma \in S^{\mathcal{G}}$, the atomic case being the hypothesis. If σ is of the form $s \cdot \varphi$, $\varphi \vee \psi$ or $\bigvee \Phi$, then φ , ψ and the members of Φ are sentences in $S^{\mathcal{G}}$ of smaller rank, on which, by the induction hypothesis, $\mathfrak{m}$ and $\mathfrak{m}'$ agree; (H2), (H5) and (H3), respectively, compute $\sigma^{\mathfrak{m}}$ and $\sigma^{\mathfrak{m}'}$ alike from these values. If σ is of the form $\sup_x \varphi$, then the sentences $\varphi(\tau)$, for the closed V_C -terms τ of sort ${}_x M$, are sentences in $S^{\mathcal{G}}$ of smaller rank, on which $\mathfrak{m}$ and $\mathfrak{m}'$ agree for the same reason; (H4) therefore gives $\sigma^{\mathfrak{m}} = \sigma^{\mathfrak{m}'}$.

(c) By (a), the $S^{\mathcal{G}}$ -type of a canonical V_C -structure is an $S^{\mathcal{G}}$ -Henkin type. Conversely, let $\mathfrak{m}$ be an $S^{\mathcal{G}}$ -Henkin type, and let α be its restriction to $\mathcal{A}$, an atomic type by (H1). The term structure $\mathfrak{N}_a$ is canonical and $\mathcal{A}\text{-tp}(\mathfrak{N}_a) = \alpha$; by (a), its $S^{\mathcal{G}}$ -type is an $S^{\mathcal{G}}$ -Henkin type that agrees with $\mathfrak{m}$ on $\mathcal{A}$, and, by (b), the two are equal. $\square$

Let α be an atomic type. By Lemma 2(a), applied to $L_{\omega_1\omega}^C$, the full type $S_{\omega_1\omega}^C\text{-tp}(\mathfrak{N}_a)$ is a Henkin type that extends α , and, by Lemma 2(b), it is the only

one. We call it the **Henkin type of α** and denote it by $\hat{\alpha}$. For every fragment $\mathcal{G}$ of $L_{\omega_1\omega}^C$, the same lemma shows that the restriction of $\hat{\alpha}$ to $S^{\mathcal{G}}$ is the only $S^{\mathcal{G}}$ -Henkin type that extends α .

Let $\mathfrak{M}$ be a V_C -structure, and let $\alpha := \mathcal{A}\text{-tp}(\mathfrak{M})$. If $\mathfrak{M}$ is canonical, then, by Lemma 2(a), the full type $S_{\omega_1\omega}^C\text{-tp}(\mathfrak{M})$ is a Henkin type that extends α , and therefore equals $\hat{\alpha}$; that is, $\varphi^{\mathfrak{M}} = \varphi^{\hat{\alpha}}$ for every $\varphi \in S_{\omega_1\omega}^C$. If $\mathfrak{M}$ is not canonical, the two values may differ, and in either direction. For example, suppose that V has a single sort descriptor M , no function symbols and, besides 1 , a single predicate symbol $p: M \rightarrow [-1, 1]$, and let $\mathfrak{M}$ be the V_C -structure with $M^{\mathfrak{M}} = \{0, 1\}$ in which every constant of C is interpreted as 0 , $p^{\mathfrak{M}}(0) = -1$ and $p^{\mathfrak{M}}(1) = 1$. For a variable x of sort M , the sentence $\sup_x p(x)$ takes the value 1 in $\mathfrak{M}$, whereas $(\sup_x p(x))^{\hat{\alpha}} = \sup_n p(c_n^M)^{\alpha} = -1$, the constants c_n^M being the only closed V_C -terms; applying the connective $s \cdot$ with $s = -1$ to this sentence reverses the inequality.

