LOCAL STABILITY IN INFINITARY REAL-VALUED LOGIC

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Abstract. In this paper we study the stability of formulas in the infinitary extension of Keisler's model theory of real-valued structures [Kei23, ADI26]. We show that four classical characterizations of stability carry over to this logic, with the types realized in elementary extensions in place of all types: the absence of the order property, the commutation of iterated limits of the values of the formula along sequences of elements, the existence of a function on pairs of types that extends the values of the formula on pairs of elements and is continuous in each variable, and the definability of types. The characterizations follow from a double limit criterion for prescribed limits, which we prove directly in the spirit of Arhangel'skii's C_p-theory; the Čech completeness of the spaces of Henkin types, which yields a model existence theorem in place of compactness, describes the realized types, for relational vocabularies, as analytic sets.

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Introduction

In this paper we study the stability of formulas in the logic $L_{\omega_1\omega}(\mathbb{R})$, the infinitary extension of Keisler's model theory of real-valued structures [Kei23, ADI26]. As a language for analysis, this logic is far more expressive than first-order logic in its two-valued and its continuous forms (Section 1). In a compact logic, a formula $\varphi(x, y)$ is stable precisely when any one (equivalently, all) of the following conditions holds [She78, BYU10].

(i) The formula φ does not have the order property.
(ii) In every model, iterated limits of the values of φ along sequences of elements commute.

Date: November 5, 2026.

2020 *Mathematics Subject Classification.* Primary 03C45, 03C75; Secondary 03C66, 54C35, 54E52, 03E15, 46B45.

This version: November 9, 2026. The most recent version of this paper is available at https://utsa-quantum-math.github.io/resources/local-stability-infinitary-logic/; it may supersede the version deposited on arXiv.

(iii) Over every model, φ extends to a function on the product of its spaces of types that is continuous in each variable.
(iv) Every type of φ over every model is definable.
(v) Over every countable model, for a countable vocabulary, the space of types of φ is separable for the uniform metric, and is countable when φ is two-valued.

The standard proofs of these classical characterizations rely heavily on compactness. We show that, although $L_{\omega_1\omega}(\mathbb{R})$ is not compact, the first four of these characterizations carry over to it. Let a countable fragment $\mathcal{F}$ of $L_{\omega_1\omega}(\mathbb{R})$, a countable structure M and a formula $\varphi(x, y)$ of $\mathcal{F}$ be given, and call φ *stable over* M when the iterated limits of the values of φ on M commute along every pair of limits of types realized in elementary extensions of M (Definition 5.1). Our principal result is the following.

Theorem A. *The stability of φ over M is equivalent to each of the following conditions.*

(1) *The formula φ does not have the order property with realized limits of types (Theorems 4.4 and 5.2).*
(2) *The values of φ on pairs of elements extend to a function on the product of the spaces of realized types of φ and of φ^*, the formula with the roles of x and y exchanged, that is continuous in each variable (Theorem 4.6).*
(3) *Each realized type of φ extends to a continuous function on the realized types of φ^*, in a manner compatible with limits (Theorem 4.7).*
(4) *Each realized type of φ is definable by a formula of the logic that induces a continuous function on the realized types and is compatible with limits. For the type of an element c, such a definition is*

$$\psi_c(y) = \bigwedge_{n<\omega} \bigvee_{k\geq n} \varphi(a_k, y),$$

where $(a_k)_{k<\omega}$ is any sequence of elements of M whose types converge to the type of c (Theorem 5.2).

In (3) and (4), a function G on the realized types of φ^* that extends a realized type p of φ is *compatible with limits* if $G(q)$ is the limit of $q(a_m)$ along $\mathcal{U}$ for every realized type q of φ^*, every sequence $(a_m)_{m<\omega}$ of elements of M and every ultrafilter $\mathcal{U}$ on ω along which the types of the elements a_m converge to p; a definition is compatible with limits when the function that the definition induces is.

Theorem A rests on a double limit criterion for prescribed limits, Theorem B, which we prove directly for countable sets. The criterion belongs to C_p-theory, the study of spaces of continuous functions with the topology of pointwise convergence. Grothendieck's double limit criterion of 1952 is one of the sources of this theory, which Arhangel'skii and his school built into a major branch of general topology [Arh92, Tka11]. A theorem of Arhangel'skii asserts that every space of countable tightness, and in particular every metrizable space, is *weakly Grothendieck* [Arh98, Tal20]: on such a space, if every infinite subset of a set of continuous functions has a continuous limit point, then the closure of the set is compact and consists of continuous functions. Theorem B carries the double limit condition from all limits to limits prescribed in advance, in spaces that need not be compact. The realized types themselves can be described by the topology of the spaces of Henkin types, the complete diagrams from which models are constructed. In $L_{\omega_1\omega}(\mathbb{R})$ these spaces are Čech-complete, and the logic therefore enjoys a model existence theorem [ADI26]. For relational vocabularies, the set of realized types is therefore a continuous image of a Polish space, hence an analytic set (Theorem 2.4), which yields the descriptive and Baire category results of Section 7.

Theorem B. *Let A and B be countable sets, let φ be a bounded real-valued function on $A \times B$, and let P be a set of pointwise limits of rows $\varphi(a, \cdot)$, $a \in A$, that includes every row, and let Q be a set of pointwise limits of columns $\varphi(\cdot, b)$, $b \in B$, that includes every column. A* ladder *for φ consists of sequences $(a_m)_{m<\omega}$ in A and $(b_n)_{n<\omega}$ in B and real numbers $t < t'$ such that $\varphi(a_m, b_n) \leq t$ for $m < n$ and $\varphi(a_m, b_n) \geq t'$ for $m \geq n$, or the same with the two inequalities exchanged. The following conditions are equivalent:*

(1) *iterated limits of φ agree whenever the limits of the rows and of the columns along which they are taken belong to P and to Q;*
(2) *there is no ladder for φ whose rows converge to a point of P and whose columns converge to a point of Q;*
(3) *the function φ extends to a function on $P \times Q$ that is continuous in each variable;*
(4) *every point p of P extends to a single continuous function on Q whose value at each point q of Q is the limit of $q(a_m)$ along $\mathcal{U}$, for every sequence $(a_m)_{m<\omega}$ in A and every ultrafilter $\mathcal{U}$ on ω along which the rows of the sequence converge to p.*

When P and Q comprise all limits, Theorem B contains Grothendieck's double limit criterion [Gro52] for countable sets; when they comprise the realized types, Theorem B yields Theorem A. In Section 3 we cast Arhangel'skii's theorem as a double limit criterion on spaces of countable tightness, and in Section 4 we prove Theorem B.

The expressive power of $L_{\omega_1\omega}(\mathbb{R})$ is nowhere more striking than in analysis. In first-order continuous logic, Casazza, Dueñez and Iovino proved that the norms of Tsirelson's space and of Schlumprecht's space cannot be defined from the approximants of their inductive constructions [CDI18]. In $L_{\omega_1\omega}(\mathbb{R})$, a single formula, the upper limit of the approximants, defines the norm of every Banach space whose norm is the pointwise limit of approximating seminorms dominated by a multiple of the norm (Proposition 1.1). Tsirelson's space, Schlumprecht's space and the higher-order Tsirelson spaces, built like Tsirelson's space from the Schreier families of higher order [AT05], are of this kind. The same formula defines even the norm of the space of Argyros and Haydon, the first Banach space on which every bounded operator is a scalar multiple of the identity plus a compact operator [AH11] (Example 1.2), a space that Gowers greeted as the new "nastiest known Banach space" [Gow09]. This logic also defines every cut of the rational numbers by a countable join of instances of the order, and it expresses in a single sentence that each element of an ordering has only finitely many predecessors (Section 1). The defining formula of the norms has the same form as the definitions of types in (4), and this expressive power is why those definitions must be compatible with limits. A formula can take the values of a type at every element of the model without defining the type: every cut of the rational numbers is given at the rational numbers by a countable join of instances of the order, although the order is unstable (Example 6.1). Continuity on the realized types does not suffice either, as an example in first-order logic over the natural numbers shows (Example 6.3).

Two features of the classical theory rest on compactness itself. The first is condition (v). Under compactness, every type of a stable formula over a countable model is defined by a uniform limit of finite combinations of instances, and there are only countably many such combinations, so that the space of types is separable for the uniform metric (Proposition 7.3). The definitions in (4), by contrast, are countable formulas, uncountably many in number, and a formula that is stable over M may possess continuum many realized types at mutual distance 2 (Example 7.5). In place of (v), descriptive set theory supplies a dichotomy: the realized types are either separable, or they contain a perfect set of types whose mutual distances are bounded away from zero (Proposition 7.6). The second is the form of the definitions in (iv) itself, which rests on the Stone–Weierstrass theorem for the compact space of all limits of types. Here a realized type is

so definable exactly when the type extends continuously to all limits of types, realized or not (Proposition 7.1), and by Proposition 7.3 the formula of Example 7.5 has realized types that do not extend in this way.

Compactness also underlies a characterization of stability of another kind, by an abstract independence relation. Harnik and Harrington showed that a theory is stable exactly when the theory carries a relation that satisfies invariance, monotonicity, symmetry, existence, stationarity over models (a type over a model has only one independent extension), finite character and local character, and that such a relation is nonforking [HH84]; Kim and Pillay characterized the simple theories in the same manner [KP97]. The existence axiom is the work of compactness, and in $L_{\omega_1\omega}(\mathbb{R})$ the axiom fails for pairs of elements: two realized types of a formula that satisfies Grothendieck's double limit condition on all limits may occur together in no elementary extension of the model, and types that occur together may do so only with a value of the formula other than the generic one (Propositions 8.3 and 8.4). Once the relation is placed on finite approximations by elements of the model, existence is supplied by approximation, and the axioms of Harnik and Harrington characterize stability in $L_{\omega_1\omega}(\mathbb{R})$ and determine the relation, whose value is the generic limit (Theorem 8.1); in the first-order case the free pairs are the nonforking ones (Proposition 8.2).

Organization of the paper. The order of the paper follows that of the proof. In Section 1 we recall the logic and display its expressive power, notably the definition of norms from their approximants. In Section 2 we define the realized types and present them through model existence. In Section 3 we give the C_p-theoretic setting; Theorem B is established directly in Section 4, and Theorem A is derived from it in Section 5. In Section 6 we show by examples why the definitions must be compatible with limits, and in Section 7 we treat the two features of the classical theory that rest on compactness itself. In Section 8 we characterize stability by an independence relation on approximations, after Harnik and Harrington. The comparison with the classical theory and the open problems close the paper, in Sections 9 and 10.

1. Countable joins define what first-order logic cannot

The interest of $L_{\omega_1\omega}(\mathbb{R})$ lies in the expressive power of its countable joins, and with that power the paper begins. In this section we recall real-valued infinitary logic, the infinitary extension of Keisler's model theory of general real-valued structures [Kei23, ADI26]. In the spirit of Keisler's book on infinitary logic [Kei71], we then show three things that the countable joins express and that lie beyond the reach of first-order logic: the norms of the most intricate Banach spaces, defined from their approximants; the cuts of the rational numbers; and the finiteness of sets of predecessors.

The structures interpret function symbols by arbitrary functions and predicate symbols by bounded real-valued functions, and the language has no equality symbol. Every atomic formula inherits the prescribed bound of its predicate symbol. Every formula θ carries a syntactic bound r_θ that bounds the values of θ in every structure. The unity symbol $\mathtt{1}$ has value and bound 1; rational scalar multiplication multiplies the bound by the absolute value of the scalar, addition adds bounds, and binary maximum $\vee$ takes the larger bound. Suprema over variables preserve bounds. Minimum and infimum are derived by negation. A countable join $\bigvee_n \theta_n$ is admitted when the formulas share a finite tuple of free variables and $\sup_n r_{\theta_n}$ is finite; this supremum is the bound of the join. The interpretation is the pointwise supremum. Countable meets obey the same bound rule. These bounds are fixed by the formation rules, rather than recalculated from the range of a formula in a particular structure.

A *countable fragment* $\mathcal{F}$ is a countable set of formulas of $L_{\omega_1\omega}(\mathbb{R})$ that contains the atomic formulas and the formula $\mathtt{1}$ and is closed under subformulas, under the finitary connectives, under

suprema over variables, and under substitution of variables for variables, with bound variables renamed when necessary to avoid capture. Every countable set of formulas of $L_{\omega_1\omega}(\mathbb{R})$ lies in a countable fragment, so the restriction to countable fragments loses nothing for statements that involve countably many formulas. All sorts of structures are assumed to be nonempty. For a structure M, a formula $\theta(\bar{u})$ and a tuple $\bar{e}$ of elements of M of the length of $\bar{u}$, we write $\theta^M(\bar{e})$, or $\theta(\bar{e})$, for the value of θ at $\bar{e}$. A structure N is an *elementary extension of M for $\mathcal{F}$*, written $M \preceq_{\mathcal{F}} N$, if M is a substructure of N and $\theta^N(\bar{e}) = \theta^M(\bar{e})$ for every formula $\theta(\bar{u})$ of $\mathcal{F}$ and every tuple $\bar{e}$ of elements of M, the empty tuple included.

Among the Banach spaces of modern analysis, the most intricate are produced by inductive constructions, in which the norm is the limit of a sequence of approximating norms. Tsirelson's space, the first Banach space found to contain no copy of c_0 or of ℓ_p, arises in this way, and so do Schlumprecht's space, the higher-order Tsirelson spaces and the space of Argyros and Haydon, on which every bounded operator is a scalar multiple of the identity plus a compact operator [AH11]. Casazza, Dueñez and Iovino proved that the Tsirelson and Schlumprecht norms admit no first-order definitions from their approximants in the structures considered there [CDI18]. Their proofs combine estimates on nonuniform convergence with local stability theory. In $L_{\omega_1\omega}(\mathbb{R})$, a single formula defines every such norm from its approximants.

Proposition 1.1 (Norms given by approximants). *Let V be a real vector space, let $|||\cdot|||$ be a norm on V, and let $(\|\cdot\|_k)_{k<\omega}$ be seminorms on V that converge pointwise to $|||\cdot|||$ and satisfy $\|x\|_k \le C|||x|||$ for some real number $C \ge 1$ and all k and x. Let the vocabulary consist of unary predicate symbols $\mathtt{N}_k$, for $k < \omega$, with a priori bound C, and let the structure whose universe is the closed unit ball of $(V, |||\cdot|||)$ interpret $\mathtt{N}_k$ by $\|\cdot\|_k$. Then the quantifier-free formula*

$$\nu(x) = \bigwedge_{n<\omega} \bigvee_{k\ge n} \mathtt{N}_k(x)$$

of $L_{\omega_1\omega}(\mathbb{R})$ satisfies $\nu(x) = |||x|||$ for every x in the closed unit ball.

