

GLEASON, KOCHEN–SPECKER, AND THE MODEL EXISTENCE GAME

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ABSTRACT. The Kochen–Specker theorem is a statement of two-valued propositional logic: in a Hilbert space of finite dimension at least three, certain finite sets of rays admit no assignment of the values 0 and 1 that gives the value 1 to exactly one member of each spanning set of mutually orthogonal rays among them. Such configurations are obtained from Gleason’s theorem by the compactness of propositional logic. We prove Gleason’s theorem and a strengthening of the Kochen–Specker theorem in infinitary real-valued logic, where the probability of a quantum proposition is the value of a formula. To this end we exhibit a countable theory of this logic in whose standard models the propositions form a lattice isomorphic to that of the closed subspaces of a separable infinite-dimensional complex Hilbert space, with probabilities given by a density operator as Gleason’s theorem requires. Every such space arises in this way, together with any density operator on it. Over this theory the Kochen–Specker hypothesis, that every probability is 0 or 1, becomes one member of a continuum of hypotheses indexed by the Borel sets of real numbers. Because the probabilities fill the unit interval in the standard models, each of these hypotheses fails there unless its set contains the whole interval. The same is true of the σ -projective sets, which are precisely the sets of values definable in those models. The Kochen–Specker hypothesis itself fails in every model, standard or not; yet every finite fragment of the corresponding theory is satisfiable, so that the compactness argument yielding the classical configuration is no longer available. In its place we employ model-theoretic forcing, cast as a two-player topological game in which every hypothesis expressible as an axiom and false in every model is refuted by a well-founded tree of finite obstructions; for the Kochen–Specker hypothesis each such tree branches infinitely. These obstructions are finer than the classical configuration, for they constrain the probabilities themselves rather than merely the values 0 and 1 and they disclose the axioms on which each refutation rests. The configuration of Kochen and Specker nonetheless survives within them as one stage of a play that a winning strategy reaches from the axioms.

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2020 *Mathematics Subject Classification.* Primary 03C75, 06C15, 81P10, 81P13, 91A44; Secondary 03C25, 03C66, 03G12.

Key words and phrases. Kochen–Specker theorem, Gleason’s theorem, hidden variables, quantum logic, orthomodular lattices, Solèr’s theorem, infinitary logic, real-valued logic, model existence theorem, model existence game, model-theoretic forcing.

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INTRODUCTION

The theorem of Kochen and Specker [KS67] has two faces. Read through Gleason’s theorem [Gle57], as J. S. Bell [Bel66] read it independently, it asserts that the lattice of closed subspaces of a Hilbert space of dimension at least three carries no probability measure with values in $\{0, 1\}$; in the form that Kochen and Specker gave it, it asserts that some finite configuration of rays admits no such assignment. That the second face follows from the first by compactness was observed by J. L. Bell [Bel96] in the language of partial Boolean algebras and later by Pitowsky [Pit98] in the language of propositional logic. In finite dimension a $\{0, 1\}$ -valued measure is a model of a propositional theory, and compactness guarantees that whenever such models are lacking, their absence is already witnessed by a finite subtheory. This finite set of Boolean constraints is what we shall call the classical finite obstruction. Hrushovski and Pitowsky [HP04] subsequently carried the analysis into first-order logic, in which probabilities can be named, and obtained in finite dimension both generalizations of the Kochen–Specker theorem and an effective form of Gleason’s theorem.

The present paper takes the classical finite obstruction as its point of departure and refines it in three directions. The notion is carried into separable infinite-dimensional Hilbert spaces with σ -additive measures and extended to hypotheses on the values of the probabilities other than the dichotomy between 0 and 1, while each obstruction is derived from the axioms by an explicit strategy. Such a refinement turns, in the first instance, on expressive power. Propositional logic apprehends probabilities only through the dichotomy between 0 and 1, whereas two-valued first-order logic can name a probability through relations such as $\mu(a) \leq r$ but never treats it as the value of a formula. Nor can either logic express the hypotheses of Gleason’s theorem: separability, the completeness of the lattice and σ -additivity lie beyond the reach of first-order logic, and separability and σ -additivity lie beyond that of any compact logic whatever. The expressive power that Gleason’s theorem demands therefore carries the

logic beyond the reach of the compactness theorem, and with it beyond the very argument that produces the configuration of Kochen and Specker. In place of that argument we employ model-theoretic forcing, cast as a two-player topological game, and it is from this game that our finite obstructions arise.

The logic in which we work is real-valued $L_{\omega_1, \omega}$, the infinitary extension of Keisler’s model theory for real-valued structures [Kei23, ADI26]. Because its formulas take real values, the probability of a quantum proposition is itself the value of a formula; because countable suprema and infima of formulas are again formulas, σ -additivity can be imposed as an axiom. Completeness demands more. By Solèr’s theorem [Sol95] it is precisely completeness that separates the lattices of closed subspaces of infinite-dimensional Hilbert spaces from the other pre-Solèr lattices, and yet by the downward Löwenheim–Skolem theorem no countable set of axioms about the lattice alone can capture it. For this reason the lattice is placed within a many-sorted structure. In the intended interpretation the *Hilbert sort* is the closed unit ball of a Hilbert space and the *Solèr sort* is the lattice of its closed subspaces, while the *sequence sort*, the closed unit ball of a space of square-summable sequences of vectors, renders every closed subspace the closed span of a sequence. A quantum probability then becomes a predicate μ on the Solèr sort, with values in $[0, 1]$. That this vocabulary recovers exactly the lattices $L(H)$ is a consequence of projective geometry: the coordinatization theorem [Bae52, MM70, Hol95] represents the atoms of the lattice by the rays of a vector space with an anisotropic hermitian form (see Theorem 3.3), and Solèr’s theorem identifies that space as a Hilbert space precisely when the lattice is complete (see Theorem 3.6).

A structure for this vocabulary is called *standard* when its Hilbert sort and its sequence sort are the closed unit balls of complex Hilbert spaces, endowed with their operations, inner products and norms (see Definition 5.2). Our axioms are closed conditions $\varphi \leq r$ or $\varphi \geq r$ with r rational, each required to hold for all values of the free variables of φ ; in them **1** denotes the sentence whose value is 1 in every structure, and $\varphi \wedge \psi$ the minimum of the values of φ and ψ . With these conventions a single countable theory axiomatizes the lattices of closed subspaces of separable infinite-dimensional complex Hilbert spaces together with σ -additive probability measures on them.