Let Φ be a subset of the set $S_{\omega_1\omega}^V$ of sentences of $L_{\omega_1\omega}^V$, and let $\mathfrak{m} \in \prod \Phi$ be given. We say that the formal type $\mathfrak{m}$ is **realized** if there is a V -structure $\mathfrak{M}$ such that $\Phi\text{-tp}(\mathfrak{M}) = \mathfrak{m}$; in that case, we say that $\mathfrak{M}$ **realizes** $\mathfrak{m}$. When V is countable, there are countably many closed V_C -terms; hence a V -structure can have a canonical expansion to V_C , to which Lemma 2 applies, only if its sorts are countable. The following form of the downward Löwenheim–Skolem theorem replaces every V -structure by one whose sorts are countable and which assigns the same values to the sentences of a given countable fragment.

Lemma 3. Assume that the vocabulary V is countable, and let $\mathcal{F}$ be a countable fragment of $L_{\omega_1\omega}^V$. For every V -structure $\mathfrak{M}$ there is a V -structure $\mathfrak{M}_0$ whose sorts are countable such that $\varphi^{\mathfrak{M}_0} = \varphi^{\mathfrak{M}}$ for every φ in the set $S^{\mathcal{F}}$ of sentences of $\mathcal{F}$.

Proof. The structure $\mathfrak{M}_0$ is obtained by closing a countable set of elements of $\mathfrak{M}$ under the functions of $\mathfrak{M}$ and under the choice of approximate witnesses for the suprema of formulas of $\mathcal{F}$. For every formula $\psi(\bar{x}y) \in \mathcal{F}$, where y is a variable not among $\bar{x}$, every tuple $\bar{a} \in \bar{x}M^{\mathfrak{M}}$ and every integer $k \geq 1$, choose an element $b_{\psi, \bar{a}, k} \in yM^{\mathfrak{M}}$ such that $\psi^{\mathfrak{M}}(\bar{a} b_{\psi, \bar{a}, k}) > (\sup_y \psi)^{\mathfrak{M}}(\bar{a}) - 1/k$. For each $M \in \mathbb{M}$, let A_0^M consist of one element of $M^{\mathfrak{M}}$, and, for each $j < \omega$, let A_{j+1}^M consist of the elements of A_j^M , the values in $M^{\mathfrak{M}}$ of the functions $f^{\mathfrak{M}}$ at tuples from A_j , and the witnesses $b_{\psi, \bar{a}, k}$ in $M^{\mathfrak{M}}$ with $\bar{a}$ from A_j ; here a tuple is *from* A_j if each of its entries lies in the set A_j^N of its sort N . Since V and $\mathcal{F}$ are countable, an induction on j shows that every set A_j^M is countable; hence so is $A^M := \bigcup_j A_j^M$, and, the sets A_j^M increasing with j , the family $(A^M)_{M \in \mathbb{M}}$ is closed under the functions $f^{\mathfrak{M}}$. Let $\mathfrak{M}_0$ be the V -structure whose sorts are the nonempty sets A^M and whose functions and predicates are the restrictions of those of $\mathfrak{M}$. By induction on the rank of $\varphi \in \mathcal{F}$ (see the proof of Lemma 2), we show that $\varphi^{\mathfrak{M}_0}(\bar{a}) = \varphi^{\mathfrak{M}}(\bar{a})$ for every tuple $\bar{a}$ from $\mathfrak{M}_0$ of the appropriate sort. For atomic φ this holds because terms take the same values in $\mathfrak{M}_0$ and in $\mathfrak{M}$, and the steps for $s \cdot$, $\vee$ and $\bigvee$ follow from the definitions, the subformulas of members of $\mathcal{F}$ being members of $\mathcal{F}$ by (F2). Let φ be of the form $\sup_y \psi(\bar{x}y)$. The induction hypothesis gives $\varphi^{\mathfrak{M}_0}(\bar{a}) \leq \varphi^{\mathfrak{M}}(\bar{a})$, since the

supremum defining $\varphi^{\mathfrak{M}_0}(\bar{a})$ ranges over a subset of ${}_y M^{\mathfrak{M}}$. Conversely, the tuple $\bar{a}$ is from A_j for some j ; for every $k \geq 1$, the witness $b_{\psi, \bar{a}, k}$ is therefore an element of $\mathfrak{M}_0$, and the induction hypothesis, applied to ψ , gives $\varphi^{\mathfrak{M}_0}(\bar{a}) \geq \psi^{\mathfrak{M}_0}(\bar{a} b_{\psi, \bar{a}, k}) = \psi^{\mathfrak{M}}(\bar{a} b_{\psi, \bar{a}, k}) > \varphi^{\mathfrak{M}}(\bar{a}) - 1/k$. The case of sentences is the lemma. $\square$