Proof. The interpretation respects the a priori bounds, since $\|x\|_k \le C|||x||| \le C$ on the closed unit ball. The atomic formulas $\mathtt{N}_k(x)$ share the free variable x and the a priori bound C, so the joins $\bigvee_{k\ge n} \mathtt{N}_k(x)$ and their meet are formulas, and the value of ν at x is $\inf_n \sup_{k\ge n} \|x\|_k$, the upper limit of a convergent sequence, which is its limit $|||x|||$. □

For Tsirelson's space, Schlumprecht's space and the higher-order Tsirelson spaces, the approximants of the inductive construction increase to the norm, so that $C = 1$ and the formula may be taken to be the single join $\bigvee_k \mathtt{N}_k(x)$. The formula ν has the form of the definitions of types in Theorem 5.2: an upper limit of formulas that share an a priori bound.

Example 1.2 (The space of Argyros and Haydon). The space of Argyros and Haydon is a Bourgain–Delbaen space: a closed subspace X of $\ell_\infty(\Gamma)$, with the supremum norm, where Γ is the union of an increasing sequence $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ of finite sets, the stages of the construction [AH11, Section 3]. The seminorms $\|x\|_k = \max_{\gamma\in\Gamma_k} |x(\gamma)|$ increase to the norm of X, so that $C = 1$ in Proposition 1.1. The single join $\bigvee_k \mathtt{N}_k(x)$ therefore defines the norm of X on the closed unit ball, from the approximants given by the stages of the construction.

The countable joins reach into the theory of orders as well. On the rational numbers every cut is defined by a countable join of instances of the order (see Example 6.1), whereas, by the o-minimality of the rational order, the subsets of the rational numbers that are first-order definable with parameters are finite unions of intervals with rational or infinite endpoints. On the natural numbers a single sentence states that every element has finitely many predecessors (see Example 6.2), whereas, by compactness, every first-order theory that has an infinite model of this kind also has a model with an element above infinitely many others.

The same expressive power shapes the theory of stability. The countable joins distinguish limits of types that first-order logic cannot tell apart, and not every limit of types is realized in an elementary extension. A theory of stability for the logic must therefore single out the limits that are realized, and in Section 2 we describe these limits through model existence.

2. Model existence takes the place of compactness

In first-order logic compactness realizes every limit of types, and the types over a model form a compact space. In $L_{\omega_1\omega}(\mathbb{R})$ the realized types are described instead by the model existence theorem. In this section we fix the setting of the rest of the paper, define the local types realized in elementary extensions of a countable structure, and show that the types of elements of the structure approximate them. For relational vocabularies we then identify the realized types with the images of the Henkin types of Aubone, Dueñez and Iovino [ADI26], from which the model existence theorem builds its models, and we conclude that the realized types form an analytic set.

For the rest of the paper, let a countable vocabulary, a countable fragment $\mathcal{F}$, a countable structure M, and a formula $\varphi(x, y)$ of $\mathcal{F}$ be given, where x and y are finite tuples of variables, and write $r = r_\varphi$. When x or y is a tuple, an element corresponding to that variable tuple means a tuple of elements of the appropriate sorts; in particular, M_x and M_y denote the corresponding sets of tuples from M. An element of an elementary extension is always understood together with the extension in which the element lies.

Definition 2.1. For an element c of an elementary extension N of M for $\mathcal{F}$, the *φ-type* of c over M, in N, is the function $\mathrm{t}(c)\colon M_y \to [-r, r]$ given by $\mathrm{t}(c)(b) = \varphi^N(c, b)$, and, for an element d of an elementary extension N, the *φ^*-type* of d over M is the function $\mathrm{t}^*(d)\colon M_x \to [-r, r]$ given by $\mathrm{t}^*(d)(a) = \varphi^N(a, d)$. The set R_x of *realized φ-types* consists of the functions $\mathrm{t}(c)$, and the set R_y of *realized φ^*-types* consists of the functions $\mathrm{t}^*(d)$, for elements c and d of elementary extensions of M for $\mathcal{F}$.

Lemma 2.2. *Let c be an element of an elementary extension N of M for $\mathcal{F}$. Then there is a sequence $(a_k)_{k<\omega}$ in M_x such that $\varphi(a_k, b) \to \varphi(c, b)$ for every $b \in M_y$.*

Proof. Enumerate M_y as $(e_i)_{i<\omega}$, with repetitions if M_y is finite, and let a positive integer k be given. Choose rational numbers t_i with $|t_i - \varphi(c, e_i)| < 1/(2k)$ for $i < k$, and let

$$\theta(x, y_0, \dots, y_{k-1}) = \bigwedge_{i<k} \Big(\tfrac{1}{k}\mathbb{1} - \big|\varphi(x, y_i) - t_i\mathbb{1}\big| \Big),$$

where $|\chi|$ abbreviates $\chi \vee (-\chi)$. The formulas θ and $\sup_x \theta$ belong to $\mathcal{F}$, by the closure properties of $\mathcal{F}$. The value of θ^N at $(c, e_0, \dots, e_{k-1})$ is at least $1/(2k)$, so $(\sup_x \theta)^N(e_0, \dots, e_{k-1}) > 0$, and elementarity gives $(\sup_x \theta)^M(e_0, \dots, e_{k-1}) > 0$. Hence there is $a_k \in M_x$ with $|\varphi(a_k, e_i) - \varphi(c, e_i)| < 3/(2k)$ for $i < k$. Every $b \in M_y$ is some e_i, so $\varphi(a_k, b) \to \varphi(c, b)$ for every $b \in M_y$. □

The downward Löwenheim–Skolem property of countable fragments, which we prove for relational vocabularies, is used below and in Section 7.

Lemma 2.3. *Let the vocabulary be relational, let N be a structure, and let A be a countable subset of N. Then there is a countable structure M with $A \subseteq M$ and $M \preceq_{\mathcal{F}} N$.*

Proof. Build countable sets $A = A_0 \subseteq A_1 \subseteq \cdots$ as follows: given A_j, for every formula $\sup_u \theta(u, \bar{v})$ of $\mathcal{F}$, every tuple $\bar{e}$ of elements of A_j and every rational number t smaller than $(\sup_u \theta)^N(\bar{e})$, add to A_{j+1} an element u_0 of N with $\theta^N(u_0, \bar{e}) > t$. Let M be the substructure of N whose universe is the union of the sets A_j, which is countable because $\mathcal{F}$ is countable. Every sort of M is nonempty, by the witnesses for the formulas $\sup_u \mathbb{1}$. By induction on the formulas of $\mathcal{F}$, every formula has the same values in M and in N at tuples of M: atomic formulas agree because M is a substructure,

the connectives and the countable joins are computed pointwise, and for a supremum over a variable the elements added in the construction give $(\sup_u \theta)^M(\bar{e}) \geq (\sup_u \theta)^N(\bar{e})$, while the reverse inequality is clear. □

For the rest of this section the vocabulary is relational, as in the examples of Sections 6 and 7.

Let V_M be the vocabulary expanded by a constant for every element of M, and let C be a set of new constants that contains countably many constants of every sort, among them a tuple e_x of constants of the sorts of x and a tuple e_y of constants of the sorts of y. Let $\mathcal{F}_{M,C}$ be the set of sentences obtained from the formulas of $\mathcal{F}$ by substituting constants of V_M and of C for the free variables. The set $\mathcal{F}_{M,C}$ is countable, and the closure properties of $\mathcal{F}$ give the closure properties of $\mathcal{F}_{M,C}$ that the results of Aubone, Dueñez and Iovino quoted below require. Let a sentence of $\mathcal{F}_{M,C}$ be given, and let $\eta(\bar{u})$, with free variables $\bar{u}$, be a subformula of the formula from which the sentence comes. Every sentence $\eta(\bar{e})$ obtained by substituting constants $\bar{e}$ of the appropriate sorts belongs to $\mathcal{F}_{M,C}$. In particular, for every sentence $\sup_u \theta(u)$ of $\mathcal{F}_{M,C}$ and every constant e of the sort of u, the sentence $\theta(e)$ belongs to $\mathcal{F}_{M,C}$. The set $\mathcal{F}_{M,C}$ is also closed under the connectives, with variables renamed apart. A *formal type* assigns to every sentence θ of $\mathcal{F}_{M,C}$ a value in $[-r_\theta, r_\theta]$, and the formal types form the compact metrizable space $\prod_{\theta \in \mathcal{F}_{M,C}} [-r_\theta, r_\theta]$. A *Henkin type* is a formal type that satisfies the recursive equations for the connectives, for the countable joins and meets, and for the suprema over variables, where the supremum over a variable is taken over the constants of the required sort [ADI26]. Let H be the set of Henkin types h that satisfy the *elementary diagram* of M, that is, $\theta(\bar{a})^h = \theta^M(\bar{a})$ for every formula $\theta(\bar{u})$ of $\mathcal{F}$ and every tuple $\bar{a}$ of elements of M, where $\bar{a}$ also denotes the tuple of the constants of the elements. Define $\pi_x(h)\colon M_y \to [-r, r]$ and $\pi_y(h)\colon M_x \to [-r, r]$ by

$$\pi_x(h)(b) = \varphi(e_x, b)^h, \qquad \pi_y(h)(a) = \varphi(a, e_y)^h.$$

We use two results of Aubone, Dueñez and Iovino [ADI26] (see also [Iov26, Proposition 6.1]). By the Henkin realization lemma, every Henkin type h is realized by a term structure: the elements of the structure are the closed terms, which for a relational vocabulary are the constants, and every sentence of $\mathcal{F}_{M,C}$ takes in the structure the value assigned to it by h. By the topology theorem of Aubone, Dueñez and Iovino, the set of Henkin types that satisfy a countable set of conditions, which in this logic are strict inequalities between sentences and real numbers, is a G_δ subset of the space of formal types, and is therefore Polish. The elementary diagram of M prescribes the value $\theta^M(\bar{a})$ of each sentence $\theta(\bar{a})$, and the requirement $\theta(\bar{a}) = \theta^M(\bar{a})$ amounts to the countably many conditions $\theta(\bar{a}) < \theta^M(\bar{a}) + 1/n$ and $\theta(\bar{a}) > \theta^M(\bar{a}) - 1/n$, for $n \geq 1$. The elementary diagram is therefore a countable set of conditions, and the space H is Polish.

Theorem 2.4 (Realized types and Henkin types). *Suppose that the vocabulary is relational. Then $R_x = \pi_x(H)$ and $R_y = \pi_y(H)$, and the maps $\pi_x\colon H \to [-r, r]^{M_y}$ and $\pi_y\colon H \to [-r, r]^{M_x}$ are continuous. Consequently, the sets R_x and R_y are analytic subsets of the compact metrizable spaces $[-r, r]^{M_y}$ and $[-r, r]^{M_x}$.*

Proof. The coordinates of $\pi_x(h)$ are coordinates of h, so π_x is continuous into the product $[-r, r]^{M_y}$.

Let $h \in H$, and let N be the reduct to the original vocabulary of the term structure that realizes h, in which the constant of an element of M is identified with that element. The sorts of N are nonempty, since C contains constants of every sort. For tuples $\bar{a}$ of elements of M, the atomic formulas have the values that the elementary diagram prescribes, so M is a substructure of N. For such tuples every formula $\theta(\bar{u})$ of $\mathcal{F}$ satisfies $\theta^N(\bar{a}) = \theta(\bar{a})^h = \theta^M(\bar{a})$, so $M \preceq_{\mathcal{F}} N$. The element c of N named by e_x satisfies $\varphi^N(c, b) = \varphi(e_x, b)^h = \pi_x(h)(b)$, so $\pi_x(h) = \mathrm{t}(c) \in R_x$.

Conversely, let c be an element of an elementary extension N of M for $\mathcal{F}$. By Lemma 2.3, there is a countable structure N' with $M \cup \{c\} \subseteq N'$ and $N' \preceq_{\mathcal{F}} N$, and then $M \preceq_{\mathcal{F}} N'$. Choose an assignment of elements of N' to the constants of C, sort by sort, that is onto N' and assigns c to e_x, and let h assign to every sentence of $\mathcal{F}_{M,C}$ the value of the sentence in N' under this assignment. The recursive equations hold because values in a structure obey them, and the equations for the suprema over variables hold because every element of N' is named by a constant. The elementary diagram of M holds because $M \preceq_{\mathcal{F}} N'$, so $h \in H$, and $\pi_x(h)(b) = \varphi^{N'}(c, b) = \mathrm{t}(c)(b)$. The statements for the variable y are proved in the same way, and a continuous image of a Polish space in a Polish space is analytic. □

Remark 2.5. Theorem 2.4 is the form that the model existence theorem takes for local types, that is, for the types of the single formula φ: a local type is realized exactly when the elementary diagram of M, together with the conditions that prescribe the local type, is satisfied by a Henkin type. In Keisler's treatment of infinitary logic, the same role is played by consistency properties [Kei71]. Through Theorem 2.4, genericity in the sense of Definition 5.1 below is expressed by the space of Henkin types, an object built from the syntax, although the proofs of Section 5 work directly with the realizing elements.

The realized types are thus designated limits in the compact spaces of all limits of types, and the question of stability becomes a question about iterated limits taken only along designated limits. In Sections 3 and 4 we answer the question in a setting from which logic is absent.

3. Arhangel'skii's theorem reduces relative compactness to countable sets

Grothendieck's criterion demands the exchange of iterated limits along all sequences of points of a compact space. Arhangel'skii's C_p-theory characterizes relative compactness in spaces of real-valued continuous functions on many spaces besides compact ones, and among its results is his theorem on spaces of countable tightness. In this section we cast that theorem as a double limit criterion that admits only the points to which sequences converge along ultrafilters. Logic does not enter, and neither do the countable sets of the later sections.