Theorem A (Gleason’s theorem in real-valued $L_{\omega_1, \omega}$). *There is a countable theory T of real-valued $L_{\omega_1, \omega}$ satisfying the following two properties.*

- (1) *In every standard model of T , the Hilbert sort is the closed unit ball of a separable infinite-dimensional complex Hilbert space H , and the map $a \mapsto S_a$, where S_a is the closed subspace of H spanned by the vectors of the Hilbert sort that belong to a , is an isomorphism of ortholattices from the Solèr sort onto the lattice $L(H)$ of closed subspaces of H ; moreover, there is a unique density operator ρ on H such that $\mu(a) = \text{Tr}(\rho P_a)$ for every element a of the Solèr sort, where P_a is the orthogonal projection onto S_a .*
- (2) *Conversely, for every separable infinite-dimensional complex Hilbert space H and every density operator ρ on H , there is a standard model of T whose Hilbert sort is the closed unit ball of H , whose Solèr sort is $L(H)$, and in which $\mu(a) = \text{Tr}(\rho P_a)$ for every $a \in L(H)$, where P_a is the orthogonal projection onto a .*

In infinite dimension a measure given by a density operator takes every value in $[0, 1]$, even when the density operator is a pure state. The hypothesis of Kochen and Specker thereby takes its place within a continuum of hypotheses, each asserting that the probabilities avoid a given Borel set of real numbers that meets $[0, 1]$, and in the standard models every one of them fails.

Theorem B (The range of the measure). *In every standard model of T , the range of μ is the whole interval $[0, 1]$. Consequently, for every Borel set $B \subseteq \mathbb{R}$ that meets $[0, 1]$, the theory T , together with an axiom saying that no value of μ lies in B (see Lemma 5.1), has no standard models.*

Once quantification over propositions is admitted, the formulas of the logic define exactly the σ -projective sets of probabilities uniformly in the standard models; the conclusion of Theorem B accordingly extends to every nonempty set of this kind (Theorem 8.6). The hypothesis of Kochen and Specker, however, admits a stronger refutation, for it fails beyond the standard models as well.

Theorem C (The Bell–Kochen–Specker theorem in real-valued $L_{\omega_1, \omega}$). *The theory T of Theorem A, together with the axiom $\mu(a) \wedge (1 - \mu(a)) \leq 0$, which, since the axioms of T confine μ to $[0, 1]$, says that μ takes only the values 0 and 1, has no models, standard or not.*

The proof, given in Section 9, depends neither on Gleason’s theorem nor on the standardness of the model. It combines σ -additivity along an orthonormal sequence named by constants of the vocabulary with the parity argument attached to the eighteen vectors of $\mathbb{R}^4$ that Cabello, Estebaranz and García-Alcaine [CEGA96] arranged in nine orthogonal bases.

Although the theory of Theorem C has no models, each of its finite subsets has one, so that its inconsistency cannot be witnessed by compactness; it is precisely at this juncture that forcing takes the place of compactness. Model-theoretic forcing [Rob71, Kei73] assembles a model from finite pieces of information, meeting each requirement on the model at some finite stage, and in real-valued $L_{\omega_1, \omega}$ it underlies the model existence theorem [ADI26]. We express that theorem as a two-player topological game of length ω , of the kind described by Väänänen [Vää11], in which one player maintains that the theory has no model while the other undertakes to construct one by committing to the values of sentences. By the theorem of Gale and Stewart [GS53] one of the two players has a winning strategy. Since the first player must challenge every axiom, her bookkeeping may be read as a Baire category argument in a Polish space of plays (Remark 10.4). The refined notion of finite obstruction is extracted from this game.

Theorem D (Finite obstructions). *Let T' be a countable theory of real-valued $L_{\omega_1, \omega}$ that consists of closed conditions and has no models. In the model existence game of Section 10, the player who claims that T' has no models has a winning strategy. For every such strategy, the tree of plays in which that player follows it is well-founded; at each leaf, the commitments made along the play, each of which says that the value of a sentence lies in a closed interval with rational endpoints, form a finite set that is satisfied in no structure. The tree branches infinitely only at the moves that concern countable suprema and infima; once the other player's answers to those moves are fixed, the plays have bounded length. For the theory of Theorem C, every finite set of axioms has a model; consequently, the tree of every winning strategy for that theory branches infinitely.*

The obstructions so obtained carry considerably more information than the classical ones. Whereas a classical obstruction is a set of Boolean constraints, an obstruction in the sense of Theorem D is a finite set of commitments, each confining the value of a sentence to a closed interval with rational endpoints. Since probabilities are among these values, such obstructions reach the hypotheses of Theorem B and Theorem 8.6, which admit no Boolean formulation at all. Because each obstruction involves only finitely many axioms, it identifies the properties of the lattice and of the measure on which the refutation depends. The strategy moreover arranges the obstructions in a well-founded tree that records how every quantifier and every countable supremum and infimum enters the inconsistency; since the tree branches infinitely only at moves concerning countable suprema and infima, it locates precisely where the infinitary axioms are used. Such obstructions exist even for theories all of whose finite subsets have models, among them the theory of Theorem C, for which compactness furnishes no unsatisfiable finite subtheory. Where compactness merely asserts that a classical obstruction exists, Theorem D exhibits each of ours as the terminal position of a play reached from the axioms by a strategy.

Nor is the classical obstruction lost in this refinement; it reappears within the finer one. For the theory of Theorem C we describe in Section 10 a winning strategy with the following property. Whenever the constructor makes the commitments that the parity argument requires, the play ends at a position whose commitments concern the probabilities of the eighteen rays of Cabello, Estebarez and García-Alcaine. These rays form nine orthogonal quadruples in which each ray lies in exactly two, and each quadruple spans the space generated by the same four consecutive vectors of the orthonormal sequence (see Table 1 and Figure 12). The configuration of Kochen–Specker type thus appears as a single stage of a play that the strategy reaches from the axioms.

These results advance the earlier work in three respects. Theorem C establishes, by an argument independent of Gleason's theorem, that the Kochen–Specker hypothesis fails in every model of a countable theory whose standard models are characterized by Theorem A. Theorem B and Theorem 8.6, in turn, refute a continuum of hypotheses of which the Kochen–Specker hypothesis is but a single instance. Finally, Theorem D carries the finite obstructions of J. L. Bell and Pitowsky beyond compact logics and finite dimension, and with them the first-order analysis of Hrushovski and Pitowsky. It applies to every countable theory of real-valued $L_{\omega_1, \omega}$ without models, even when all of its finite fragments are satisfiable.