Proposition 4 (Henkin Realizability). Assume that the vocabulary V is countable, and let $\Phi \subseteq S_{\omega_1 \omega}^V$ and $\mathfrak{m} \in \prod \Phi$ be given.

- (a) The formal type $\mathfrak{m}$ is realized by a V -structure whose sorts are countable if and only if it extends to a Henkin type in $\prod S_{\omega_1 \omega}^C$.
- (b) If $\Phi \subseteq S^{\mathcal{F}}$ for a countable fragment $\mathcal{F}$ of $L_{\omega_1 \omega}^V$, then $\mathfrak{m}$ is realized if and only if it extends to an $S_C^{\mathcal{F}}$ -Henkin type, where $S_C^{\mathcal{F}}$ is the set of sentences of the fragment $\mathcal{F}_C$ of $L_{\omega_1 \omega}^C$ generated by $\mathcal{F}$.

Proof. The idea is that, since V is countable, a V -structure has countable sorts exactly when it has a canonical expansion to V_C , and, by Lemma 2, the types of canonical V_C -structures are exactly the Henkin types; in part (b), Lemma 3 first replaces the realizing structure by one whose sorts are countable. (a) Suppose first that a V -structure $\mathfrak{M}$ whose sorts are countable realizes $\mathfrak{m}$. Since every sort of $\mathfrak{M}$ is countable and nonempty, there is an expansion $\mathfrak{M}[C]$ of $\mathfrak{M}$ to V_C in which, for each $M \in \mathbb{M}$, the map $n \mapsto (c_n^M)^{\mathfrak{M}[C]}$ is a surjection of ω onto $M^{\mathfrak{M}}$; this expansion is canonical. By Lemma 2(a), applied to $L_{\omega_1 \omega}^C$, the full type $S_{\omega_1 \omega}^C\text{-tp}(\mathfrak{M}[C])$ is a Henkin type, and it extends $\mathfrak{m}$, because the value of a V -sentence in $\mathfrak{M}[C]$ is its value in $\mathfrak{M}$. Conversely, let $\mathfrak{m}' \in \prod S_{\omega_1 \omega}^C$ be a Henkin type that extends $\mathfrak{m}$. By Lemma 2(c), applied to $L_{\omega_1 \omega}^C$, the type $\mathfrak{m}'$ is the full type of a canonical V_C -structure $\mathfrak{N}$, whose V -reduct therefore realizes $\mathfrak{m}$. The sorts of this reduct are countable: every element of $\mathfrak{N}$ is the value of a closed V_C -term, and there are only countably many closed V_C -terms, since V_C is countable together with V .

(b) If a V -structure $\mathfrak{M}$ realizes $\mathfrak{m}$, then, by Lemma 3, so does a V -structure $\mathfrak{M}_0$ whose sorts are countable, since $\Phi \subseteq S^{\mathcal{F}}$; by Lemma 2(a), applied to $\mathcal{F}_C$, the $S_C^{\mathcal{F}}$ -type of the canonical expansion $\mathfrak{M}_0[C]$ constructed in (a) is an $S_C^{\mathcal{F}}$ -Henkin type, and it extends $\mathfrak{m}$. Conversely, let $\mathfrak{m}'$ be an $S_C^{\mathcal{F}}$ -Henkin type that extends $\mathfrak{m}$. By Lemma 2(c), applied to $\mathcal{F}_C$, the type $\mathfrak{m}'$ is the $S_C^{\mathcal{F}}$ -type of a canonical V_C -structure, whose V -reduct realizes $\mathfrak{m}$. $\square$

10. TOPOLOGY ON THE SPACE OF HENKIN TYPES

The omitting types theorem of Section 11 is obtained from the Baire category theorem, which requires a Čech-complete space; the space in question consists of the Henkin types for the sentences of a fragment of the expanded language, regarded as a subspace of the compact Hausdorff space of all formal types for these sentences. We show that, when the vocabulary and the fragment are countable, the Henkin types form a G_δ subset of the space of formal types, and therefore a Čech-complete space and a Baire space.