For a topological space Z and a real number $r \geq 0$, we regard sets of functions from Z to $[-r, r]$ as subspaces of the compact Hausdorff space $[-r, r]^Z$, with the product topology, the topology of pointwise convergence. We write $C_p(Z)$ for the space of continuous real-valued functions on Z with the topology of pointwise convergence. For a sequence $(f_n)_{n<\omega}$ of functions from Z to $[-r, r]$ and an ultrafilter $\mathcal{U}$ on ω, the limit $\lim_{n\to\mathcal{U}} f_n$ exists in $[-r, r]^Z$ and is computed pointwise. A space Z has *countable tightness* when every point in the closure of a subset S of Z lies in the closure of a countable subset of S. First countable spaces, metrizable spaces among them, have countable tightness. A point z of Z lies in the closure of the set of terms of a sequence $(z_m)_{m<\omega}$ in Z exactly when the sequence converges to z along some ultrafilter on ω. If z lies in the closure, the sets $\{m < \omega : z_m \in O\}$, for the neighborhoods O of z, are nonempty and closed under finite intersections, and every ultrafilter that contains them all will serve.

Theorem 3.1 (Arhangel'skii's theorem as a double limit criterion). *Let Z be a space of countable tightness, let $r \geq 0$, and let $\mathcal{A}$ be a set of continuous functions from Z to $[-r, r]$. The following conditions are equivalent:*

(a) *for all sequences $(f_n)_{n<\omega}$ in $\mathcal{A}$ and $(z_m)_{m<\omega}$ in Z, all ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on ω, and every point z of Z to which $(z_m)_{m<\omega}$ converges along $\mathcal{V}$,*

$$\lim_{n\to\mathcal{U}} \lim_{m\to\mathcal{V}} f_n(z_m) = \lim_{m\to\mathcal{V}} \lim_{n\to\mathcal{U}} f_n(z_m);$$

(b) *for every sequence $(f_n)_{n<\omega}$ in $\mathcal{A}$ and every ultrafilter $\mathcal{U}$ on ω, the function $\lim_{n\to\mathcal{U}} f_n$ is continuous;*

(c) *every function in the closure of $\mathcal{A}$ in $[-r, r]^Z$ is continuous, that is, $\mathcal{A}$ is relatively compact in $C_p(Z)$.*

Proof. Suppose (b), let sequences, ultrafilters and a point z be as in (a), and let $g = \lim_{n\to\mathcal{U}} f_n$, which is continuous. Each f_n is continuous, so $\lim_{m\to\mathcal{V}} f_n(z_m) = f_n(z)$, and the left side of (a) is $\lim_{n\to\mathcal{U}} f_n(z) = g(z)$. The right side is $\lim_{m\to\mathcal{V}} g(z_m) = g(z)$, by continuity of g.

Suppose that (b) fails, and let $(f_n)_{n<\omega}$ and $\mathcal{U}$ be such that $g = \lim_{n\to\mathcal{U}} f_n$ is discontinuous, at a point z say. Then there is $\epsilon > 0$ such that z lies in the closure of the set of points z' with $|g(z') - g(z)| > \epsilon$. By countable tightness, z lies in the closure of a countable subset of that set, which is nonempty because z does not belong to the set. Enumerate the countable subset as a sequence $(z_m)_{m<\omega}$, repetitions allowed, and let $\mathcal{V}$ be an ultrafilter on ω along which the sequence converges to z. As in the first paragraph, the left side of (a) is $g(z)$, while the right side is $\lim_{m\to\mathcal{V}} g(z_m)$, which differs from $g(z)$ by at least ϵ, so (a) fails.

Condition (c) implies (b), since every limit of a sequence from $\mathcal{A}$ along an ultrafilter belongs to the closure of $\mathcal{A}$ in $[-r, r]^Z$. Suppose (b), let f belong to the closure of $\mathcal{A}$ in $[-r, r]^Z$, and suppose that f is discontinuous at a point z of Z. As in the preceding paragraph, there are $\epsilon > 0$ and a countable set D of points z' with $|f(z') - f(z)| > \epsilon$ such that z lies in the closure of D. The set $Z_0 = \{z\} \cup D$ is countable, so the restriction of f to Z_0 belongs to the closure of the restrictions of the members of $\mathcal{A}$ in the metrizable space $[-r, r]^{Z_0}$. Hence there are $f_n \in \mathcal{A}$ whose restrictions to Z_0 converge to the restriction of f. For a nonprincipal ultrafilter $\mathcal{U}$, the function $g = \lim_{n\to\mathcal{U}} f_n$ is continuous by (b), and $g = f$ on Z_0. Continuity of g at z yields a point z' of D with $|g(z') - g(z)| < \epsilon$, that is, with $|f(z') - f(z)| < \epsilon$, contrary to the choice of D. Hence f is continuous. The closure of $\mathcal{A}$ in $[-r, r]^Z$ is then a compact subset of $C_p(Z)$, which is (c). Conversely, a compact subset of $C_p(Z)$ containing $\mathcal{A}$ is closed in the Hausdorff space $\mathbb{R}^Z$ and contains the closure of $\mathcal{A}$ in $[-r, r]^Z$, since $[-r, r]^Z$ is closed in $\mathbb{R}^Z$. □

Condition (a) restricts the double limit condition to the points to which sequences converge along ultrafilters. The implication from (b) to (c) is an instance of a theorem of Arhangel'skii: every space Z of countable tightness is *weakly Grothendieck* [Arh98], that is, every set of continuous functions on Z whose infinite subsets have limit points in $C_p(Z)$ has compact closure in $C_p(Z)$. Tall gives a self-contained proof and a comprehensive discussion [Tal20]. Condition (b) asserts this property of $\mathcal{A}$ in a strong form, since a limit along a nonprincipal ultrafilter of a sequence of distinct members of $\mathcal{A}$ is a limit point of the set of terms of the sequence. The proof above gives the implication directly. The contribution of Theorem 3.1 is the double limit condition (a), which the next section develops into Theorem B. Double limits along restricted families of sequences go back to Pryce [Pry71] and to König and Kuhn [KK87]. For a compact space Z of countable tightness, every sequence of points converges along every ultrafilter, and condition (a) is the double limit condition along all sequences, as in Grothendieck's criterion [Gro52].

The one-sided criterion of the next section, Theorem 4.7, asserts that the exchange condition for prescribed sets P and Q of admissible limits holds exactly when each prescribed limit of

rows p determines a single continuous function on Q, the same for every sequence of rows and every ultrafilter along which the rows converge to p. This is the counterpart, for prescribed limits, of the continuity of limit functions in Theorem 3.1, and in Section 4 we prove it directly: exchange against the columns suffices to construct the unique separately continuous extension (Theorems 4.5 to 4.7).

4. Prescribed limits govern the exchange of iterated limits

Within this C_p-theoretic setting, we now prescribe the limits themselves. In this section we prove Theorem B, which characterizes the exchange of iterated limits along prescribed limits, for a bounded function on the product of two countable sets. Logic does not enter here either.

In this section, and in the parts of Section 7 that do not mention logic, let nonempty countable sets A and B, a real number $r \geq 0$, and a function $\varphi\colon A \times B \to [-r, r]$ be given. In Section 5 these letters are specialized to the formula fixed in Section 2. For $a \in A$, the *row* of a is the function $\rho_a\colon B \to [-r, r]$ given by $\rho_a(b) = \varphi(a, b)$, and, for $b \in B$, the *column* of b is the function $\kappa_b\colon A \to [-r, r]$ given by $\kappa_b(a) = \varphi(a, b)$. Let X be the closure of the set of rows in the product $[-r, r]^B$, and let Y be the closure of the set of columns in the product $[-r, r]^A$, with the product topologies. The sets A and B are countable, so the spaces X and Y are compact and metrizable, and we fix metrics d_X and d_Y for their topologies. Every point of X is the limit of a sequence of rows, and every point of Y is the limit of a sequence of columns. The *uniform metrics* are $\delta_X(p, p') = \sup_{b\in B} |p(b) - p'(b)|$ on X and $\delta_Y(q, q') = \sup_{a\in A} |q(a) - q'(a)|$ on Y.

Definition 4.1. Sets $P \subseteq X$ and $Q \subseteq Y$ of *admissible limits* are sets that contain all rows and all columns, respectively. For a sequence $(s_m)_{m<\omega}$ of real numbers, or of points of X or of Y, and an ultrafilter $\mathcal{U}$ on ω, we write $\lim_{m\to\mathcal{U}} s_m$ for the limit along $\mathcal{U}$. Limits in X and in Y along ultrafilters exist by compactness of the product spaces and are computed pointwise.

Definition 4.2. Let sets P and Q of admissible limits be given. The *exchange condition* for (P, Q) holds if

$$\lim_{m\to\mathcal{U}} q(a_m) = \lim_{n\to\mathcal{V}} p(b_n)$$

for all sequences $(a_m)_{m<\omega}$ in A and $(b_n)_{n<\omega}$ in B and all ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on ω such that $p = \lim_{m\to\mathcal{U}} \rho_{a_m}$ belongs to P and $q = \lim_{n\to\mathcal{V}} \kappa_{b_n}$ belongs to Q.

Since $q(a_m) = \lim_{n\to\mathcal{V}} \varphi(a_m, b_n)$ and $p(b_n) = \lim_{m\to\mathcal{U}} \varphi(a_m, b_n)$, the exchange condition says that the iterated limits $\lim_{m\to\mathcal{U}} \lim_{n\to\mathcal{V}} \varphi(a_m, b_n)$ and $\lim_{n\to\mathcal{V}} \lim_{m\to\mathcal{U}} \varphi(a_m, b_n)$ agree whenever the limits of the rows and of the columns are admissible. For $P = X$ and $Q = Y$, the exchange condition is Grothendieck's double limit condition in the ultrafilter form of Ben Yaacov [BY14]. If $\mathcal{U}$ is principal, at m_0 say, then $p = \rho_{a_{m_0}}$, and both limits equal $q(a_{m_0})$. The same holds when $\mathcal{V}$ is principal, so only nonprincipal ultrafilters matter.

Definition 4.3. Let sets P and Q of admissible limits be given. An *admissible ladder* consists of sequences $(a_m)_{m<\omega}$ in A and $(b_n)_{n<\omega}$ in B and real numbers $t < t'$ such that the rows ρ_{a_m} converge in X to a point of P, the columns κ_{b_n} converge in Y to a point of Q, and either

(i) $\varphi(a_m, b_n) \leq t$ for $m < n$ and $\varphi(a_m, b_n) \geq t'$ for $m \geq n$, or
(ii) $\varphi(a_m, b_n) \geq t'$ for $m < n$ and $\varphi(a_m, b_n) \leq t$ for $m \geq n$.

For $P = X$ and $Q = Y$, every ladder in the usual sense yields an admissible ladder: the ladder can be restricted to one subsequence of indices, used for both sequences, along which the rows and the columns converge in the compact metrizable spaces X and Y, and the restriction keeps the pattern of the ladder.

Theorem 4.4 (Ladders). *Let sets P and Q of admissible limits be given. The exchange condition for (P, Q) holds if and only if there is no admissible ladder.*

Proof. Suppose that an admissible ladder in case (i) is given, with limits $p \in P$ and $q \in Q$. A convergent sequence converges to the same point along every nonprincipal ultrafilter, so $p = \lim_{m\to\mathcal{U}} \rho_{a_m}$ and $q = \lim_{n\to\mathcal{V}} \kappa_{b_n}$ for all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$. For each m, the value $q(a_m)$ is the limit of $\varphi(a_m, b_n)$ as $n \to \infty$, which is at most t, so $\lim_{m\to\mathcal{U}} q(a_m) \le t$. For each n, the value $p(b_n)$ is the limit of $\varphi(a_m, b_n)$ as $m \to \infty$, which is at least t', so $\lim_{n\to\mathcal{V}} p(b_n) \ge t'$. Hence the exchange condition fails. Case (ii) is symmetric.

Conversely, suppose that the exchange condition fails, and let sequences, ultrafilters and limits $p \in P$ and $q \in Q$ be as in Definition 4.2, with $\alpha = \lim_{m\to\mathcal{U}} q(a_m)$ different from $\beta = \lim_{n\to\mathcal{V}} p(b_n)$. By the remark after Definition 4.2, both ultrafilters are nonprincipal. Suppose that $\alpha < \beta$, and choose real numbers $t_- < t < t' < t_+$ with $\alpha < t_-$ and $t_+ < \beta$. Enumerate B as $(e_k)_{k<\omega}$ and A as $(e'_k)_{k<\omega}$, with repetitions if necessary. We choose indices $n_0 < n_1 < \cdots$ and $m_0 < m_1 < \cdots$ in the order $n_0, m_0, n_1, m_1, \ldots$, as follows. At the stage of n_j, consider the set of indices n with $p(b_n) > t_+$, with $\varphi(a_{m_i}, b_n) < t$ for every $i < j$, and with $|\kappa_{b_n}(e'_k) - q(e'_k)| < 1/(j+1)$ for every $k \le j$. This set belongs to $\mathcal{V}$, since $\lim_{n\to\mathcal{V}} p(b_n) = \beta > t_+$, since $\lim_{n\to\mathcal{V}} \varphi(a_{m_i}, b_n) = q(a_{m_i}) < t_-$ by the choice of m_i below, and since $q = \lim_{n\to\mathcal{V}} \kappa_{b_n}$. The set is infinite, because $\mathcal{V}$ is nonprincipal, and we choose n_j in the set larger than the indices already chosen. At the stage of m_i, consider the set of indices m with $q(a_m) < t_-$, with $\varphi(a_m, b_{n_j}) > t'$ for every $j \le i$, and with $|\rho_{a_m}(e_k) - p(e_k)| < 1/(i+1)$ for every $k \le i$. This set belongs to $\mathcal{U}$ for the same reasons, since $\lim_{m\to\mathcal{U}} \varphi(a_m, b_{n_j}) = p(b_{n_j}) > t_+$, and we choose m_i in the set larger than the indices already chosen. Then $\varphi(a_{m_i}, b_{n_j}) < t$ for $i < j$ and $\varphi(a_{m_i}, b_{n_j}) > t'$ for $i \ge j$, the rows $\rho_{a_{m_i}}$ converge to p, and the columns $\kappa_{b_{n_j}}$ converge to q, so that the sequences $(a_{m_i})_{i<\omega}$ and $(b_{n_j})_{j<\omega}$ form an admissible ladder in case (i). If $\alpha > \beta$, the same construction, with the inequalities reversed, gives case (ii). □

The proof applies to arbitrary sets P and Q: the ultrafilters supply the large sets from which the ladder is extracted, and the approximation of p and q on finitely many points at each stage keeps the limits of the extracted sequences admissible.