Organization of the paper. The background occupies Sections 1–4 and is drawn from lattice theory, the theory of hermitian forms and projective geometry. Its principal sources

are MacNeille completions [Mac37], the coordinatization of projective geometries [Bae52, MM70, Hol95] and the theorems of Amemiya and Araki [AA66] and of Solèr [Sol95] on orthomodular spaces. These lead first to the Solèr lattices, which up to isomorphism are the lattices of closed subspaces of infinite-dimensional real, complex and quaternionic Hilbert spaces. They lead also to the pre-Solèr lattices, in which every condition defining a Solèr lattice holds except possibly completeness. In the standard models of T the Solèr sort is a Solèr lattice, whereas in the standard models of the part of T that mentions neither the sequence sort nor the measure it need only be a pre-Solèr lattice. The logic and the theory T are presented in Sections 5 and 6, and Sections 7–9 contain the proofs of Theorems A, B and C together with Theorem 8.6, which identifies the sets of values that formulas define in the standard models. Section 10 is devoted to the model existence game, whose analysis rests on the determinacy theorem of Gale and Stewart [GS53], and to the proof of Theorem D. The optional Section 4 and the remarks of Section 11 may be omitted on a first reading, and in Appendix A we compare the game with the treatment of the Henkin construction in the space of Henkin types [ADI26].

1. LATTICES

Birkhoff and von Neumann [BvN36] observed that the experimentally verifiable propositions about a quantum system correspond to the closed subspaces of a Hilbert space, and that these subspaces form a lattice in which the distributive law fails; the logic of quantum mechanics is therefore not that of a Boolean algebra. In place of the distributive law they proposed the weaker modular law, which holds in finite dimensions but fails for the closed subspaces of an infinite-dimensional Hilbert space; what holds in every dimension is the still weaker orthomodular law. In this section we collect the notions from lattice theory that we need: orthocomplementation and orthomodularity; atoms, together with the covering property and irreducibility; and the MacNeille completion, which supplies the joins and meets that an incomplete lattice lacks.

Definition 1.1. A *partially ordered set* is a set with a reflexive, antisymmetric and transitive relation $\leq$. For a subset S of a partially ordered set, an *upper bound* of S is an element b with $a \leq b$ for all $a \in S$, and the *join* $\bigvee S$ of S is its least upper bound, when it exists; lower bounds and the *meet* $\bigwedge S$ are defined dually. A *lattice* is a partially ordered set in which every pair of elements a, b has a join $a \vee b$ and a meet $a \wedge b$. A lattice is *bounded* when it has a least element 0 and a greatest element 1, and *complete* when every subset has a join.

In a lattice, the order can be recovered from the meet: for all elements a and b , the relation $a \leq b$ holds if and only if $a \wedge b = a$; it is in this form that the axioms of Section 6 express the order. Completeness, likewise, can be stated through meets: a lattice is complete if and only if every subset has a meet, since, when every subset has a join, the meet of a subset is the join of its lower bounds, and dually.

Definition 1.2. An *orthocomplementation* on a bounded lattice Λ is a map $a \mapsto a^\perp$ from Λ to Λ such that, for all $a, b \in \Lambda$, the equations $a^{\perp\perp} = a$, $a \wedge a^\perp = 0$ and $a \vee a^\perp = 1$ hold, and the inequality $a \leq b$ implies $b^\perp \leq a^\perp$. A bounded lattice with an orthocomplementation is an *ortholattice*. Two elements a, b of an ortholattice are *orthogonal*, written $a \perp b$, when $a \leq b^\perp$. An ortholattice is *orthomodular* when, for all elements a and b , the inequality $a \leq b$ implies $b = a \vee (b \wedge a^\perp)$.

In the lattice $L(H)$ of closed subspaces of a Hilbert space H , ordered by inclusion and orthocomplemented by $M \mapsto M^\perp$, orthomodularity says that every closed subspace N containing a closed subspace M is the orthogonal sum of M and $N \cap M^\perp$. Orthomodular or

not, every ortholattice satisfies the De Morgan law $(\bigvee S)^\perp = \bigwedge \{a^\perp : a \in S\}$ for every subset S whose join exists, since $a \mapsto a^\perp$ is an order-reversing bijection that is its own inverse; it is through this law that orthocomplements of joins are computed below, in the MacNeille completion and in the coordinatization theorem of Section 3.

Definition 1.3. An *atom* of a bounded lattice is a nonzero element p such that every element b with $b \leq p$ is equal to 0 or to p . A bounded lattice is *atomistic* when every element a is the join of the atoms below it (this join being required to exist). It has the *covering property* when, for every atom p and every element a with $p \not\leq a$, no element lies strictly between a and $a \vee p$. It is *irreducible* when, for any two distinct atoms p and q , some atom r distinct from p and q lies below $p \vee q$.

For complete orthomodular lattices that are atomistic and have the covering property, irreducibility in this sense is equivalent to the absence of a decomposition as a direct product of two lattices, each with more than one element (see Maeda and Maeda [MM70] and Holland [Hol95, §4]); we adopt the definition above because it is the form in which irreducibility enters the coordinatization theorem (see Theorem 3.3).

Definition 1.4. A *sub-ortholattice* of an ortholattice Λ is a subset that contains 0 and 1 and is closed under the operations $\vee$, $\wedge$ and $^\perp$ of Λ . An *embedding* of ortholattices is an injective map that preserves $\vee$, $\wedge$, $^\perp$, 0 and 1.

The pre-Solèr lattices of Section 3, among them the Solèr sorts of the standard models of the part of T that mentions neither the sequence sort nor the measure, need not be complete (see Example 7.6); in Theorems 4.1 and 7.5 below we show that the lattice of orthogonally closed subspaces of a hermitian space into which such a lattice embeds is its MacNeille completion. This completion, which we construct presently, is characterized by a notion of density that concerns the order alone; no topology is involved. Let Γ be a complete lattice, and let Λ be a subset of Γ . We say that Λ is *join-dense* in Γ when every $g \in \Gamma$ satisfies $g = \bigvee \{a \in \Lambda : a \leq g\}$, the join being computed in Γ , and that Λ is *meet-dense* in Γ when every $g \in \Gamma$ satisfies $g = \bigwedge \{a \in \Lambda : a \geq g\}$. For instance, the set $\mathbb{Q}$ of rational numbers is join-dense and meet-dense in the extended real line $[-\infty, \infty]$.