Let $\mathcal{F}$ be a fragment of $L_{\omega_1\omega}^V$, and recall from Section 6 that $S_C^{\mathcal{F}}$ is the set of sentences of the fragment $\mathcal{F}_C$ of $L_{\omega_1\omega}^C$ generated by $\mathcal{F}$; we denote the set of $S_C^{\mathcal{F}}$ -Henkin types by $\mathcal{H}_C^{\mathcal{F}}$. As recalled in Section 2, the product $\prod S_C^{\mathcal{F}} = \prod_{\varphi \in S_C^{\mathcal{F}}} [-r_\varphi, r_\varphi]$ of Section 8, where r_φ is the a priori bound of φ , endowed with the product topology, is a compact Hausdorff space. Since the product topology is generated by the projections (see [Eng89, Section 2.3 and Proposition 1.4.8]), and since the open rays with rational endpoints form a subbasis of the topology of each interval $[-r_\varphi, r_\varphi]$, the sets

$$\begin{aligned} \langle\langle \varphi > r \rangle\rangle &:= \left\{ \mathbf{m} \in \prod S_C^{\mathcal{F}} : \varphi^{\mathbf{m}} > r \right\} \\ \langle\langle \varphi < r \rangle\rangle &:= \left\{ \mathbf{m} \in \prod S_C^{\mathcal{F}} : \varphi^{\mathbf{m}} < r \right\} \end{aligned}$$

where $\varphi \in S_C^{\mathcal{F}}$ and $r \in \mathbb{Q}$, form a subbasis of the product topology. For an $S_C^{\mathcal{F}}$ -theory T , we write $\langle\langle T \rangle\rangle$ for the set of formal types $\mathbf{m} \in \prod S_C^{\mathcal{F}}$ that satisfy every condition of T , in the sense of Section 8, and we say that $\mathbf{m}$ **satisfies** T if $\mathbf{m} \in \langle\langle T \rangle\rangle$; thus $\langle\langle T \rangle\rangle$ is the intersection of the subbasic sets given by the conditions of T , and $\langle\langle \emptyset \rangle\rangle = \prod S_C^{\mathcal{F}}$. For each $\varphi \in S_C^{\mathcal{F}}$, the projection $\mathbf{m} \mapsto \varphi^{\mathbf{m}}$ is continuous on $\prod S_C^{\mathcal{F}}$ (see [Eng89, Proposition 1.4.8]); hence, for a closed condition $\varphi \geq r$ with $\varphi \in S_C^{\mathcal{F}}$, the set $\langle\langle \varphi \geq r \rangle\rangle = \{ \mathbf{m} \in \prod S_C^{\mathcal{F}} : \varphi^{\mathbf{m}} \geq r \}$ is closed, being the preimage of the closed ray $[r, \infty)$ under this projection. The sets that occur in the proof of Proposition 5 below are likewise defined by equations and inequalities between finitely many coordinates: if $\sigma_1, \dots, \sigma_k \in S_C^{\mathcal{F}}$ and $g : \mathbb{R}^k \rightarrow \mathbb{R}$ is continuous, then the set $\{ \mathbf{m} : g(\sigma_1^{\mathbf{m}}, \dots, \sigma_k^{\mathbf{m}}) > 0 \}$ is open, and the set $\{ \mathbf{m} : g(\sigma_1^{\mathbf{m}}, \dots, \sigma_k^{\mathbf{m}}) = 0 \}$, which is the intersection of the open sets $\{ \mathbf{m} : |g(\sigma_1^{\mathbf{m}}, \dots, \sigma_k^{\mathbf{m}})| < 1/n \}$ with $n \in \mathbb{Z}_+$, is a G_δ set.