Theorem 4.5 (The limit function). *Suppose that the exchange condition for (P, Q) holds, and let $p \in P$ and $q \in Q$ be given. For all sequences $(a_k)_{k<\omega}$ in A with $\rho_{a_k} \to p$ and $(b_n)_{n<\omega}$ in B with $\kappa_{b_n} \to q$, the limits $\lim_k q(a_k)$ and $\lim_n p(b_n)$ exist and are equal, and the common value $D(p, q)$ does not depend on the sequences. Moreover:*

(1) *$D(\rho_a, q) = q(a)$ and $D(p, \kappa_b) = p(b)$ for $a \in A$ and $b \in B$;*
(2) *$D(p, q) = \inf_n \sup_{k\ge n} q(a_k)$ for every sequence $(a_k)_{k<\omega}$ in A with $\rho_{a_k} \to p$;*
(3) *$|D(p, q) - D(p, q')| \le \delta_Y(q, q')$ and $|D(p, q) - D(p', q)| \le \delta_X(p, p')$ for $p, p' \in P$ and $q, q' \in Q$.*

Proof. For all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$, the exchange condition gives $\lim_{k\to\mathcal{U}} q(a_k) = \lim_{n\to\mathcal{V}} p(b_n)$. The left side does not depend on $\mathcal{V}$ and the right side does not depend on $\mathcal{U}$, so all limits of the bounded sequences $(q(a_k))_{k<\omega}$ and $(p(b_n))_{n<\omega}$ along nonprincipal ultrafilters have one common value L. Every cluster point of a bounded sequence is the limit of the sequence along some nonprincipal ultrafilter, so both sequences converge to L. If $(a'_k)_{k<\omega}$ is another sequence with $\rho_{a'_k} \to p$, the same argument with the same sequence $(b_n)_{n<\omega}$ gives $\lim_k q(a'_k) = \lim_n p(b_n) = L$, and likewise for another sequence of columns, so L does not depend on the sequences. Part (1) follows from the constant sequences, part (2) holds because the upper limit of a convergent sequence is the limit, and part (3) follows from $|q(a_k) - q'(a_k)| \le \delta_Y(q, q')$ and from $|p(b_n) - p'(b_n)| \le \delta_X(p, p')$ by passage to the limit. □

Theorem 4.6 (Separately continuous extension). *Let sets P and Q of admissible limits be given. The exchange condition for (P, Q) holds if and only if there is a function $F\colon P \times Q \to [-r, r]$ that is continuous in each variable, for the topologies of P and Q as subspaces of X and Y, and satisfies $F(\rho_a, \kappa_b) = \varphi(a, b)$ for $a \in A$ and $b \in B$. In that case F is unique, and $F = D$.*

Proof. Suppose that the exchange condition holds, and let $F = D$, which extends φ by part (1) of Theorem 4.5. Fix $p \in P$, and let $(q_j)_{j<\omega}$ converge to q in Q. For each j, choose a sequence of columns converging to q_j. By Theorem 4.5, the values of p along the sequence converge to $D(p, q_j)$, so there is $b_j \in B$ with $d_Y(\kappa_{b_j}, q_j) < 1/(j+1)$ and $|p(b_j) - D(p, q_j)| < 1/(j+1)$. Then $\kappa_{b_j} \to q$, so $p(b_j) \to D(p, q)$ by Theorem 4.5, and therefore $D(p, q_j) \to D(p, q)$. Thus $D(p, \cdot)$ is continuous on the metrizable space Q, and the symmetric argument shows that $D(\cdot, q)$ is continuous on P.

Conversely, let F be as in the statement. For $a \in A$ and $q \in Q$, let $(b_n)_{n<\omega}$ be a sequence with $\kappa_{b_n} \to q$. Continuity of $F(\rho_a, \cdot)$ gives $F(\rho_a, q) = \lim_n \varphi(a, b_n) = q(a)$. Symmetrically, $F(p, \kappa_b) = p(b)$ for $p \in P$ and $b \in B$. Now let sequences, ultrafilters and limits $p \in P$ and $q \in Q$ be as in Definition 4.2. The rows ρ_{a_m} converge to p along $\mathcal{U}$, and $F(\cdot, q)$ is continuous on P, so $\lim_{m\to\mathcal{U}} q(a_m) = \lim_{m\to\mathcal{U}} F(\rho_{a_m}, q) = F(p, q)$, and, in the same way, $\lim_{n\to\mathcal{V}} p(b_n) = \lim_{n\to\mathcal{V}} F(p, \kappa_{b_n}) = F(p, q)$. The two limits are equal, so the exchange condition holds. Finally, for $p \in P$, $q \in Q$ and a sequence $(a_k)_{k<\omega}$ with $\rho_{a_k} \to p$, continuity of $F(\cdot, q)$ gives $F(p, q) = \lim_k F(\rho_{a_k}, q) = \lim_k q(a_k) = D(p, q)$. □

The next criterion is one-sided in the sense of Grothendieck, since it concerns only the limits of rows, evaluated at the admissible limits of columns.

Theorem 4.7 (One-sided criterion). *Let sets P and Q of admissible limits be given. The exchange condition for (P, Q) holds if and only if, for every $p \in P$, there is a continuous function $G_p\colon Q \to [-r, r]$ such that*

$$\lim_{m\to\mathcal{U}} q(a_m) = G_p(q)$$

for every sequence $(a_m)_{m<\omega}$ in A, every ultrafilter $\mathcal{U}$ on ω with $\lim_{m\to\mathcal{U}} \rho_{a_m} = p$, and every $q \in Q$. In that case $G_p = D(p, \cdot)$.

Proof. If the exchange condition holds, let $G_p = D(p, \cdot)$, which is continuous by Theorem 4.6. For $(a_m)_{m<\omega}$, $\mathcal{U}$ and q as in the statement, and for a sequence $(b_n)_{n<\omega}$ with $\kappa_{b_n} \to q$, the exchange condition and Theorem 4.5 give $\lim_{m\to\mathcal{U}} q(a_m) = \lim_n p(b_n) = D(p, q)$. Conversely, let the functions G_p be given. Taking $q = \kappa_b$ gives $G_p(\kappa_b) = \lim_{m\to\mathcal{U}} \varphi(a_m, b) = p(b)$. For sequences, ultrafilters and limits p and q as in Definition 4.2, the hypothesis gives $\lim_{m\to\mathcal{U}} q(a_m) = G_p(q)$, and continuity of G_p on Q gives $\lim_{n\to\mathcal{V}} p(b_n) = \lim_{n\to\mathcal{V}} G_p(\kappa_{b_n}) = G_p(q)$. The last assertion follows from the first half of the proof and from the uniqueness of the limits. □

The function G_p is required to give the limits of rows at every admissible limit of columns, and not only to extend the function $b \mapsto p(b)$ continuously from the columns. The weaker requirement, even imposed on both sides, does not imply the exchange condition.

Example 4.8 (The half graph). Let $A = B = \mathbb{N}$, and let $\varphi(m, n)$ be 1 if $m < n$ and -1 otherwise. The rows converge to the constant function $p_\infty = -1$ on B, and the columns converge to the constant function $q_\infty = 1$ on A. The space X consists of the rows and p_∞, the space Y consists of the columns and q_∞, and the rows and the columns are isolated points. Let $P = X$ and $Q = Y$. Every point of X extends continuously from the columns to Y: the row ρ_m by the

evaluation $q \mapsto q(m)$, and p_∞ by the constant -1. Symmetrically, every point of Y extends continuously from the rows to X. The exchange condition nevertheless fails, since $q_\infty(m) = 1$ for every m and $p_\infty(n) = -1$ for every n. The function required by Theorem 4.7 for p_∞ would take the value -1 at the columns, hence at q_∞ by continuity, and the value $\lim_m q_\infty(m) = 1$ at q_∞.

5. Stability is the definability of realized types

The two strands of the argument now meet: the realized types of Section 2 supply the designated limits, and Theorem B supplies the criterion. In this section we prove Theorem A.

We apply Section 4, and later Section 7, with $A = M_x$ and $B = M_y$, the sets of tuples of elements of M of the lengths of x and of y, and with the restriction of φ to $A \times B$. For $a \in M_x$ and $b \in M_y$, the types $\mathrm{t}(a)$ and $\mathrm{t}^*(b)$, computed in M, are then the row ρ_a and the column κ_b. By Lemma 2.2, the realized types lie in the closures X and Y of the rows and of the columns, and they contain the rows and the columns, which are the types of elements of M, realized in M itself. Hence R_x and R_y are sets of admissible limits.

Definition 5.1. The formula φ satisfies the *generic double limit condition* over M if the exchange condition for (R_x, R_y) holds, and φ has the *generic order property* over M if there is an admissible ladder for (R_x, R_y).

In terms of elements, the generic double limit condition says that $\lim_{m\to\mathcal{U}} \varphi(a_m, d) = \lim_{n\to\mathcal{V}} \varphi(c, b_n)$ whenever c realizes the limit of the types $\mathrm{t}(a_m)$ along $\mathcal{U}$ and d realizes the limit of the types $\mathrm{t}^*(b_n)$ along $\mathcal{V}$, since $\varphi(a_m, d) = \mathrm{t}^*(d)(a_m)$ and $\varphi(c, b_n) = \mathrm{t}(c)(b_n)$. The value $\varphi(c, d)$ for elements of different extensions is never used.

When φ satisfies the generic double limit condition over M, Theorem 4.5 applies with $P = R_x$ and $Q = R_y$, and we call $D(p, q)$ the *generic value* of φ at (p, q): the limit of $q(a_k)$ along any sequence $(a_k)_{k<\omega}$ in M_x whose rows converge to p, and of $p(b_n)$ along any sequence $(b_n)_{n<\omega}$ in M_y whose columns converge to q.

A *formula in the instances* is a formula of $L_{\omega_1\omega}(\mathbb{R})$, in the vocabulary expanded by constants for the elements of M, built from the formula $\mathtt{1}$ and from instances $\varphi(a, y)$, with $a \in M_x$, by the connectives of the logic, countable joins and meets included, without quantifiers. The value of such a formula θ at an element d of an elementary extension depends only on the values $\varphi(a, d)$, so θ induces a function $\tilde\theta$ on R_y with $\tilde\theta(\mathrm{t}^*(d)) = \theta(d)$. A realized φ-type p is *generically definable* if there is a formula θ in the instances such that $\tilde\theta$ is continuous on R_y and $\lim_{m\to\mathcal{U}} \varphi(a_m, d) = \theta(d)$ for every sequence $(a_m)_{m<\omega}$ in M_x, every ultrafilter $\mathcal{U}$ with $\lim_{m\to\mathcal{U}} \mathrm{t}(a_m) = p$, and every element d of an elementary extension. Such a formula θ is called a *definition* of p.

Theorem 5.2 (Stability and definability of realized types). *The following conditions are equivalent:*

1. *φ satisfies the generic double limit condition over M;*
2. *φ does not have the generic order property over M;*
3. *every realized φ-type over M is generically definable.*

In that case, for every element c of an elementary extension and every sequence $(a_k)_{k<\omega}$ in M_x with $\varphi(a_k, b) \to \varphi(c, b)$ for every $b \in M_y$, the formula

$$\psi_c(y) = \bigwedge_{n<\omega} \bigvee_{k\geq n} \varphi(a_k, y)$$

is a definition of $\mathrm{t}(c)$: the formula takes the value $\varphi(c, b)$ at every $b \in M_y$ and the value $D(\mathrm{t}(c), \mathrm{t}^*(d))$ at every element d of an elementary extension, and every definition θ of $\mathrm{t}(c)$ satisfies $\tilde\theta = D(\mathrm{t}(c), \cdot)$.

Proof. Conditions (1) and (2) are equivalent by Theorem 4.4. The formulas $\varphi(a_k, y)$ have a priori bound r, so the joins $\bigvee_{k\geq n} \varphi(a_k, y)$ and their meet are formulas in the instances, and the value of ψ_c at d is $\inf_n \sup_{k\geq n} \mathrm{t}^*(d)(a_k)$. If (1) holds, then, by Theorem 4.5, this value is $D(\mathrm{t}(c), \mathrm{t}^*(d))$, which is $\varphi(c, b)$ for $d = b \in M_y$. The induced function $D(\mathrm{t}(c), \cdot)$ is continuous on R_y by Theorem 4.6, and Theorem 4.7 gives $\lim_{m\to\mathcal{U}} \varphi(a_m, d) = D(\mathrm{t}(c), \mathrm{t}^*(d)) = \psi_c(d)$ whenever $\lim_{m\to\mathcal{U}} \mathrm{t}(a_m) = \mathrm{t}(c)$. Hence ψ_c is a definition, and (3) holds. If (3) holds, the functions $\tilde{\theta}$ induced by the definitions satisfy the hypothesis of Theorem 4.7 for (R_x, R_y), so (1) holds, and Theorem 4.7 gives $\tilde{\theta} = D(\mathrm{t}(c), \cdot)$. □

For elements c and d of one elementary extension, the value $D(\mathrm{t}(c), \mathrm{t}^*(d))$ need not equal $\varphi(c, d)$. For instance, in the first-order fragment, let M carry an equivalence relation E, two-valued, with infinitely many infinite classes, let $\varphi = E$, and let c and d be elements of one new class of an elementary extension. Then $\varphi(c, d) = 1$, while $\mathrm{t}(c)$ and $\mathrm{t}^*(d)$ take only the value -1, so that $D(\mathrm{t}(c), \mathrm{t}^*(d)) = -1$. As with the definitions of types in first-order logic, the value D may be read heuristically as the value of φ for an element realizing $\mathrm{t}(c)$ placed generically with respect to d.

6. Expressive power demands definitions compatible with limits

Theorem A asks more of a definition than agreement with the type on the model: continuity on the realized types, and compatibility with limits. The examples of this section show that each demand is forced by the expressive power of the countable joins, and that the passage to realized limits can render stable a formula that first-order logic declares unstable.