A join-dense and meet-dense subset determines, up to isomorphism, the complete lattice that contains it (see Proposition 1.5 below), and that lattice can therefore be built from the subset alone. The construction, due to MacNeille [Mac37], is as follows. For a partially ordered set Λ and a subset X of Λ , let X^u and X^l be the sets of upper and lower bounds of X in Λ , and call X a *cut* when $X = (X^u)^l$. The cuts, ordered by inclusion, form a complete lattice $\text{DM}(\Lambda)$, the *MacNeille completion* of Λ , into which the map $a \mapsto \{b \in \Lambda : b \leq a\}$ embeds Λ , preserving all joins and meets that exist in Λ . The letters DM recall its other name, the Dedekind–MacNeille completion.

When Λ is an ortholattice, its orthocomplementation induces one on the cuts. For a cut X , let $X^\perp$ be the set $\{b \in \Lambda : b \leq a^\perp \text{ for every } a \in X\}$ of lower bounds of $\{a^\perp : a \in X\}$; as a set of lower bounds, it is a cut, and the map $X \mapsto X^\perp$ reverses inclusion. An element b lies in $X^\perp$ exactly when $b^\perp$ is an upper bound of X , whence $X^{\perp\perp} = (X^u)^l = X$; moreover, $X \cap X^\perp = \{0\}$, because an element b with $b \leq b^\perp$ satisfies $b = b \wedge b^\perp = 0$. Since $X \mapsto X^\perp$ is an order-reversing bijection that is its own inverse, it carries the meet $X \cap X^\perp$ to the join of $X^\perp$ and X , which is therefore $\{0\}^\perp = \Lambda$. Thus $X \mapsto X^\perp$ is an orthocomplementation of $\text{DM}(\Lambda)$; it extends that of Λ , since it sends the cut $\{b \in \Lambda : b \leq c\}$ to the cut $\{b \in \Lambda : b \leq c^\perp\}$, and with it we regard $\text{DM}(\Lambda)$ as an ortholattice.

Proposition 1.5 (Banaschewski [Ban56], Schmidt [Sch56]). *Let Λ be a partially ordered set, and let Γ be a complete lattice that contains Λ as a join-dense and meet-dense subset,*

the order of Γ inducing on Λ the order of Λ . Then the map that sends each $g \in \Gamma$ to the set $\{a \in \Lambda : a \leq g\}$ is an isomorphism of Γ onto $\text{DM}(\Lambda)$ that restricts on Λ to the canonical embedding $a \mapsto \{b \in \Lambda : b \leq a\}$. If Λ is an ortholattice, then the lattice Γ has at most one orthocomplementation that extends that of Λ ; any such orthocomplementation is given by $g^\perp = \bigwedge \{a^\perp : a \in \Lambda, a \leq g\}$, and the isomorphism carries it to the orthocomplementation $X \mapsto X^\perp$ of $\text{DM}(\Lambda)$.

Proof. For $g \in \Gamma$, write $\Lambda_{\leq g} = \{a \in \Lambda : a \leq g\}$; we show that each $\Lambda_{\leq g}$ is a cut, that the map $g \mapsto \Lambda_{\leq g}$ is an order embedding, and that every cut lies in its image, after which the assertions on orthocomplementations follow from the De Morgan law. By join-density, we have $g = \bigvee \Lambda_{\leq g}$; hence the upper bounds of $\Lambda_{\leq g}$ in Λ are the elements $b \in \Lambda$ with $b \geq g$. By meet-density, the common lower bounds of these elements are the elements $a \in \Lambda$ with $a \leq g$; thus $\Lambda_{\leq g}$ is a cut. For $g, g' \in \Gamma$, join-density also shows that $g \leq g'$ if and only if $\Lambda_{\leq g} \subseteq \Lambda_{\leq g'}$; hence the map is an order embedding, and on Λ it is the canonical embedding, because the order of Γ induces that of Λ . If X is a cut and $g = \bigvee X$ in Γ , then the cut X is contained in $\Lambda_{\leq g}$; conversely, an element $a \in \Lambda$ with $a \leq g$ lies below every $b \in X^u$, since each such b is an upper bound of X in Γ , and therefore $a \in X = (X^u)^l$. Thus $X = \Lambda_{\leq g}$, and the map is onto $\text{DM}(\Lambda)$. The formula for $g^\perp$ follows from $g = \bigvee \Lambda_{\leq g}$ and the De Morgan law. By that formula, an element $b \in \Lambda$ lies below $g^\perp$ exactly when $b \leq a^\perp$ for every $a \in \Lambda_{\leq g}$; hence $\Lambda_{\leq g^\perp} = (\Lambda_{\leq g})^\perp$. 

 **Historical note.** Letters that von Neumann wrote to Birkhoff in 1935, while the two were preparing their paper of 1936, show why they rejected the lattice of closed subspaces of an infinite-dimensional Hilbert space as the logic of quantum mechanics: that lattice is not modular, whereas von Neumann wanted the probabilities of quantum propositions to behave like a dimension function, that is, like a strictly increasing valuation, and only a modular lattice carries such a function [Réd07]. In their paper they postulated instead an abstract projective geometry, whose lattice of subspaces is modular [BvN36]. A year later Husimi observed that the closed subspaces of a Hilbert space satisfy the orthomodular law [Hus37], which was rediscovered several times in the 1950s and 1960s. In his thesis of 1964, Piron took completeness, orthomodularity, atoms and the covering property as axioms for the propositions about a quantum system, and he showed that the irreducible lattices with these properties are represented by hermitian forms over a division ring when their dimension is at least four [Pir64].

2. HERMITIAN SPACES

The coordinatization theorem, which we state in Section 3, turns the atoms of a suitable ortholattice into the one-dimensional subspaces of a vector space over a division ring, and its orthocomplementation into orthogonality with respect to a hermitian form. In this section we collect the facts about such spaces that we need. The most important of them, on which Sections 3 and 4 depend, is that orthomodularity of the lattice of orthogonally closed subspaces is equivalent to a property of the space itself: every orthogonally closed subspace and its orthogonal complement have the whole space as their sum (see Proposition 2.8).

Definition 2.1. A *division ring* is a ring with $1 \neq 0$ in which every nonzero element is invertible; multiplication need not be commutative. An *involution* of a division ring Δ is a map $\alpha \mapsto \alpha^*$ such that $(\alpha + \beta)^* = \alpha^* + \beta^*$, $(\alpha\beta)^* = \beta^*\alpha^*$ and $\alpha^{**} = \alpha$ for all $\alpha, \beta \in \Delta$. Let E be a left vector space over Δ . A *hermitian form* on E is a map $\langle \cdot | \cdot \rangle : E \times E \rightarrow \Delta$ such that $\langle \alpha x + \beta y | z \rangle = \alpha \langle x | z \rangle + \beta \langle y | z \rangle$ and $\langle y | x \rangle = \langle x | y \rangle^*$ for all $x, y, z \in E$ and $\alpha, \beta \in \Delta$;

it is *anisotropic* when $\langle x | x \rangle \neq 0$ for every nonzero vector $x \in E$. A *hermitian space* is a left vector space over a division ring with an involution, together with an anisotropic hermitian form.