Proposition 5. If the vocabulary V and the fragment $\mathcal{F}$ are countable, then the set $\mathcal{H}_C^{\mathcal{F}}$ of $S_C^{\mathcal{F}}$ -Henkin types is a G_δ subset of $\prod S_C^{\mathcal{F}}$.

Proof. The idea is that the Henkin conditions consist of countably many equations, each either between finitely many coordinates or between one coordinate and the supremum of countably many others, and that each such equation is equivalent to countably many requirements defining G_δ sets. All the index sets below are countable: since V is countable, so are V_C , the set of variables and the set of closed V_C -terms, and the fragment $\mathcal{F}_C$ is countable by Section 6.

By the continuity of the projections, the following sets are G_δ subsets of $\prod S_C^{\mathcal{F}}$, and, for each $i \in \{1, \dots, 5\}$, a formal type belongs to all the sets listed under (Hi) if and only if it satisfies the Henkin condition (Hi):

- (H1) the set $\{ \mathbf{m} : 1^{\mathbf{m}} = 1 \}$;
- (H2) for $\varphi \in S_C^{\mathcal{F}}$ and $s \in \mathbb{Q}$, the set $\{ \mathbf{m} : (s \cdot \varphi)^{\mathbf{m}} = s \varphi^{\mathbf{m}} \}$;
- (H3) for each sentence $\bigvee \Phi \in S_C^{\mathcal{F}}$, the closed sets $\{ \mathbf{m} : (\bigvee \Phi)^{\mathbf{m}} \geq \varphi^{\mathbf{m}} \}$ with $\varphi \in \Phi$, and the set

$$\bigcap_{n \in \mathbb{Z}_+} \bigcup_{\varphi \in \Phi} \left\{ \mathbf{m} : (\bigvee \Phi)^{\mathbf{m}} < \varphi^{\mathbf{m}} + \frac{1}{n} \right\},$$

which is a G_δ set because each union is open;

- (H4) for each variable x and each formula $\varphi \in \mathcal{F}_C$ whose free variables are among x , the closed sets $\{\mathfrak{m} : (\sup_x \varphi)^\mathfrak{m} \geq \varphi(\tau)^\mathfrak{m}\}$ with τ a closed V_C -term of sort ${}_x M$, and the set

$$\bigcap_{n \in \mathbb{Z}_+} \bigcup_{\tau} \left\{ \mathfrak{m} : (\sup_x \varphi)^\mathfrak{m} < \varphi(\tau)^\mathfrak{m} + \frac{1}{n} \right\};$$

- (H5) for $\varphi, \psi \in \mathcal{S}_C^F$, the set $\{\mathfrak{m} : (\varphi \vee \psi)^\mathfrak{m} = \max\{\varphi^\mathfrak{m}, \psi^\mathfrak{m}\}\}$.

Each set listed under (H1), (H2) and (H5), and each closed set listed under (H3) and (H4), is the zero set of a continuous function of finitely many coordinates, such as $\mathfrak{m} \mapsto \max\{\varphi^\mathfrak{m} - (\bigvee \Phi)^\mathfrak{m}, 0\}$, and each union listed is a union of sets of the form $\{\mathfrak{m} : g(\sigma_1^\mathfrak{m}, \dots, \sigma_k^\mathfrak{m}) > 0\}$; the remark preceding the proposition therefore applies. The family of all these sets, being indexed by countable sets, is countable, and its intersection is $\mathcal{H}_C^F$; since a countable intersection of G_δ sets is a G_δ set, the proposition follows. $\square$

Corollary 6. If the vocabulary V and the fragment $\mathcal{F}$ are countable, then $\mathcal{H}_C^F$ is Čech-complete.

Proof. By Proposition 5, the space $\mathcal{H}_C^F$ is a G_δ subset of the compact Hausdorff space $\prod \mathcal{S}_C^F$; as recalled in Section 2, every such subset is Čech-complete. $\square$

Corollary 7. If the vocabulary V and the fragment $\mathcal{F}$ are countable, then $\mathcal{H}_C^F$ is a Baire space.