Example 6.1 (The double limit condition fails for an unstable formula). Let the vocabulary consist of one binary predicate symbol $<$ with a priori bound 1, let $\mathcal{F}$ be the set of first-order formulas, let M be the ordered set $\mathbb{Q}$, in which $a < b$ has the value 1 if $a < b$ and -1 otherwise, and let $\varphi(x, y)$ be the formula $x < y$. For every first-order formula θ there are a finite set V_θ of real numbers, depending only on θ, and, for each $v \in V_\theta$, a formula θ_v of classical first-order logic in the symbol $<$, such that in every structure in which $<$ takes only the values 1 and -1, the formula θ takes values in V_θ and takes the value v exactly where θ_v holds. The claim is proved by induction on θ. For a supremum over a variable u, the value w is taken exactly where $\exists u\, \theta_w \wedge \bigwedge_{v\in V_\theta,\, v>w} \neg\exists u\, \theta_v$ holds. The ordered set $\mathbb{R}$, with the same convention, is an elementary extension of $\mathbb{Q}$ in classical first-order logic, since the theory of dense linear orders without endpoints is model complete, and therefore $\mathbb{Q} \preceq_{\mathcal{F}} \mathbb{R}$. Let $(a_m)_{m<\omega}$ and $(b_n)_{n<\omega}$ be increasing sequences of rational numbers with limit $\sqrt{2}$, and let $c = d = \sqrt{2}$ in $\mathbb{R}$. For a rational number b, the value $\varphi(a_m, b)$ converges to $\varphi(\sqrt{2}, b)$, and for a rational number a, the value $\varphi(a, b_n)$ converges to $\varphi(a, \sqrt{2})$. Hence the element $\sqrt{2}$ realizes the limits of the types of both sequences along all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$. Nevertheless $\varphi(a_m, \sqrt{2}) = 1$ for every m and $\varphi(\sqrt{2}, b_n) = -1$ for every n, so the generic double limit condition fails. The type of the cut at $\sqrt{2}$ is still defined on $\mathbb{Q}$ by the formula $\bigvee_{a\in\mathbb{Q},\, a>\sqrt{2}}(a < y)$ of $L_{\omega_1\omega}(\mathbb{R})$, whose terms have the common a priori bound 1: at a rational number b, the formula has the value 1 exactly when some rational number $a > \sqrt{2}$ satisfies $a < b$, that is, exactly when $\sqrt{2} < b$.

Example 6.2 (The generic condition holds where the double limit condition fails). Let the vocabulary be as in Example 6.1, let M be the ordered set $\mathbb{N}$ with the same convention, and let $\varphi(x, y)$ be the formula $x < y$. The sequences $a_m = m$ and $b_n = n$ give $\lim_m \lim_n \varphi(a_m, b_n) = 1$ and $\lim_n \lim_m \varphi(a_m, b_n) = -1$, so Grothendieck's double limit condition fails in M.

Let $\pi_0 = \mathtt{1}$, and, for a positive integer n, let

$$\pi_n(x) = \sup_{x_1} \cdots \sup_{x_n} \big((x_1 < x_2) \wedge \cdots \wedge (x_{n-1} < x_n) \wedge (x_n < x)\big),$$

so that $\pi_n^M(m)$ is 1 if $m \geq n$ and -1 otherwise. Let

$$\sigma = \inf_x \bigvee_{n<\omega} (-\pi_{n+1}(x)), \qquad \tau = \inf_u \inf_v ((u < v) \vee -(u < v)),$$

where the join is permitted because all of the terms have a priori bound 1. For natural numbers n and b, let ϵ be 1 if $n < b$ and -1 otherwise, let ϵ^* be 1 if $b < n$ and -1 otherwise, and let

$$\zeta_{n,b}(y) = \inf_x (-\pi_n(x) \vee \pi_{n+1}(x) \vee \epsilon\,(x < y)), \qquad \zeta^*_{n,b}(y) = \inf_x (-\pi_n(x) \vee \pi_{n+1}(x) \vee \epsilon^*\,(y < x)).$$

Let $\mathcal{F}$ be the smallest countable fragment that contains σ, τ, and the formulas $\zeta_{n,b}$ and $\zeta^*_{n,b}$ for all natural numbers n and b. In M the sentences σ and τ have the value 1, and so do the formulas $\zeta_{n,b}$ and $\zeta^*_{n,b}$ at the element b: at an element $x \neq n$ either $\pi_n(x) = -1$ or $\pi_{n+1}(x) = 1$, and at $x = n$ the last term has the value 1 by the choice of ϵ and of ϵ^*.

Let e be an element of an elementary extension N of M for $\mathcal{F}$. Since $\tau^N = 1$, the predicate $<$ takes only the values 1 and -1 in N, and so do the formulas π_n. Since $\sigma^N = 1$, some formula π_{n+1} has the value -1 at e. Let n be the least such number, so that $\pi_n^N(e) = 1$. Since $\zeta^N_{n,b}(b) = 1$, the value of the last term at $x = e$ is 1, that is, $(e < b)^N = \epsilon = (n < b)^M$, and likewise $(b < e)^N = (b < n)^M$. Hence every realized φ-type and every realized φ^*-type over M is the type of an element of M.

The generic double limit condition holds. Let sequences and ultrafilters be given as in Definition 4.2, with limits in R_x and R_y realized by elements c and d, and let m_0 and n_0 be natural numbers with $\mathrm{t}(c) = \mathrm{t}(m_0)$ and $\mathrm{t}^*(d) = \mathrm{t}^*(n_0)$. Along $\mathcal{U}$ the values $\varphi(a_m, m_0)$ converge to $\varphi(m_0, m_0) = -1$, and the values $\varphi(a_m, m_0 + 1)$ converge to $\varphi(m_0, m_0 + 1) = 1$. These values lie in $\{-1, 1\}$, so $m_0 \leq a_m < m_0 + 1$, that is, $a_m = m_0$, for $\mathcal{U}$-almost all m, and likewise $b_n = n_0$ for $\mathcal{V}$-almost all n. Both limits of Definition 4.2 are therefore equal to the value of $m_0 < n_0$.

Example 6.3 (The half graph in first-order logic). This example realizes Example 4.8 in first-order logic. Let the vocabulary, the structure $M = \mathbb{N}$ and the formula $\varphi(x, y) = x < y$ be as in Example 6.2, and let $\mathcal{F}$ be the set of first-order formulas. The sentence τ and the formulas π_n and $\zeta_{n,b}$, $\zeta^*_{n,b}$ of Example 6.2 are first-order. Let e be an element of an elementary extension N of M for $\mathcal{F}$. The predicate $<$ is two-valued in N because $\tau^N = 1$. If $\pi^N_{n+1}(e) = -1$ for some n, then, for the least such n, the argument of Example 6.2 gives $\mathrm{t}(e) = \mathrm{t}(n)$ and $\mathrm{t}^*(e) = \mathrm{t}^*(n)$. Otherwise $\pi^N_{b+1}(e) = 1$ for every $b \in \mathbb{N}$, and the first-order formulas $\inf_x(-\pi_{b+1}(x) \vee (y < x))$ and $\inf_x(-\pi_{b+1}(x) \vee -(x < y))$, which have the value 1 at $y = b$ in M, give $(b < e)^N = 1$ and $(e < b)^N = -1$. Such elements exist, for instance in an ultrapower of M by a nonprincipal ultrafilter, which is an elementary extension by Łoś's theorem [Kei23]. Hence R_x consists of the types $\mathrm{t}(m)$ and the constant function $p_\infty = -1$, and R_y consists of the types $\mathrm{t}^*(n)$ and the constant function $q_\infty = 1$. The types $\mathrm{t}(m)$ converge to p_∞ and the types $\mathrm{t}^*(n)$ converge to q_∞. Every realized φ-type has a continuous definition in the sense of Section 10: a formula in the instances that agrees with the type on M_y and induces a continuous function on R_y, with no requirement of compatibility with limits. For $\mathrm{t}(m)$ such a definition is the instance $\varphi(m, y)$, whose value at q_∞ is 1, and for p_∞ it is the formula -1. Symmetrically, every realized φ^*-type has a continuous definition on R_x: the instance $\varphi(x, n)$ for $\mathrm{t}^*(n)$, with value -1 at p_∞, and the formula 1 for q_∞. The generic double limit condition nevertheless fails: for nonstandard elements c and d and the sequences $a_m = m$ and $b_n = n$, the limit of $\varphi(a_m, d)$ is 1 and the limit of $\varphi(c, b_n)$ is -1. The type p_∞ is not generically definable: a definition θ of p_∞ would satisfy $\tilde{\theta}(\mathrm{t}^*(n)) = p_\infty(n) = -1$, hence $\tilde{\theta}(q_\infty) = -1$ by continuity, while compatibility with the limits of the instances $\varphi(m, y)$ at q_∞ forces $\tilde{\theta}(q_\infty) = 1$. Continuity cannot be dropped from the definition of generic definability either. The formula $\bigwedge_{a\in\mathbb{N}} \varphi(a, y)$, whose terms have the common a priori bound 1, takes the value -1 at every $\mathrm{t}^*(n)$ and the value 1 at q_∞, and it is compatible with the

limits for p_∞, as the instances $\varphi(m, y)$ are for the types $\mathrm{t}(m)$. The function that the formula induces on R_y is, however, discontinuous at q_∞.

7. Finite definitions and the counting of types rest on compactness

Two classical characterizations of stability have no counterpart in $L_{\omega_1\omega}(\mathbb{R})$: the definability of types by uniform limits of finite combinations of instances, and the separability of the spaces of types over countable models. In this section we determine exactly when the first holds, show by an example that the second has no analogue, and put in its place a dichotomy from descriptive set theory, together with consequences of Baire category for the Henkin types. We close with the predicates that formulas define on the realized types when these are few.

A *finite combination of columns* is a function on Y of the form $q \mapsto g(q(a_1), \dots, q(a_k))$, where $a_1, \dots, a_k \in A$ and g is built from the coordinates and the constant 1 by rational scalar multiplication, addition, maximum and minimum. Such a function is continuous on Y.

Proposition 7.1. *Let $p \in X$ be given. The following conditions are equivalent:*

1. *there is a continuous function G on Y with $G(\kappa_b) = p(b)$ for every $b \in B$;*
2. *for every real number $\epsilon > 0$ there is a finite combination of columns h with $|h(\kappa_b) - p(b)| \leq \epsilon$ for every $b \in B$.*

Proof. Suppose (1). The finite combinations of columns form a set of continuous functions on the compact space Y that contains the rational constants and is closed under rational scalar multiplication, addition, maxima and minima. For points $q \neq q'$ of Y, real numbers α and β, and $\epsilon > 0$, choose $a \in A$ with $q(a) \neq q'(a)$ and rational numbers s and u with $|sq(a) + u - \alpha| < \epsilon$ and $|sq'(a) + u - \beta| < \epsilon$. The finite combination $q \mapsto sq(a) + u$ takes values within ϵ of α and β at q and q'. The proof of the lattice form of the Stone–Weierstrass theorem [Sto48] then applies with approximate interpolation. For $q, q' \in Y$ choose $h_{q,q'}$ with $|h_{q,q'} - G| < \epsilon$ at q and at q', a rational constant if $q = q'$. For fixed q, compactness gives finitely many q' such that $h_q = \min_{q'} h_{q,q'}$ satisfies $h_q < G + \epsilon$ on Y and $h_q > G - \epsilon$ at q, and compactness again gives finitely many q such that $\max_q h_q$ is within ϵ of G on Y. This function is a finite combination of columns, which gives (2).

Suppose (2), and choose finite combinations h_k with $|h_k(\kappa_b) - p(b)| \leq 1/k$ for every $b \in B$. The continuous functions h_k satisfy $|h_k - h_l| \leq 1/k + 1/l$ on the dense set of columns, hence on Y, so they converge uniformly to a continuous function G on Y with $G(\kappa_b) = p(b)$. □

Corollary 7.2. *Suppose that the exchange condition for (P, Q) holds and that $Q = Y$. Then every $p \in P$ satisfies the equivalent conditions of Proposition 7.1.*

Proof. By Theorem 4.6, the function $D(p, \cdot)$ is continuous on $Q = Y$, and $D(p, \kappa_b) = p(b)$ by Theorem 4.5. □

For $P = X$ and $Q = Y$, Corollary 7.2 is the countable form of Grothendieck's criterion, with the conclusion of the Stone–Weierstrass theorem.

When $\mathcal{F}$ consists of first-order formulas, every point q of Y is realized. Let $\kappa_{b_n} \to q$, and let $\mathcal{V}$ be a nonprincipal ultrafilter on ω. By Łoś's theorem for first-order real-valued logic [Kei23], the ultrapower of M by $\mathcal{V}$ is an elementary extension of M through the diagonal embedding, and the class d of the sequence $(b_n)_{n<\omega}$ satisfies $\varphi(a, d) = \lim_{n \to \mathcal{V}} \varphi(a, b_n) = q(a)$. Then $R_y = Y$ and, symmetrically, $R_x = X$, and Corollary 7.2 recovers the passage from the double limit condition to definability by finite combinations of instances, for countable models.

A set $S \subseteq X$ is *few* if S is separable for the uniform metric δ_X. For a function φ with values in $\{-1, 1\}$, distinct points of X are at δ_X-distance 2, and the few sets are then exactly the countable ones.

Proposition 7.3. *If every point of a set $S \subseteq X$ satisfies the equivalent conditions of Proposition 7.1, then S is few.*

Proof. For $p \in S$, the continuous function $\hat{p}$ on Y of condition (1) is unique, since the columns are dense in Y, and the supremum of $|\hat{p} - \hat{p}'|$ over Y equals the supremum over the columns, which is $\delta_X(p, p')$. The map $p \mapsto \hat{p}$ is therefore an isometric embedding of (S, δ_X) into the space of continuous functions on the compact metrizable space Y with the supremum norm. This space is separable, and every subset of a separable metric space is separable. □

The exchange condition alone does not make the admissible limits few.

Example 7.4 (Continuum many admissible limits). Let $B = \mathbb{N}$, let A be the countable set of functions $a\colon \mathbb{N} \to \{-1, 1\}$ that take the value 1 at only finitely many points, and let $\varphi(a, n) = a(n)$. The rows are the elements of A, and X is the set of all functions $\mathbb{N} \to \{-1, 1\}$. The columns κ_n, given by $\kappa_n(a) = a(n)$, are pairwise distinct, and Y consists of the columns and the constant function -1, their only cluster point. For every n_0, let $s_{n_0} \in A$ take the value 1 exactly at n_0. Let $P = X$, and let Q be the set of columns. The exchange condition holds: if $\kappa_{b_n} \to \kappa_{n_0}$ along $\mathcal{V}$, then $s_{n_0}(b_n) \to 1$, so $b_n = n_0$ for $\mathcal{V}$-almost all n, and both limits of Definition 4.2 equal $p(n_0)$. The set P consists of continuum many points at mutual δ_X-distance 2, so P is not few, and, by Proposition 7.3, some point of P does not satisfy the conditions of Proposition 7.1. Since the columns are isolated in Y and converge to the constant function -1, the points of P that satisfy them are exactly the eventually constant ones.