It follows from these identities that $\langle x | \alpha y \rangle = \langle x | y \rangle \alpha^*$ for all $x, y \in E$ and $\alpha \in \Delta$. The bracket is linear in its first argument, as is customary in mathematics; in Dirac notation, which physicists use, it is linear in the second. Nothing in the definition forces the scalars to be real or complex: over $\mathbb{Q}$ with the identity involution, the form $\sum_n x_n y_n$, for finitely supported rational sequences $x = (x_n)_{n \in \mathbb{N}}$ and $y = (y_n)_{n \in \mathbb{N}}$, is anisotropic, since a sum of rational squares vanishes only when every square does.

Definition 2.2. In a hermitian space E , two vectors x and y are *orthogonal*, written $x \perp y$, when $\langle x | y \rangle = 0$. For $S \subseteq E$, let $S^\perp$ be the set $\{x \in E : \langle x | y \rangle = 0 \text{ for all } y \in S\}$. A subspace M is *orthogonally closed* when $M^{\perp\perp} = M$.

Orthogonality is symmetric, since $\langle y | x \rangle = \langle x | y \rangle^*$ and the involution fixes 0; in particular, for every subset S of E , we have $S \subseteq S^{\perp\perp}$.

Definition 2.3 (The lattice $\mathcal{C}(E)$). For a hermitian space E , we write $\mathcal{C}(E)$ for the set of orthogonally closed subspaces of E , ordered by inclusion.

Every set of the form $S^\perp$ is a subspace, and it is orthogonally closed: applying $^\perp$, which reverses inclusion, to $S \subseteq S^{\perp\perp}$ gives $S^{\perp\perp\perp} \subseteq S^\perp$, and the reverse inclusion is the same fact for $S^\perp$. If $(M_j)_{j \in J}$ is a family of orthogonally closed subspaces, indexed by an arbitrary set J , then their intersection, which equals $\bigcap_j M_j^{\perp\perp} = (\bigcup_j M_j^\perp)^\perp$, is orthogonally closed. Hence $\mathcal{C}(E)$ is a complete lattice with least element $\{0\}$ and greatest element E ; the meet of the family is $\bigcap_j M_j$, and its join is $(\sum_j M_j)^{\perp\perp}$, which may be strictly larger than $\sum_j M_j$. For $M \in \mathcal{C}(E)$, anisotropy gives $M \cap M^\perp = \{0\}$, whence $M \vee M^\perp = (M^\perp \cap M)^\perp = E$; since the map $M \mapsto M^\perp$ also reverses inclusion and is its own inverse on $\mathcal{C}(E)$, the lattice $\mathcal{C}(E)$, with this map, is a complete ortholattice.

Definition 2.4. A hermitian space E is *orthomodular* when $M + M^\perp = E$ for every $M \in \mathcal{C}(E)$, where $M + M^\perp$ is the set of sums $v + y$ with $v \in M$ and $y \in M^\perp$.

The lattice join $M \vee M^\perp$ is always E ; orthomodularity asks that the algebraic sum be E as well.

Definition 2.5. A *Hilbert space* is a hermitian space H over the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$, with the usual involution (the identity, complex conjugation or quaternionic conjugation), such that $\langle x | x \rangle > 0$ for every nonzero vector x and the space is complete in the norm $\|x\| = \langle x | x \rangle^{1/2}$.

Let H be a Hilbert space. By the projection theorem, every closed subspace M of H satisfies $M + M^\perp = H$, and every subspace M satisfies $M^{\perp\perp} = \overline{M}$. Hence the orthogonally closed subspaces of H are exactly its closed subspaces, which form the lattice $L(H) = \mathcal{C}(H)$; the space H is orthomodular, and so, by Proposition 2.8 below, is the lattice $L(H)$.

Lemma 2.6. *In a hermitian space E , every finite-dimensional subspace F is orthogonally closed and satisfies $F + F^\perp = E$.*

Proof. The idea is to write down the orthogonal projection onto F . Let n be the dimension of F . By anisotropy, the Gram–Schmidt process gives F an orthogonal basis $w_1, \dots, w_n$ with $\langle w_i | w_i \rangle \neq 0$ for $i = 1, \dots, n$. For $z \in E$, the vector $Pz = \sum_i \langle z | w_i \rangle \langle w_i | w_i \rangle^{-1} w_i$ lies in F , and the difference $z - Pz$ lies in $F^\perp$; hence $F + F^\perp = E$. If $y \in F^{\perp\perp}$, the vector $y - Py$ lies in $F^\perp \cap F^{\perp\perp}$ and is therefore orthogonal to itself; by anisotropy it is 0, whence $y \in F$. $\color{violet}{\odot}$

Lemma 2.7. *In a hermitian space E over a division ring Δ , the subspace $M + \Delta x$ belongs to $\mathcal{C}(E)$ for every $M \in \mathcal{C}(E)$ and every $x \in E$.*

Proof. If $x \in M$, there is nothing to prove; we therefore assume that $x \notin M$. Since $M + \Delta x$ is contained in $(M + \Delta x)^{\perp\perp}$, it suffices to show that every vector of $(M + \Delta x)^{\perp\perp}$ differs from a multiple of x by a vector of M . The idea is that such a vector induces on $M^\perp$, through the form, a functional obtained from the one induced by x by multiplication on the right by a scalar. Put $W = M^\perp$, and let ξ_x be the functional $z \mapsto \langle z | x \rangle$ on W ; it is nonzero, since $x \notin W^\perp = M$, and we may choose $z_0 \in W$ with $\xi_x(z_0) = 1$. Let y be an element of $(M + \Delta x)^{\perp\perp}$, and let ξ_y be the functional $z \mapsto \langle z | y \rangle$ on W . Since $W \cap x^\perp = (M + \Delta x)^\perp$, the functional ξ_y vanishes on the kernel of ξ_x ; as $z - \xi_x(z)z_0$ lies in that kernel for every $z \in W$, we obtain $\xi_y(z) = \xi_x(z)\alpha$ with $\alpha = \xi_y(z_0)$. Because $\langle z | \alpha^*x \rangle = \langle z | x \rangle\alpha$, the vector $y - \alpha^*x$ is orthogonal to W and therefore lies in $W^\perp = M$. Thus $y \in M + \Delta x$. $\textcolor{violet}{\smileyemote}$