Proof. By Corollary 6, the space $\mathcal{H}_C^F$ is Čech-complete, and the Baire category theorem for Čech-complete spaces (see Theorem 1) shows that it is a Baire space. $\square$

In Section 11 we express in terms of conditions that a set of Henkin types has empty interior; for this purpose we use a basis of the topology of $\mathcal{H}_C^F$ described by finite theories. Since $\langle\langle S \rangle\rangle$ is a finite intersection of subbasic open sets whenever S is a finite $\mathcal{S}_C^F$ -theory, the sets $\mathcal{H}_C^F \cap \langle\langle S \rangle\rangle$ with S finite form a basis of the topology of $\mathcal{H}_C^F$ (see [Eng89, Sections 1.1 and 2.1]).

11. OMITTING TYPES THEOREM

The Henkin types that satisfy a theory are the types of the canonical structures, those in which every element is the value of a closed term, whose reducts are models of the theory; to omit countably many types in a model of the theory, it therefore suffices to find such a Henkin type that realizes none of them. In this section we show that, when the vocabulary and the fragment are countable and the theory has a model, these Henkin types form a nonempty Baire space, and that the required Henkin type exists as soon as, for each instance of each of the types, the Henkin types that satisfy the theory and the instance form a nowhere dense subset of this space: those that realize one of the types then form a meager subset, and the Baire category theorem provides a Henkin type outside it.

Henceforth, assume that the vocabulary V and the fragment $\mathcal{F}$ of Section 10 are countable, and fix an $S^{\mathcal{F}}$ -theory B (a theory whose conditions are on sentences of $\mathcal{F}$) that has a model. Recall from Section 10 that $\mathcal{H}_C^{\mathcal{F}}$ is the set of $S_C^{\mathcal{F}}$ -Henkin types, where $S_C^{\mathcal{F}}$ is the set of sentences of the fragment $\mathcal{F}_C$ of $L_{\omega_1, \omega}^C$ generated by $\mathcal{F}$. The set $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle$ of $S_C^{\mathcal{F}}$ -Henkin types that satisfy B (see Section 10) is nonempty. Indeed, by Lemma 3, the theory B has a model $\mathfrak{M}$ whose sorts are countable; the canonical expansion $\mathfrak{M}[C]$ of $\mathfrak{M}$ constructed in the proof of Proposition 4(a) satisfies B , since the values of V -sentences in $\mathfrak{M}[C]$ are their values in $\mathfrak{M}$, and its $S_C^{\mathcal{F}}$ -type belongs to $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle$ by Lemma 2(a). Conversely, by Lemma 2(c), every member of $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle$ is the $S_C^{\mathcal{F}}$ -type of a canonical V_C -structure, whose V -reduct is a model of B ; thus $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle$ is the set of $S_C^{\mathcal{F}}$ -types of the canonical V_C -structures whose V -reducts are models of B . Since $S^{\mathcal{F}}$ is countable, so is the theory B , whence the set $\langle\langle B \rangle\rangle$ is a countable intersection of open sets; by Proposition 5, the set $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle$ is therefore a G_δ subset of the compact Hausdorff space $\prod S_C^{\mathcal{F}}$. As recalled in Section 2, such a set is Čech-complete, and, by Theorem 1, the space $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle$ is a nonempty Baire space.