Example 7.5 (Continuum many realized types). This example realizes the phenomenon of Example 7.4, continuum many admissible limits under the exchange condition, with realized types. The space Y may have more cluster points than in Example 7.4. Let the vocabulary have two sorts, *points* and *indices*, together with a binary predicate symbol $<$ on indices and a predicate symbol $\mathtt{P}$ on pairs of a point and an index, both with a priori bound 1. Let N_0 be the structure whose indices are the natural numbers, with $<$ as in Example 6.2, and whose points are all functions $f\colon \mathbb{N} \to \{-1, 1\}$, with $\mathtt{P}(f, i) = f(i)$. Let $\varphi(x, y)$ be the formula $\mathtt{P}(x, y)$, with x a point variable and y an index variable. Let π_n and σ be the formulas of Example 6.2, on the sort of indices, let τ be the sentence $\inf_u \inf_v |u < v| \wedge \inf_x \inf_v |\mathtt{P}(x, v)|$, and, for every natural number n, let

$$\chi_n(x) = \inf_u \inf_v \bigl(-\pi_n(u) \vee \pi_{n+1}(u) \vee -\pi_n(v) \vee \pi_{n+1}(v) \vee \epsilon(x, u, v)\bigr),$$

where $\epsilon(x, u, v) = (\mathtt{P}(x, u) \wedge \mathtt{P}(x, v)) \vee (-\mathtt{P}(x, u) \wedge -\mathtt{P}(x, v))$ has the value 1 exactly when $\mathtt{P}(x, u) = \mathtt{P}(x, v)$ in a structure in which $\mathtt{P}$ is two-valued. Let $\mathcal{F}$ be the smallest countable fragment that contains σ, τ and the formulas χ_n. These sentences and formulas have the value 1 in N_0, at every point in the case of χ_n, since exactly one index has exactly n predecessors. For every natural number n, let s_n be the point with $s_n(n) = 1$ and $s_n(i) = -1$ for $i \neq n$. By Lemma 2.3, there is a countable structure $M \preceq_{\mathcal{F}} N_0$ that contains all indices and all points s_n.

Every function $f\colon \mathbb{N} \to \{-1, 1\}$ is a point of N_0, and $\mathrm{t}(f)(n) = f(n)$. The realized φ-types over M are therefore continuum many, at mutual distance 2, and they are not few.

Let e be an index of an elementary extension N of M for $\mathcal{F}$. As in Example 6.2, the sentences σ and τ give a natural number n such that e has exactly n predecessors in the sense of the formulas π_n and π_{n+1}. For every point a of M, the value $\chi_n^N(a) = 1$, taken at $u = e$ and $v = n$, then gives $\mathtt{P}(a, e) = \mathtt{P}(a, n)$. Hence every realized φ^*-type over M is the type of an index of M.

The generic double limit condition holds. Let sequences and ultrafilters be given as in Definition 4.2, with limits in R_x and R_y realized by elements c and d, and let n_0 be a natural number with $\mathrm{t}^*(d) = \mathrm{t}^*(n_0)$. At the point s_{n_0} of M, the values $\mathtt{P}(s_{n_0}, b_n)$ converge along $\mathcal{V}$ to $\mathtt{P}(s_{n_0}, n_0) = 1$, so $b_n = n_0$ for $\mathcal{V}$-almost all n, and $\lim_{n \to \mathcal{V}} \varphi(c, b_n) = \varphi(c, n_0)$. On the other side, $\lim_{m \to \mathcal{U}} \varphi(a_m, d) = \lim_{m \to \mathcal{U}} \varphi(a_m, n_0) = \varphi(c, n_0)$, since c realizes the limit of the types $\mathrm{t}(a_m)$

along $\mathcal{U}$ and n_0 is an index of M. By Proposition 7.3, some realized φ-type is not definable by finite combinations of instances. The points of Y outside R_y are the cluster points of the types $\mathrm{t}^*(n)$. None of these cluster points is realized: every realized φ^*-type is some $\mathrm{t}^*(n_0)$, which takes the value 1 at s_{n_0}, while $\mathrm{t}^*(n)(s_{n_0}) = -1$ for $n \neq n_0$.

When P is analytic, descriptive set theory supplies a dichotomy in place of compactness, of the kind that Morley found for the number of countable models of a sentence of $L_{\omega_1\omega}$ [Mor70], without the intermediate alternative of Morley's theorem.

Proposition 7.6 (Few or a perfect set). *Let $S \subseteq X$ be a nonempty analytic set. Then exactly one of the following holds:*

(1) *the set S is few;*
(2) *there are a real number $\epsilon > 0$ and a subset of S homeomorphic, in the topology of X, to the Cantor space, any two distinct points of which are at δ_X-distance greater than ϵ.*

Proof. A set as in (2) is uncountable and ϵ-separated for δ_X, so (1) and (2) exclude each other. The metric δ_X is the supremum of the continuous functions $(p, p') \mapsto |p(b) - p'(b)|$ on $X \times X$, and is therefore lower semicontinuous, so that, for every $\epsilon > 0$, the set Γ_ϵ of pairs (p, p') with $\delta_X(p, p') > \epsilon$ is open in $X \times X$. There is a continuous map f from the Baire space ω^ω onto S [Kec95, Section 14.A], and the set Γ'_ϵ of pairs (s, s') of $\omega^\omega \times \omega^\omega$ with $(f(s), f(s')) \in \Gamma_\epsilon$ is an open symmetric irreflexive relation on ω^ω. By Feng's theorem [Fen93, Theorem 1.1], either ω^ω is the union of countably many sets C_i none of which contains a pair of Γ'_ϵ, or there is a nonempty perfect set $K' \subseteq \omega^\omega$ any two distinct points of which form a pair of Γ'_ϵ.

In the second case, the map f is injective on K', since $\delta_X(f(s), f(s')) > \epsilon$ for distinct $s, s' \in K'$. The set K' is closed in ω^ω, hence a nonempty perfect Polish space, so K' contains a compact subset K homeomorphic to the Cantor space [Kec95, Theorem 6.2], and f maps K homeomorphically onto a subset of S as in (2). If the second case fails for every $\epsilon = 1/k$, then for every k the sets $f(C_i)$ cover S, and any two points of one of them are at δ_X-distance at most $1/k$. Choosing one point in each nonempty set $f(C_i)$, for all k and i, gives a countable subset of S that is dense for δ_X, which is (1). □

For a function with values in $\{-1, 1\}$, Proposition 7.6 reduces to the perfect set theorem for analytic sets, and in Example 7.4 the second case holds, together with the exchange condition.

Baire category gives joint continuity of the extension at many points when the admissible limits are parametrized by Polish spaces.

Proposition 7.7. *Suppose that the exchange condition for (P, Q) holds, and let H and H' be Polish spaces with continuous maps π of H onto P and π' of H' onto Q. Then the function $(h, k) \mapsto D(\pi(h), \pi'(k))$ on $H \times H'$ is continuous in each variable, and is jointly continuous at every point of a comeager subset of $H \times H'$.*

Proof. The function is continuous in each variable by Theorem 4.6, since π and π' are continuous. A real-valued function on the product of two metrizable spaces that is continuous in each variable is a pointwise limit of continuous functions, by Lebesgue's theorem in the generality given by Rudin [Rud81], and therefore of Baire class 1 on the metrizable space $H \times H'$. The points of continuity of a function of Baire class 1 on a Polish space form a comeager set [Kec95, Theorem 24.14]. □

The set of points of joint continuity is a G_δ set, so the Kuratowski–Ulam theorem [Kec95, Theorem 8.41] applies: for comeager many $h \in H$, the function of Proposition 7.7 is jointly continuous at the point (h, k) for comeager many $k \in H'$.

Corollary 7.8. *Suppose that the vocabulary is relational. Then the dichotomy of Proposition 7.6 holds for the realized φ-types, and, if φ satisfies the generic double limit condition over M, the*

function $(h,k) \mapsto D(\pi_x(h), \pi_y(k))$ on $H \times H$ is continuous in each variable and jointly continuous at every point of a comeager subset of $H \times H$.

Proof. The set R_x is nonempty and analytic by Theorem 2.4, so Proposition 7.6 applies, and Proposition 7.7 applies with $H' = H$, $\pi = \pi_x$ and $\pi' = \pi_y$. □

The analytic sets of Theorem 2.4 cannot be replaced by Borel sets, even under the generic double limit condition.

Proposition 7.9 (Realized types that are not Borel). *There are a relational vocabulary, a countable fragment $\mathcal{F}$, a countable structure M and a formula $\varphi(x,y)$ that satisfies the generic double limit condition over M, such that R_x is an analytic subset of X that is not Borel and every realized φ^*-type over M is a column.*

Proof. Let $\Sigma = \{-1,1\}^{\mathbb{N}}$, let $A_0 \subseteq \Sigma$ be an analytic set that is not Borel, let Fin be the countable set of points of Σ that take the value 1 only finitely often, and let $A = A_0 \cup \mathrm{Fin}$. The set A is analytic and not Borel, since A_0 is obtained from A by removing a subset of the countable set Fin. There is a tree T of pairs (s,t) of finite sequences of the same length, with values in $\{-1,1\}$ and in $\mathbb{N}$, such that A is the projection of the set $[T]$ of infinite branches (u,v) of T [Kec95, Section 14.A].

Let the vocabulary have two sorts, *points* and *indices*, a predicate symbol $\mathtt{P}$ on pairs of a point and an index, predicate symbols B_n and $W_{n,k}$ on points and I_n on indices, for natural numbers n and k, all with a priori bound 1. Let N_0 be the structure whose indices are the natural numbers and whose points are the branches $(u,v) \in [T]$, with $\mathtt{P}((u,v),j) = u(j)$, $B_n((u,v)) = u(n)$, $W_{n,k}((u,v)) = 1$ if $v(n) = k$ and -1 otherwise, and $I_n(j) = 1$ if $j = n$ and -1 otherwise. Let $\varphi(x,y)$ be the formula $\mathtt{P}(x,y)$, with x a point variable and y an index variable. Let $\mathcal{F}$ be the smallest countable fragment that contains the following sentences, all of which have the value 1 in N_0: the sentences $\inf_x \inf_u |\mathtt{P}(x,u)|$, $\inf_x |B_n(x)|$, $\inf_x |W_{n,k}(x)|$ and $\inf_u |I_n(u)|$, which make the symbols two-valued; the sentences $\inf_u(-I_m(u) \vee -I_n(u))$ and $\inf_x(-W_{n,k}(x) \vee -W_{n,l}(x))$, for $m \neq n$ and $k \neq l$; the sentences $\inf_u \bigvee_n I_n(u)$ and $\inf_x \bigvee_k W_{n,k}(x)$, for every n; the sentences $\inf_x \inf_u(-I_n(u) \vee (\mathtt{P}(x,u) \wedge B_n(x)) \vee (-\mathtt{P}(x,u) \wedge -B_n(x)))$, which say that $\mathtt{P}(x,n) = B_n(x)$; and the sentence

$$\inf_x \bigwedge_{(s,t) \notin T} \bigvee_{i<|s|} (-s(i)B_i(x) \vee -W_{i,t(i)}(x)),$$

which says that every initial segment of the pair coded by x belongs to T. For every natural number n, let s_n be the point of Σ with $s_n(n) = 1$ and $s_n(i) = -1$ for $i \neq n$, and choose $z_n \in [T]$ with first coordinate s_n, which exists since $s_n \in \mathrm{Fin} \subseteq A$. By Lemma 2.3, there is a countable structure $M \preceq_{\mathcal{F}} N_0$ that contains all indices and all points z_n.

For $u \in A$ and $(u,v) \in [T]$, the point (u,v) of N_0 has $\mathrm{t}((u,v)) = u$. Conversely, let c be a point of an elementary extension N of M for $\mathcal{F}$. The sentences above have the value 1 in N, so for every n there is exactly one $k = v(n)$ with $W^N_{n,k}(c) = 1$; with $u(n) = B^N_n(c)$, the last sentence gives $(u,v) \in [T]$, and the sentences on I_n give $\varphi(c,n) = u(n)$. Hence $R_x = A$. The space X is all of Σ, since $R_x \subseteq X$ by Lemma 2.2 and A contains the dense set Fin, so that R_x is not a Borel subset of X, and in particular is not Polish.

On the other side, every index e of N satisfies $I^N_n(e) = 1$ for some n, and then $\varphi(a,e) = B_n(a) = \varphi(a,n)$ for every point a of M; every realized φ^*-type is a column κ_n. The argument of Example 7.5, with z_{n_0} in place of s_{n_0}, shows that φ satisfies the generic double limit condition over M. □

Proposition 7.10. *Suppose that the vocabulary is relational. Then the set of Henkin types $h \in H$ such that $\pi_x(h)$ is a limit, for the uniform metric δ_X, of a sequence of rows contains a dense G_δ*

subset of H, and is therefore comeager in H. For every such h, the type $\pi_x(h)$ is definable by finite combinations of instances.

Proof. For $a \in M_x$ let λ_a be the sentence $\sup_y |\varphi(e_x, y) - \varphi(a, y)|$ of $\mathcal{F}_{M,C}$. The supremum over the constants of the sort of y includes the constants of the elements of M, so $\lambda_a^h \geq \delta_X(\pi_x(h), \rho_a)$ for every $h \in H$. For $\epsilon > 0$, the set W_ϵ of $h \in H$ with $\lambda_a^h < \epsilon$ for some $a \in M_x$ is open in H. The set W_ϵ is dense. Let a nonempty basic open subset of H be given by finitely many conditions $\theta_i \in (\alpha_i, \beta_i)$ with rational endpoints α_i and β_i, for $i < k$, on sentences $\theta_i = \theta_i'(e_x, \bar{w}, \bar{m})$ of $\mathcal{F}_{M,C}$, where $\bar{w}$ lists the other constants of C that occur and $\bar{m}$ the constants of elements of M, and let h belong to the set. Let $\Theta(x, \bar{u}, \bar{m}) = \bigwedge_{i<k}((\theta_i' - \alpha_i 1) \wedge (\beta_i 1 - \theta_i'))$, a formula of $\mathcal{F}$ with parameters from M. In the structure N realizing h, which is an elementary extension of M by Theorem 2.4, the value of $\sup_x \sup_{\bar{u}} \Theta$ at $\bar{m}$ is positive, so the same holds in M, and there are $a \in M_x$ and a tuple $\bar{a}'$ of elements of M with $\Theta^M(a, \bar{a}', \bar{m}) > 0$. Let h' be the Henkin type obtained, as in the proof of Theorem 2.4, from M itself and an assignment of the constants of C onto M that sends e_x to a and $\bar{w}$ to $\bar{a}'$. Then $h' \in H$, h' satisfies the conditions $\theta_i \in (\alpha_i, \beta_i)$, and $\lambda_a^{h'} = 0$, so $h' \in W_\epsilon$. Hence $\bigcap_k W_{1/k}$ is a dense G_δ subset of H, contained in the set of the statement, since for h in this set there are $a_k \in M_x$ with $\delta_X(\pi_x(h), \rho_{a_k}) < 1/k$. The coordinate functions $q \mapsto q(a_k)$ are finite combinations of columns that approximate $\pi_x(h)$ within $1/k$ on the columns, which gives condition (2) of Proposition 7.1. □

Baire-generic Henkin types therefore give local types that are approximately realized in M, with or without the generic double limit condition. More precisely, the set of $h \in H$ for which $\pi_x(h)$ cannot be approximated uniformly by rows is meager in H.