Proposition 2.8. *For a hermitian space E , the lattice $\mathcal{C}(E)$ is orthomodular if and only if E is an orthomodular space.*

Proof. Suppose first that E is orthomodular, and let $M \subseteq M'$ be elements of $\mathcal{C}(E)$. Every vector $z \in M'$ can be written as $v + y$ with $v \in M$ and $y \in M^\perp$; since $y = z - v$ lies in M' , we have $M' = M + (M' \cap M^\perp) \subseteq M \vee (M' \wedge M^\perp) \subseteq M'$, which is the orthomodular law. Conversely, suppose that $\mathcal{C}(E)$ is orthomodular, let Δ be the division ring of scalars of E , let M be an element of $\mathcal{C}(E)$, and let x be a vector of $E \setminus M$. By Lemma 2.7, the subspace $M' = M + \Delta x$ belongs to $\mathcal{C}(E)$, and orthomodularity gives $M' = M \vee (M' \cap M^\perp)$. Since $M' \neq M$, the subspace $M' \cap M^\perp$ is nonzero; as it meets M only in 0, while M has codimension one in M' , it is one-dimensional, spanned by a vector u . Then $M \subsetneq M + \Delta u \subseteq M'$, and the codimension count gives $M' = M + \Delta u$; hence $x \in M + M^\perp$. $\textcolor{violet}{\smileyemote}$

$\textcolor{violet}{\boxtimes}$ **Historical note.** In 1964 Piron published a theorem asserting that every orthomodular complex inner product space is complete, but his proof contained an error [Pir64]. Amemiya and Araki gave a correct proof in 1966, and their method also covers real and quaternionic scalars [AA66]. It was then speculated that, among infinite-dimensional hermitian spaces over arbitrary division rings, orthomodularity alone characterizes the Hilbert spaces over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Keller ended the speculation in 1980 with a “non-classical Hilbert space”, an infinite-dimensional orthomodular hermitian space over a division ring other than these three [Kel80]. Keller was a student of Herbert Gross, whose papers and book on forms in infinite-dimensional vector spaces shaped the theory; the story is told in Holland’s survey of 1995 [Hol95].

3. PROJECTIVE GEOMETRY, SOLÈR LATTICES AND PRE-SOLÈR LATTICES

In this section we explain how an ortholattice with a few combinatorial properties can be represented by orthogonally closed subspaces of a hermitian space, a vector space over a division ring with an anisotropic hermitian form, and how Solèr’s condition, together with completeness, forces the scalars to be real, complex or quaternionic. The ortholattices that have these properties, Solèr’s condition and completeness are the Solèr lattices; those that have them all except, possibly, completeness are the pre-Solèr lattices, which recur throughout the paper, because no axiom about the lattice alone can enforce completeness (see Section 5). We assume familiarity with Hilbert spaces, but none with projective geometry.

3.1. Points and lines. In a complex Hilbert space H , the atoms of the lattice $L(H)$ of closed subspaces are the *rays*, the one-dimensional subspaces $\mathbb{C}x$ spanned by nonzero vectors $x \in H$; the join of two distinct rays is the two-dimensional subspace, or *plane*, that they span. More generally, we call the one-dimensional subspaces of any vector space its *rays*. Which

rays lie in which planes is information about the lattice alone (see Figure 1); for a vector space of dimension at least four, this information determines the space and its division ring of scalars, in the sense made precise by Theorem 3.2 below.

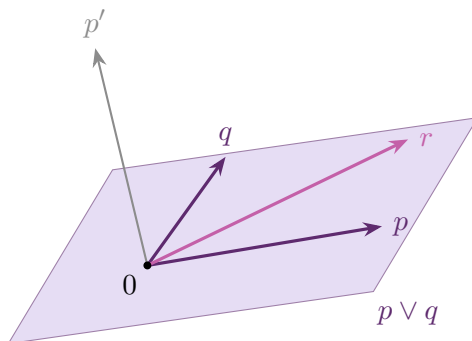


FIGURE 1. In the lattice $L(H)$ of closed subspaces of a complex Hilbert space H , the atoms are the rays. The join $p \vee q$ of two rays p and q is the plane they span; the ray r in that plane is an atom below $p \vee q$, while the ray p' is not.

Definition 3.1. A *projective geometry* is a set of *points* together with a family of subsets, called *lines*, such that:

- (PG1) any two distinct points lie on exactly one line;
- (PG2) every line contains at least three points;
- (PG3) (*the Veblen–Young axiom*) if a, b, a', b' are distinct points and the line through a and b meets the line through a' and b' , then the line through a and a' meets the line through b and b' .

A set of points is a *projective subspace* when it contains the whole line through any two of its points; the projective subspace *generated* by a set of points is the intersection of all projective subspaces containing it, and a *projective plane* is the projective subspace generated by three points not on one line. Axiom (PG3) says that two lines lying in a common projective plane always meet (see Figure 2); a projective geometry has no parallel lines. The word *projective* comes from perspective drawing, in which parallel rails, projected onto the canvas, meet at the horizon.

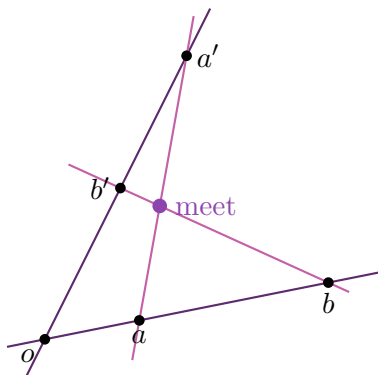


FIGURE 2. The Veblen–Young axiom. The lines ab and $a'b'$ meet at o ; then the lines aa' and bb' meet as well.

The basic example comes from linear algebra. Let V be a left vector space of dimension at least two over a division ring Δ . Take as points the rays of V , and as lines the sets of rays contained in a two-dimensional subspace. Axiom (PG1) holds because two distinct rays span a unique two-dimensional subspace, and axiom (PG2) because the two-dimensional subspace spanned by linearly independent vectors x and y contains the three distinct rays Δx , Δy and $\Delta(x + y)$. For (PG3), suppose that the two-dimensional subspaces spanned by a, b and by a', b' have a ray in common; then the rays a, b, a', b' lie in a subspace of dimension at most three, and, by counting dimensions, any two two-dimensional subspaces of that subspace have a ray in common, in particular those spanned by a, a' and by b, b' .