Let $\bar{x}$ be a finite tuple of distinct variables. An $\mathcal{F}$ -**type** in $\bar{x}$ is a set $T(\bar{x})$ of expressions $\varphi(\bar{x}) \geq r$, where $r \in \mathbb{Q}$ and $\varphi \in \mathcal{F}$ is a formula whose free variables are among $\bar{x}$; substituting closed terms for $\bar{x}$ turns each of them into a closed condition in the sense of Section 7. For a tuple $\bar{\tau}$ of closed V_C -terms of sort $\bar{x}M$, the **instance** $T(\bar{\tau})$ of $T(\bar{x})$ is the union of the closed conditions $\varphi(\bar{\tau}) \geq r$, each a countable theory, for the members $\varphi(\bar{x}) \geq r$ of $T(\bar{x})$; it is an $S_C^{\mathcal{F}}$ -theory because the fragment $\mathcal{F}_C$ is closed under substitution of terms (see (F4) in Section 6). A Henkin type $\mathfrak{m} \in \mathcal{H}_C^{\mathcal{F}}$ **realizes** $T(\bar{x})$ if it satisfies some instance $T(\bar{\tau})$, that is, if $\mathfrak{m} \in \langle\langle T(\bar{\tau}) \rangle\rangle$ for some tuple $\bar{\tau}$; otherwise, it **omits** $T(\bar{x})$. Similarly, if $\mathfrak{M}$ is a V -structure, a tuple $\bar{a} \in \bar{x}M^{\mathfrak{M}}$ **satisfies** $T(\bar{x})$ in $\mathfrak{M}$ if $\varphi^{\mathfrak{M}}(\bar{a}) \geq r$ for every member $\varphi(\bar{x}) \geq r$ of $T(\bar{x})$. We say that $T(\bar{x})$ **has nonempty interior in $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle$** if there exist a tuple $\bar{\tau}$ of closed V_C -terms of sort $\bar{x}M$ and a finite $S_C^{\mathcal{F}}$ -theory S such that

$$\emptyset \neq \mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle \cap \langle\langle S \rangle\rangle \subseteq \langle\langle T(\bar{\tau}) \rangle\rangle.$$

Otherwise, we say that $T(\bar{x})$ **has empty interior in $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle$** . Since the sets $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle \cap \langle\langle S \rangle\rangle$, with S a finite $S_C^{\mathcal{F}}$ -theory, form a basis of the topology of $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle$ (see Section 10), the type $T(\bar{x})$ has nonempty interior in this space exactly when, for some tuple $\bar{\tau}$, the closed subset $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle \cap \langle\langle T(\bar{\tau}) \rangle\rangle$ of this space has nonempty interior. The members of an $\mathcal{F}$ -type are closed conditions so that this subset is closed, being an intersection of the closed sets $\langle\langle \varphi(\bar{\tau}) \geq r \rangle\rangle$ (see Section 10); when it has empty interior, its complement in $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle$ is therefore a dense open set, and the Baire category theorem applies to countable intersections of such sets, as the proof of Theorem 8 shows.

Theorem 8 (Omitting Types Theorem). Assume that the vocabulary V and the fragment $\mathcal{F}$ are countable, and let B be an $S^{\mathcal{F}}$ -theory that has a model. If $(T_n(\bar{x}_n))_{n < \omega}$ is a sequence of $\mathcal{F}$ -types each of which has empty interior in $\mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle$, then there is an $S_C^{\mathcal{F}}$ -Henkin type $\mathfrak{m} \in \mathcal{H}_C^{\mathcal{F}} \cap \langle\langle B \rangle\rangle$ that omits every

$T_n(\bar{x}_n)$; consequently, the theory B has a model whose sorts are countable and in which, for every $n < \omega$, no tuple of elements satisfies $T_n(\bar{x}_n)$.

Proof. The idea is to write the set of Henkin types that satisfy B and omit every $T_n(\bar{x}_n)$ as a countable intersection of open dense subsets of the nonempty Baire space $\mathcal{H}_C^F \cap \langle\langle B \rangle\rangle$. Fix $n < \omega$, let $\bar{x} := \bar{x}_n$ and $T := T_n$, and let $\bar{\tau}$ be a tuple of closed V_C -terms of sort $\bar{x}M$. Since the set $\langle\langle \varphi(\bar{\tau}) \geq r \rangle\rangle$ is closed for every member $\varphi(\bar{x}) \geq r$ of $T(\bar{x})$ (see Section 10), the set $\mathcal{H}_C^F \cap \langle\langle B \rangle\rangle \cap \langle\langle T(\bar{\tau}) \rangle\rangle$ is closed in the space $\mathcal{H}_C^F \cap \langle\langle B \rangle\rangle$; by the definition of empty interior, its interior in this space is empty. Its complement