In first-order logic the definable predicates on a space of types are the continuous functions on that space. When the realized types are few, the Stone–Weierstrass theorem for separable metric spaces of the companion paper [Iov26] gives the infinitary form of this fact, describing every function on the realized φ^*-types that a formula in the instances defines, and not only the definitions of types.

Proposition 7.11 (Definable predicates on few realized types). *Suppose that $r > 0$ and that the metric space (R_y, δ_Y) is separable. Then the functions $\tilde{\theta}$ induced by the formulas θ in the instances are exactly the bounded Lipschitz functions on (R_y, δ_Y), and the uniform closure of the set of these functions is the space of bounded uniformly continuous functions on (R_y, δ_Y). If, moreover, φ satisfies the generic double limit condition over M, then, for every realized φ-type p, the function $q \mapsto D(p, q)$ on R_y is $\tilde{\theta}$ for such a formula θ.*

Proof. Let $(a_i)_{i<\omega}$ enumerate M_x. The coordinate functions $q \mapsto q(a_i)$ take values in $[-r, r]$, separate the points of R_y, and have coordinate metric $\sup_i |q(a_i) - q'(a_i)| = \delta_Y(q, q')$. Reading the instance $\varphi(a_i, y)$ as the ith nullary predicate symbol of the coordinate vocabulary of the companion paper [Iov26] turns the formulas in the instances into the sentences for that vocabulary, and $\tilde{\theta}$ into the function that the sentence θ defines. The first assertion then follows from Theorems 2.3 and 3.1 of the companion paper [Iov26]. The second follows from the first, since $q \mapsto D(p, q)$ is bounded by r and 1-Lipschitz for δ_Y, by Theorem 4.5. □

The Lipschitz property in Proposition 7.11 refers to the uniform metric δ_Y, whose topology on R_y is in general finer than the topology of R_y as a subspace of Y. A function induced by a formula in the instances need not be continuous for the second topology, as the formula $\bigwedge_{a \in \mathbb{N}} \varphi(a, y)$ of Example 6.3 shows. Continuity for the second topology, together with compatibility with limits, is what generic definability adds.

Remark 7.12 (The role of Beth's theorem). The generic double limit condition determines the function $q \mapsto D(\mathrm{t}(c), q)$ on R_y implicitly, and the function is 1-Lipschitz for the uniform

metric δ_Y, by Theorem 4.5. When (R_y, δ_Y) is separable, Proposition 7.11 turns the implicit definition into an explicit one, by a formula of $L_{\omega_1\omega}(\mathbb{R})$ in the instances. This passage is the Lipschitz form of the approximate Beth property of the companion paper [Iov26, Corollary 5.7], and the proof there writes the formula as the Lipschitz envelope of McShane, an infimum of cones over a dense sequence of realized types. The formula ψ_c is an explicit definition obtained without the hypothesis of separability. For a relational vocabulary, separability cannot be omitted from the description of all definable predicates in Proposition 7.11. If (R_y, δ_Y) is not separable, the dichotomy of Proposition 7.6, with the roles of rows and columns exchanged and applied to the analytic set R_y of Theorem 2.4, gives $\epsilon > 0$ and a set $K \subseteq R_y$ homeomorphic to the Cantor space whose points are at mutual δ_Y-distance greater than ϵ. Each of the $2^{\mathfrak{c}}$ functions from K to $\{0, \epsilon\}$ is 1-Lipschitz for δ_Y and extends, by McShane's formula truncated at 0 and ϵ, to a bounded 1-Lipschitz function on R_y, and distinct functions have distinct extensions. There are only $\mathfrak{c}$ formulas in the instances, so some of these extensions are induced by no formula.

8. Free approximations characterize stability

In first-order logic, stability has a characterization of a different kind, by an abstract independence relation. Harnik and Harrington proved that a theory is stable exactly when the theory carries a relation, written with Makkai's symbol $\downarrow$, that satisfies invariance, monotonicity, symmetry, existence, stationarity over models, finite character and local character, and that such a relation is nonforking [HH84]; Kim and Pillay later characterized the simple theories in the same manner [KP97]. The axioms determine the relation. In this section we give the local form of this characterization in $L_{\omega_1\omega}(\mathbb{R})$. The existence axiom of the characterization is the work of compactness, and in $L_{\omega_1\omega}(\mathbb{R})$ the axiom fails for pairs of elements: two realized types of a formula that satisfies Grothendieck's double limit condition on all limits may occur together in no elementary extension of M, and types that do occur together may occur only with a value of φ other than the generic one (Propositions 8.3 and 8.4). The relation is therefore placed on finite approximations by elements of M, and existence is supplied by Lemma 2.2 in place of compactness.

A *context* is a triple $\gamma = (\epsilon, B, A)$, where $\epsilon > 0$ and $B \subseteq M_y$ and $A \subseteq M_x$ are finite; a context is *finer* than another if it has a smaller ϵ and larger sets. For $p \in R_x$ and $q \in R_y$, a pair $(a, b) \in M_x \times M_y$ *approximates* (p, q) *in* γ if $|\varphi(a, b') - p(b')| < \epsilon$ for $b' \in B$ and $|\varphi(a', b) - q(a')| < \epsilon$ for $a' \in A$. The set B thus carries the instances at which p is tested, and A those at which q is tested.

Theorem 8.1 (Stability and independence). *The formula φ satisfies the generic double limit condition over M if and only if there is a relation $a \overset{p}{\underset{\gamma}{\perp}} \overset{q}{} b$, read "$(a,b)$ is a free approximation of (p,q) in γ", holding only of pairs (a,b) that approximate (p,q) in γ, with the following properties:*

(1) monotonicity*: a free approximation of (p,q) in a context is free in every coarser context;*
(2) existence*: every $(p,q) \in R_x \times R_y$ has a free approximation in every context;*
(3) stationarity*: for every (p,q) and every $\eta > 0$ there is a context in which any two free approximations (a,b) and (a',b') of (p,q) satisfy $|\varphi(a,b) - \varphi(a',b')| < \eta$;*
(4) base*: $F(\rho_a, q) = q(a)$ and $F(p, \kappa_b) = p(b)$ for $a \in M_x$ and $b \in M_y$;*
(5) finite character*: for every (p,q) and every $\eta > 0$ there are a finite $B_0 \subseteq M_y$ and $\delta > 0$ such that $|F(p',q) - F(p,q)| < \eta$ whenever $|p'(b) - p(b)| < \delta$ for $b \in B_0$, and symmetrically in q.*

Here $F(p,q)$, the free value *of (p,q), is the limit of the values of φ at free approximations of (p,q) along finer contexts, which exists by (1)–(3). Every such relation has free value $F = D$, the generic value of Section 5.*

Proof. If such a relation exists, then F is defined on $R_x \times R_y$, extends φ by (4), and is continuous in each variable by (5); Theorem 4.6 gives the generic double limit condition and $F = D$.

Conversely, suppose that φ satisfies the generic double limit condition, and declare $a \overset{p}{\underset{\gamma}{\perp}} \overset{q}{} b$, for a context $\gamma = (\epsilon, B, A)$, when (a,b) approximates (p,q) in γ and either $|\varphi(a,b) - q(a)| < \epsilon$, so that b is chosen after a, or $|\varphi(a,b) - p(b)| < \epsilon$, so that a is chosen after b. Monotonicity is clear. For existence, choose by Lemma 2.2 an element a whose row is within ϵ of p on B, and then an element b whose column is within ϵ of q on $A \cup \{a\}$. By Theorem 4.6, the function $D(\cdot, q)$ is continuous on R_x, and $D(\rho_a, q) = q(a)$ by Theorem 4.5; hence in a fine enough context every free approximation of the first kind satisfies

$$|\varphi(a,b) - D(p,q)| \leq |\varphi(a,b) - q(a)| + |q(a) - D(p,q)| < \eta,$$

and symmetrically for the second kind. Stationarity follows, with $F = D$, and base and finite character then follow from Theorems 4.5 and 4.6. The last assertion is the first part of the proof. □

The axioms are those of Harnik and Harrington, stated locally. Symmetry is built in, since the axioms are invariant under the exchange of the two sides, and local character is replaced by the approximation of every realized type by elements of the countable model (Lemma 2.2). The one departure is that existence concerns approximations rather than pairs of elements, and Propositions 8.3 and 8.4 show that existence cannot be strengthened. The determination of the relation comes, as in their theorem, from base and finite character, here through the uniqueness of separately continuous extensions (Theorem 4.6): the free value is the generic value D.

Finite character cannot be required jointly in the two variables, even in first-order logic. Let φ be a binary predicate symbol interpreted in a countably infinite structure M as equality, with the value 1 on the diagonal and -1 elsewhere, and let $\mathcal{F}$ be first-order. The limits of rows are the rows and the constant type $p_\infty = -1$, the limits of columns are the columns and $q_\infty = -1$, all of them are realized, and the exchange condition holds, with $D(p_\infty, q_\infty) = -1$ and $D(\rho_a, \kappa_a) = \varphi(a,a) = 1$. For finite sets $A_0 \subseteq M_x$ and $B_0 \subseteq M_y$ and an element a of M outside both, the row ρ_a agrees with p_∞ on B_0 and the column κ_a agrees with q_∞ on A_0, so D is not jointly continuous at (p_∞, q_∞). Joint continuity holds only generically, at the points given by Corollary 7.8.

The relation determines which pairs of elements are free. Suppose that φ satisfies the generic double limit condition over M, and call a pair (c,d) of elements of one elementary extension of M *free* if $\varphi(c,d) = D(\mathrm{t}(c), \mathrm{t}^*(d))$.

Proposition 8.2. *Suppose that φ satisfies the generic double limit condition over M, and let c and d be elements of an elementary extension N of M, with $p = \mathrm{t}(c)$ and $q = \mathrm{t}^*(d)$.*

1. *The pair (c,d) is free if and only if for every $\epsilon > 0$ and every finite $B \subseteq M_y$ there is $a \in M_x$ with $|\varphi(a,b) - p(b)| < \epsilon$ for $b \in B$ and $|\varphi(a,d) - \varphi(c,d)| < \epsilon$, and if and only if the symmetric condition holds with an element of M_y.*
2. *If $c \in M_x$ or $d \in M_y$, then (c,d) is free.*
3. *If D is jointly continuous at (p,q), then (c,d) is free.*
4. *If $\mathcal{F}$ consists of first-order formulas, then every $(p,q) \in R_x \times R_y$ is realized by a free pair, and (c,d) is free exactly when $\varphi(c,d)$ is the value given at d by the definition of p over M.*

Proof. (1) Enumerate M_y as $(e_i)_{i<\omega}$. If the pair is free, Lemma 2.2 gives elements a_k whose rows converge to p, and $\varphi(a_k,d) = q(a_k) \to D(p,q) = \varphi(c,d)$ by Theorem 4.5. Conversely, the condition with $\epsilon = 1/k$ and $B = \{e_0, \dots, e_k\}$ gives elements a_k whose rows converge to p with $q(a_k) = \varphi(a_k,d) \to \varphi(c,d)$, while $q(a_k) \to D(p,q)$ by Theorem 4.5. The other form is symmetric.

(2) is part (1) of Theorem 4.5.

(3) Enumerate M_x as $(e'_i)_{i<\omega}$, and for $k \geq 1$ choose rational numbers t_i, s_i and u within $1/(2k)$ of $\varphi(c,e_i)$, $\varphi(e'_i,d)$ and $\varphi(c,d)$, for $i < k$. The formula

$$\sup_x \sup_y \Big(\bigwedge_{i<k} \big(\tfrac{1}{k}\mathtt{1} - |\varphi(x,e_i) - t_i\mathtt{1}|\big) \wedge \bigwedge_{i<k} \big(\tfrac{1}{k}\mathtt{1} - |\varphi(e'_i,y) - s_i\mathtt{1}|\big) \wedge \big(\tfrac{1}{k}\mathtt{1} - |\varphi(x,y) - u\mathtt{1}|\big)\Big)$$

of $\mathcal{F}$, with parameters in M, is positive in N, hence in M, which gives $a_k \in M_x$ and $b_k \in M_y$ with $\rho_{a_k} \to p$, $\kappa_{b_k} \to q$ and $\varphi(a_k,b_k) \to \varphi(c,d)$. Since $\varphi(a_k,b_k) = D(\rho_{a_k}, \kappa_{b_k})$, joint continuity gives $\varphi(c,d) = D(p,q)$.

(4) Let d realize q in N, let $\rho_{a_k} \to p$, and let $\mathcal{U}$ be a nonprincipal ultrafilter on ω. In the ultrapower $N^{\mathcal{U}}$, an elementary extension of N by Łoś's theorem [Kei23], the class c' of $(a_k)_{k<\omega}$ realizes p and satisfies $\varphi(c',d) = \lim_{\mathcal{U}} q(a_k) = D(p,q)$. When $\mathcal{F}$ is first-order every point of Y is realized (Section 7), so that $D(p,\cdot)$ is the continuous extension of p to the compact space of all φ^*-types, that is, the definition of p over M. □

In the first-order case, Proposition 8.2(4) identifies the free pairs with those in which the type of c over $M \cup \{d\}$ agrees at d with the nonforking extension of p, and Theorem 8.1 becomes the local form of the theorem of Harnik and Harrington. In $L_{\omega_1\omega}(\mathbb{R})$ free pairs are guaranteed only at the points of joint continuity of D, and the next two propositions show that they may fail to exist elsewhere. Both use structures of *patterns*, elements σ of $C = \{-1,1\}^\omega$ with the product topology, and two-valued unary predicate symbols $\mathtt{P}_n$, for $n < \omega$, interpreted by $\mathtt{P}_n(\sigma) = \sigma(n)$.