Points are *independent* when none of them lies in the projective subspace generated by the others; in the basic example, rays are independent exactly when nonzero vectors spanning them, one in each ray, are linearly independent. Two projective geometries are *isomorphic* when there is a bijection between their sets of points under which the lines of the one correspond exactly to the lines of the other. A *semilinear isomorphism* from a vector space E over a division ring Δ onto a vector space E' over a division ring Δ' is an additive bijection $f: E \rightarrow E'$ for which there is an isomorphism $\sigma: \Delta \rightarrow \Delta'$ with $f(\alpha x) = \sigma(\alpha)f(x)$ for all $\alpha \in \Delta$ and $x \in E$. The following theorem says that, as soon as there are four independent points, the basic example is, up to isomorphism, the only one.

Theorem 3.2 (Coordinatization of projective geometries; see Baer [Bae52]). *Every projective geometry that contains four independent points is isomorphic to the geometry of rays of a left vector space E over a division ring Δ . The pair (Δ, E) is unique up to a semilinear isomorphism.*

The division ring Δ is thus recovered from the geometry alone, whatever it may be: every division ring occurs, among them the rational numbers, the non-archimedean ordered fields (fields with infinitesimals, such as fields of formal Laurent series) and the quaternions. The hypothesis of four independent points, on the other hand, cannot be dispensed with: a projective plane, regarded as a projective geometry in its own right, need not come from a division ring. The planes that do are exactly those in which Desargues's theorem holds (if two triangles are in perspective from a point, the three intersection points of corresponding sides lie on one line). Planes in which it fails, the *non-Desarguesian* planes, exist; in a geometry with four independent points, by contrast, Desargues's theorem follows from the axioms.

3.2. From lattices to geometries. To apply Theorem 3.2 to an orthomodular lattice that is atomistic and has the covering property, we take its atoms as the points of a geometry whose lines are the sets $\{r : r \text{ an atom with } r \leq p \vee q\}$, for distinct atoms p and q ; in $L(H)$, these are the rays in the plane spanned by p and q . The covering property gives this geometry (PG1) and, with orthomodularity, (PG3); irreducibility in the sense of Definition 1.3 is exactly (PG2) (see Maeda and Maeda [MM70, §§33–34] and Holland [Hol95, §4]).

Orthocomplementation adds a relation between points: by the definition in Section 1, atoms p and q are orthogonal when $p \leq q^\perp$. In $L(H)$ this is orthogonality of rays, and it determines the inner product up to the ambiguity described at the end of Theorem 3.3 below. The following properties of an ortholattice Λ enter the theorems of this section:

- (C) the lattice is complete;
- (OM) it is orthomodular;
- (A) it is atomistic;
- (CP) it has the covering property;
- (I) it is irreducible;
- (D) it contains four mutually orthogonal atoms.

Condition (D) supplies the four independent points of Theorem 3.2, since mutually orthogonal atoms are independent. The coordinatization theorem below, which goes back to Birkhoff and von Neumann [BvN36] in finite dimensions, shows that every ortholattice with (OM), (A), (CP), (I) and (D) is represented in the manner of $L(H)$: its atoms correspond to the rays of a vector space, and orthogonality of atoms to orthogonality of rays with respect to an anisotropic hermitian form.

Theorem 3.3 (Coordinatization; Baer [Bae52], Maeda and Maeda [MM70, §34], Holland [Hol95, §3]). *Let Λ be an ortholattice that satisfies (OM), (A), (CP), (I) and (D), but not necessarily (C). Then there are a division ring Δ with an involution, a hermitian space E over Δ , and a bijection $p \mapsto R_p$ from the atoms of Λ onto the rays of E , such that two atoms p and q are orthogonal exactly when $R_p \perp R_q$. The map*

$$\gamma: \Lambda \rightarrow \mathcal{C}(E), \quad \gamma(a) = \{0\} \cup \bigcup \{R_p : p \text{ an atom with } p \leq a\},$$

where $\mathcal{C}(E)$ is the lattice of orthogonally closed subspaces of E , is an embedding of ortholattices whose image contains every finite-dimensional subspace of E and the orthogonal complement of each. The triple formed by Δ , E and the form is unique up to a semilinear isomorphism followed by a rescaling $\langle x \mid y \rangle' = \langle x \mid y \rangle \zeta$, where $\zeta^* = \zeta \neq 0$, with the new involution $\alpha \mapsto \zeta^{-1} \alpha^* \zeta$.

Proof of the statements about γ . The orthocomplementation of Λ induces a *polarity* on the geometry of its atoms, in the sense of Holland [Hol95, Definition 3.2]: the map that sends each projective subspace to the set of points orthogonal to all its points. This map reverses inclusion, sends the set of all points to the empty set, and sends each point p to the set of atoms below $p^\perp$, a hyperplane (that is, a maximal proper projective subspace) whose image is again $\{p\}$ (see Holland [Hol95, §4]). By Theorem 3.2, the atoms become the rays of a vector space E , and by Baer's theorem [Bae52] (see Holland [Hol95, Theorem 3.6]), every polarity of this kind under which no point is orthogonal to itself is orthogonality for an anisotropic hermitian form on E . If two atoms lie below a , every atom on the line they determine lies below a ; hence $\gamma(a)$ is a subspace. For an atom r , the following are equivalent: $r \leq a^\perp$; $a \leq r^\perp$; $p \leq r^\perp$ for every atom $p \leq a$ (because a is the join of these atoms, by (A)); $R_r \perp R_p$ for every atom $p \leq a$. Thus $\gamma(a^\perp) = \gamma(a)^\perp$, and applying this twice gives $\gamma(a) = \gamma(a)^{\perp\perp}$. By (A), the relation $a \leq b$ holds if and only if $\gamma(a) \subseteq \gamma(b)$; hence γ is injective. Meets are preserved because an atom lies below $a \wedge b$ exactly when it lies below a and b , and joins are preserved by the De Morgan law. Finally, let F be a finite-dimensional subspace of E with basis $x_1, \dots, x_k$, and let p_i be the atom with $R_{p_i} = \Delta x_i$, for $i = 1, \dots, k$. The join $p_1 \vee \dots \vee p_k$ exists, as every finite join does in a lattice, and, because γ preserves joins, its image is $(\Delta x_1 + \dots + \Delta x_k)^{\perp\perp} = F^{\perp\perp}$, which is F by Lemma 2.6. The orthogonal complement $F^\perp$ is therefore $\gamma((p_1 \vee \dots \vee p_k)^\perp)$. 

Piron [Pir64] proved this representation for complete lattices; completeness is not needed for the embedding, and in Theorem 3.6 below it serves only to make γ surjective.