$$O_{T, \bar{\tau}} := \{ \mathfrak{m} \in \mathcal{H}_C^F \cap \langle\langle B \rangle\rangle : \mathfrak{m} \text{ does not satisfy } T(\bar{\tau}) \}$$

is therefore an open dense subset of $\mathcal{H}_C^F \cap \langle\langle B \rangle\rangle$. Since V is countable, there are only countably many tuples $\bar{\tau}$; consequently, the set $O_T := \bigcap_{\bar{\tau}} O_{T, \bar{\tau}}$ of members of $\mathcal{H}_C^F \cap \langle\langle B \rangle\rangle$ that omit $T(\bar{x})$ is a countable intersection of open dense subsets of $\mathcal{H}_C^F \cap \langle\langle B \rangle\rangle$. As n was arbitrary, the same holds for each set O_{T_n} , and hence for the set $O := \bigcap_{n < \omega} O_{T_n}$ of members of $\mathcal{H}_C^F \cap \langle\langle B \rangle\rangle$ that omit $T_n(\bar{x}_n)$ for every $n < \omega$. Since $\mathcal{H}_C^F \cap \langle\langle B \rangle\rangle$ is a nonempty Baire space, the set O is dense in it, and in particular nonempty.

Let $\mathfrak{m}$ be a member of O . By Lemma 2(c), the type $\mathfrak{m}$ is the S_C^F -type of a canonical V_C -structure $\mathfrak{N}$, whose V -reduct $\mathfrak{M}$ is a model of B ; the sorts of $\mathfrak{M}$ are countable, since every element of $\mathfrak{N}$ is the value of one of the countably many closed V_C -terms. Now fix $n < \omega$ and a tuple $\bar{a} \in \bar{x}_n M^{\mathfrak{M}}$. Since $\mathfrak{N}$ is canonical, we may write $\bar{a} = \bar{\tau}^{\mathfrak{N}}$ for a tuple $\bar{\tau}$ of closed V_C -terms, and the substitution lemma (see the proof of Lemma 2(a)), which holds for tuples of closed terms by the same induction, gives $\varphi^{\mathfrak{M}}(\bar{a}) = \varphi^{\mathfrak{N}}(\bar{\tau}^{\mathfrak{N}}) = \varphi(\bar{\tau})^{\mathfrak{m}}$ for every formula $\varphi \in \mathcal{F}$ whose free variables are among $\bar{x}_n$. Since $\mathfrak{m}$ omits $T_n(\bar{x}_n)$, it does not satisfy $T_n(\bar{\tau})$, and the tuple $\bar{a}$ therefore does not satisfy $T_n(\bar{x}_n)$ in $\mathfrak{M}$. $\square$

REFERENCES

- [Čec37] E. Čech, *On bicomcompact spaces*, Ann. of Math. (2) **38** (1937), no. 4, 823–844.
- [CI14] X. Caicedo and J. Iovino, *Omitting uncountable types, and the strength of $[0, 1]$ -valued logics*, Ann. Pure Appl. Logic **165** (2014), no. 6, 1169–1200; arXiv:1202.5981.
- [Eag14] C. J. Eagle, *Omitting types for infinitary $[0, 1]$ -valued logic*, Ann. Pure Appl. Logic **165** (2014), no. 3, 913–932.
- [Eng89] R. Engelking, *General topology*, revised and completed ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989.
- [Hen49] L. Henkin, *The completeness of the first-order functional calculus*, J. Symbolic Logic **14** (1949), no. 3, 159–166.
- [Kei23] H. J. Keisler, *Model theory for real-valued structures*, Beyond First Order Model Theory, Vol. II (J. Iovino, ed.), Chapman & Hall/CRC, Boca Raton, FL, 2023, Chapter 2.
- [Mor74] M. Morley, *Applications of topology to $L_{\omega_1\omega}$* , Proceedings of the Tarski Symposium (L. Henkin et al., eds.), Proc. Sympos. Pure Math., vol. 25, Amer. Math. Soc., Providence, RI, 1974, pp. 233–240.

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