Proposition 8.3 (Types that never meet). *There are a countable relational vocabulary, a countable fragment $\mathcal{F}$, a countable structure M and a formula φ that satisfies Grothendieck's double limit condition on all limits of types over M, together with $p \in R_x$ and $q \in R_y$ that are realized by no pair of elements of one elementary extension of M.*

Proof. Add two-valued binary predicate symbols $\mathtt{G}_n$, interpreted by $\mathtt{G}_n(\sigma,\tau) = 1$ exactly when $\sigma(n) = \tau(n)$, and let

$$J(u,v) = \bigvee_{n<\omega} \mathtt{G}_n(u,v), \qquad \varphi(u,v) = \bigvee_{n<\omega} 2^{-n}\mathtt{G}_n(u,v),$$

so that, on patterns, $J(\sigma,\tau) = 1$ if σ and τ agree somewhere and -1 otherwise, and $\varphi(\sigma,\tau) = 2^{-m}$, where m is the first place of agreement ($2^{-\infty} = 0$). The function φ is continuous on $C \times C$. Let

Σ consist of $\inf_u \inf_v J(u,v)$ and of the sentences saying that the $\mathtt{P}_n$ and $\mathtt{G}_n$ are two-valued and that $\mathtt{G}_n(u,v) = 1$ exactly when $\mathtt{P}_n(u) = \mathtt{P}_n(v)$, and let $\mathcal{F}$ be the smallest fragment containing Σ and φ. Let S be the countable set of patterns that take the value 1 somewhere and -1 at all but finitely many places, and let M be the structure on S. The sets S, $S \cup \{\mathbf{1}\}$ and $S \cup \{-\mathbf{1}\}$, where $\mathbf{1}$ and $-\mathbf{1}$ are the constant patterns, are dense in C and *intersecting*: any two of their members agree somewhere. The structures on them therefore satisfy Σ.

The only countable joins in $\mathcal{F}$ are the substitution instances of J and φ. Interpret every formula of $\mathcal{F}$ on C by replacing J with the constant 1 and taking suprema over C; by induction on formulas, these functions are continuous, since a supremum over the compact space C of a continuous function is continuous, and on an intersecting dense set they give the values in the corresponding structure, since the supremum over a dense set of a continuous function is the supremum over C. Hence M is an elementary substructure of the structures on $S \cup \{\pm\mathbf{1}\}$.

Let p be the φ-type of $\mathbf{1}$ and q the φ^*-type of $-\mathbf{1}$. For $k < \omega$, let $e_k \in S$ take the value 1 only at k, and $f_k \in S$ the value 1 exactly at $0, \dots, k$; then $p(e_k) = 2^{-k}$ and $q(f_k) = 2^{-(k+1)}$. If c and d in an elementary extension N of M realized p and q, then N would satisfy Σ; from $\varphi(c, e_k) = 2^{-k}$ and two-valuedness, $\mathtt{G}_n(c, e_k) = -1$ for $n < k$, so that $\mathtt{P}_n(c) = 1$ for all n, and likewise $\mathtt{P}_n(d) = -1$ for all n, contrary to $\inf_u \inf_v J(u,v) = 1$.

For $j < \omega$ and $\tau \in \{-1, 1\}^{j+1}$, let $b_\tau \in S$ agree with τ at $0, \dots, j$, take the value 1 at $j+1$ and -1 beyond; a pattern σ agrees with $-\tau$ at $0, \dots, j$ exactly when $\varphi(\sigma, b_\tau) \leq 2^{-(j+1)}$. Since the values of φ have a gap between $2^{-(j+1)}$ and 2^{-j}, pointwise convergence of rows or of columns forces coordinatewise convergence of the patterns, and joint continuity of φ on $C \times C$ gives equal iterated limits along all sequences and ultrafilters. □

Proposition 8.4 (Types that meet). *There are a countable relational vocabulary, a countable fragment $\mathcal{F}$, a countable structure M, a formula φ that satisfies Grothendieck's double limit condition on all limits of types over M, and $(p, q) \in R_x \times R_y$ realized by a pair of elements of one elementary extension of M, such that every such pair (c, d) satisfies $\varphi(c, d) \neq D(p, q)$.*

Proof. For a pattern σ, let $\mathrm{fm}(\sigma)$ and $\mathrm{fp}(\sigma)$ in $\omega + 1$ be the first places where σ takes the values -1 and 1, respectively. Add a two-valued binary predicate symbol $\mathtt{H}$, let $E_n(u)$ be the finitary formula saying that $\mathtt{P}_i(u) = 1$ for $i < n$ and $\mathtt{P}_n(u) = -1$, let $E'_n(v)$ say that $\mathtt{P}_i(v) = -1$ for $i < n$ and $\mathtt{P}_n(v) = 1$, and let

$$\Theta(u, v) = \bigwedge_{n<\omega} \big(E_n(u) \leftrightarrow E'_n(v)\big),$$

with $\leftrightarrow$ the two-valued biconditional, so that $\Theta(\sigma, \tau) = 1$ exactly when $\mathrm{fm}(\sigma) = \mathrm{fp}(\tau)$, both values ∞ included. Let Σ consist of the sentences saying that the $\mathtt{P}_n$ and $\mathtt{H}$ are two-valued and of $\inf_u \inf_v (\mathtt{H}(u, v) \leftrightarrow \Theta(u, v))$, let $\mathcal{F}$ be the smallest fragment containing Σ, and let $\varphi = \mathtt{H}$. A set of patterns becomes a model of Σ with $\mathtt{H}(\sigma, \tau) = 1$ exactly when $\mathrm{fm}(\sigma) = \mathrm{fp}(\tau)$. Let S be the countable set of eventually constant patterns that take both values, let M be the structure on S, and let N be the structure on $T = S \cup \{\mathbf{1}, -\mathbf{1}\}$. On S both invariants are finite, one of them is 0, and each attains every finite value.

A substructure M of N is elementary if, for every formula $\psi(z, \bar{w})$ of $\mathcal{F}$ and every tuple $\bar{m}$ from M, the supremum of $\psi^N(z, \bar{m})$ over $z \in N$ equals its supremum over $z \in M$; this is the Tarski–Vaught test, with the usual proof by induction on formulas. In M and in N the formula Θ coincides with $\mathtt{H}$, so every formula of $\mathcal{F}$ is equivalent there to a first-order formula in $\mathtt{H}$ and finitely many $\mathtt{P}_n$, say with $n < n_0$. Two elements of N take the same values at all these atomic formulas, against every element of N, exactly when they agree at $0, \dots, n_0 - 1$ and have the same invariants fm and fp, since each value in $\omega + 1$ is attained by each invariant on T; as the logic has no equality, the value of a formula depends only on these classes. Given $\bar{m}$, choose $K \geq n_0$ larger than the invariants of $\bar{m}$. The exchange of the values K and ∞ of both invariants

is a bijection of the classes realized in N, preserves $\mathtt{P}_n$ for $n < n_0$ and $\mathtt{H}$, fixes the classes of $\bar{m}$, and sends the class of $\mathbf{1}$ to that of the pattern $u_K \in S$ that equals 1 before K and -1 from K on, and the class of $-\mathbf{1}$ to that of the pattern $v_K = -u_K$. Hence $\psi^N(\mathbf{1}, \bar{m}) = \psi^N(u_K, \bar{m})$ and $\psi^N(-\mathbf{1}, \bar{m}) = \psi^N(v_K, \bar{m})$, and M is an elementary substructure of N.

The φ-type of an element c of an elementary extension is determined by $\mathrm{fm}(c) \in \omega + 1$, and convergence of such types is convergence of fm in $\omega + 1$; likewise for φ^*-types and fp. Let p be the type of $\mathbf{1}$ and q the φ^*-type of $-\mathbf{1}$, both constant with value -1; the pair $(\mathbf{1}, -\mathbf{1})$ realizes them in N. Any pair (c, d) realizing (p, q) in an elementary extension, which satisfies Σ, has $\mathrm{fm}(c) = \mathrm{fp}(d) = \infty$, and therefore $\varphi(c, d) = \Theta(c, d) = 1$. For sequences in S with $\mathrm{fm}(a_m) \to k$ and $\mathrm{fp}(b_n) \to l$ in $\omega + 1$, both iterated limits of $\varphi(a_m, b_n)$ equal 1 if $k = l$ is finite and -1 otherwise, since the invariants on S are finite. Grothendieck's double limit condition therefore holds on all limits, and $D(p, q) = -1$. □

In Proposition 8.4 the pairs (u_K, v_K), which approximate both sides at once, have value 1, while the free approximations of Theorem 8.1 have value -1: the function D is discontinuous at (p, q). Together with Proposition 8.2(3), this locates exactly where pairs of elements can be free, and shows that the approximations of Theorem 8.1 cannot be replaced by pairs of elements.

9. What extends and what is new

For the reader versed in the first-order theory, we gather here what extends to $L_{\omega_1\omega}(\mathbb{R})$ and what is new to it.

The local stability theory of first-order logic extends as follows, with the realized types in place of all types.

	First-order logic	$L_{\omega_1\omega}(\mathbb{R})$
Stability	Exchange of iterated limits along all limits of types; equivalent to the absence of a ladder	Exchange along realized limits; equivalent to the absence of a ladder whose limits are realized (Definition 5.1 and Theorems 4.4 and 5.2)
Extension to types	Separately continuous, on the compact spaces of types	Unique on the realized types, continuous in each variable (Theorem 4.6), and jointly continuous on a comeager set of pairs of Henkin types (Corollary 7.8)
Definability of types	By uniform limits of finite combinations of instances	Every realized type by the single countable formula ψ_c, an upper limit of instances, continuous on the realized types and compatible with limits; equivalent to the generic double limit condition (Theorem 5.2)
Beth's theorem	Turns implicit definitions into explicit ones	The Lipschitz form of Beth's theorem when the realized types are few; the formula ψ_c gives an explicit construction for individual types without this hypothesis (Remark 7.12)
Value at a new parameter	Given by the defining scheme to the nonforking extension	The generic value D (Theorems 4.5 and 5.2)
Independence	Stability is the existence of an independence relation with the axioms of Harnik and Harrington, which is then nonforking	Stability is the existence of such a relation on finite approximations by elements of the model, which is then determined, with value D; pairs of elements are free at the points of joint continuity of D, and need not be free elsewhere (Theorem 8.1 and Propositions 8.2 to 8.4)
Definable predicates	The continuous functions on the compact space of types	When the realized types are few, the bounded Lipschitz functions for the uniform metric, and their uniform limits the bounded uniformly continuous functions; the generic definitions are those among them that are continuous for the topology of pointwise convergence and compatible with limits (Proposition 7.11)

When every limit of types is realized, as in first-order logic, the exchange condition yields continuous definitions on the compact type spaces and hence uniform approximation by finite combinations of instances (Corollary 7.2). This recovers the classical passage from stability to definability of types. Formulas with countable joins still have their greater expressive power, as Example 6.3 illustrates.

What is new to $L_{\omega_1\omega}(\mathbb{R})$ begins in analysis. For Tsirelson's and Schlumprecht's spaces, Casazza, Dueñez and Iovino established first-order nondefinability from their approximants [CDI18]. A single infinitary formula defines these norms and also the norm of the space of Argyros and Haydon, on which every bounded operator is a scalar multiple of the identity plus a compact operator, and the norms of the higher-order Tsirelson spaces (Proposition 1.1 and Example 1.2).

Within stability theory, definability by finite combinations of instances, which every type of a stable first-order formula enjoys, becomes a property of individual types: a realized type has it exactly when the type extends continuously from the columns to all limits of columns (Proposition 7.1). The property is distinct from the generic double limit condition (Examples 7.4 and 7.5), and Baire-generic Henkin types give types that have it (Proposition 7.10).

The count of types becomes richer in the same measure: over a countable model, where a stable first-order formula has few types, a formula that satisfies the generic double limit condition may have continuum many realized types at mutual distance 2 (Example 7.5). Such realized types are either few or contain a perfect set of types at mutual distances bounded below (Proposition 7.6 and Corollary 7.8), and definitions by finite combinations make them few (Proposition 7.3). For relational vocabularies they form an analytic set, which need not be Borel (Theorem 2.4 and Proposition 7.9).

The characterization of stability by independence survives in approximate form: existence, which compactness provides for pairs of elements in first-order logic, is supplied by approximation from the countable model, and its failure for pairs of elements (Propositions 8.3 and 8.4) shows that the approximate form is the correct one (Theorem 8.1).

Directions for further work are gathered in Section 10.

10. Problems

Two implications of the first-order theory remain open in $L_{\omega_1\omega}(\mathbb{R})$. Say that a realized φ-type p over a countable model M' has a *continuous definition* if some formula θ of $L_{\omega_1\omega}(\mathbb{R})$ in the instances, with parameters from M', satisfies $\theta(b) = p(b)$ for every $b \in M'_y$ and induces a continuous function on the realized φ^*-types over M'; no compatibility with limits is required. In first-order logic, continuous definitions over every countable model give the double limit condition, by Shelah's theorem [She78], its continuous form due to Ben Yaacov and Usvyatsov [BYU10], and Propositions 7.1 and 7.3; in $L_{\omega_1\omega}(\mathbb{R})$ they do not suffice over a single model (Example 6.3). Definability by finite combinations of instances makes the realized types few (Proposition 7.3), and for abstract admissible limits the converse fails under the exchange condition (Example 7.4).

(1) If, for every countable model M' of the $\mathcal{F}$-theory of M, every realized φ-type over M' has a continuous definition, does φ satisfy the generic double limit condition over every such model?

(2) If φ satisfies the generic double limit condition over M and the realized φ-types are separable for the uniform metric δ_X, is every realized φ-type over M uniformly approximable on M_y by finite combinations of instances, as in Proposition 7.1?

The independence relation of Theorem 8.1 is attached to a single model, and its passage between models $M \preceq_{\mathcal{F}} M'$, the counterpart of transitivity in the theorem of Harnik and Harrington [HH84], holds when $\mathcal{F}$ is first-order and in general meets the failure of amalgamation of Proposition 8.3. Three broader directions remain. The realized types are analytic for relational vocabularies and need not be Borel (Theorem 2.4 and Proposition 7.9), which leaves the hypotheses on the fragment under which they are Borel or Polish. Over uncountable models the topological part of Theorem B persists, with ultrafilters on the sets of rows and of columns in place of sequences, but a limit of types need not be the limit of a sequence of rows, and nets do not give formulas of a countable fragment; the counting of types may then depend on set-theoretic hypotheses, as the analogue of Feng's theorem for generalized Baire spaces does [SS23]. Finally, the model existence game of Aubone and Iovino [AI26] realizes the generic limits, and a characterization of the generic double limit condition by strategies in the game would connect the game with stability.

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