3.3. Harmonic conjugates and Solèr's condition. The hypotheses of Theorem 3.3 do not force the division ring Δ to be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$: the lattice of all subspaces of $\mathbb{Q}^4$, orthocomplemented by orthogonality for the form $\sum_{i < 4} x_i y_i$, satisfies them, with $\Delta = \mathbb{Q}$. Solèr's condition, together with completeness, singles out the real numbers, the complex numbers and the quaternions; it is phrased through the harmonic conjugate, a classical construction of projective geometry that uses joins and meets alone. In the hermitian space of Theorem 3.3, the harmonic conjugate of a ray of a plane spanned by two orthogonal rays, with respect

to these two, coincides with the orthocomplement of that ray in the plane exactly when the ray is spanned by the sum of two vectors of equal “length”, one from each of the orthogonal rays, as we show below (see Figure 4); agreement along an infinite orthogonal sequence of atoms therefore yields, after a rescaling of the form, an infinite orthonormal sequence, which, together with orthomodularity, is what Solèr’s theorem requires (see Lemma 3.4 and Theorem 3.6).

Let p and q be distinct points of a projective geometry, write $p + q$ for the line through them (in the basic example, the rays of the plane spanned by p and q), and let c be a point of $p + q$ other than p and q . Two distinct lines in a common projective plane meet in exactly one point, by (PG1) and (PG3); by a slight abuse of notation, we denote that point by the intersection of the two lines. Choose a point r not on $p + q$ and a point r' on $r + p$ other than r and p ; put $c' = (c + r') \cap (q + r)$ and $c'' = (p + c') \cap (q + r')$. The point $h = (r + c'') \cap (p + q)$ is the *harmonic conjugate* of c with respect to p and q (see Figure 3). In the basic example it does not depend on the choices of r and r' , as the computation below shows; by Theorem 3.2, the same holds in every projective geometry with four independent points. In an ortholattice that satisfies (OM), (A), (CP) and (I), whose atoms form a projective geometry as in the preceding subsection, the same construction applies to atoms, with $\vee$ in place of $+$ and $\wedge$ in place of $\cap$; the relation “ h is the harmonic conjugate of c with respect to p and q ” is then expressed by a first-order formula in the language of lattices.

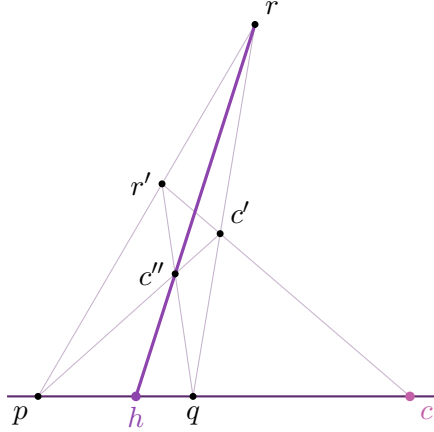


FIGURE 3. The construction of the harmonic conjugate h of a point c of the line through p and q , with respect to p and q , by way of the auxiliary points r , r' , c' and c'' .

In the basic example, the construction is simple in coordinates. Let u and v be vectors spanning p and q , respectively, scaled so that $c = \Delta(u + v)$; this is possible because c lies on $p + q$ and differs from p and q . Likewise, let w be a vector spanning r , scaled so that $r' = \Delta(u + w)$, which is possible because r' lies on $r + p$ and differs from r and p . A direct computation gives $c' = \Delta(v - w)$, $c'' = \Delta(u - v + w)$ and $h = \Delta(u - v)$. All coefficients are ± 1 , so the computation is valid over every division ring:

$$\text{the harmonic conjugate of } \Delta(u + v) \text{ with respect to } \Delta u \text{ and } \Delta v \text{ is } \Delta(u - v). \quad (1)$$

Suppose now that the vector space carries an anisotropic hermitian form, as the space E of Theorem 3.3 does, and that $p \perp q$; then $u \perp v$. The orthocomplement of c within the plane $p \vee q$, the ray $c^\perp \wedge (p \vee q)$, consists of the vectors $\alpha u + \beta v$ with $\langle \alpha u + \beta v \mid u + v \rangle = \alpha \langle u \mid u \rangle + \beta \langle v \mid$

$v\rangle = 0$. It coincides with the harmonic conjugate $\Delta(u - v)$ exactly when $\langle u | u \rangle = \langle v | v \rangle$, that is, when u and v have the same “length” (see Figure 4). In characteristic 2, that is, when $1 + 1 = 0$ in Δ , the vectors $u + v$ and $u - v$ coincide, and the equality $\langle u | u \rangle = \langle v | v \rangle$ would make $u + v$ isotropic, since then $\langle u + v | u + v \rangle = (1 + 1)\langle u | u \rangle = 0$. Hence, when the harmonic conjugate of c coincides with its orthocomplement in the plane $p \vee q$, the characteristic of Δ is not 2.

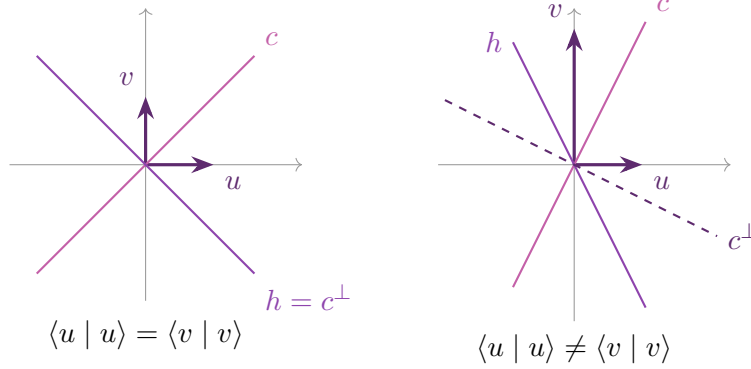


FIGURE 4. The ray $c = \Delta(u + v)$ spanned by the sum of orthogonal vectors u and v , its harmonic conjugate $h = \Delta(u - v)$ with respect to Δu and Δv , and its orthocomplement $c^\perp$ in the plane of u and v . The rays h and $c^\perp$ coincide when u and v have equal length (left), and differ otherwise (right).

Solèr's condition, in the form we use, asks for an infinite orthogonal sequence of atoms along which the orthocomplement and the harmonic conjugate agree at every step:

- (S) There are mutually orthogonal atoms p_n , for $n \in \mathbb{N}$, and atoms $c_n \leq p_n \vee p_{n+1}$ with $c_n \neq p_n, p_{n+1}$, such that, for every $n \in \mathbb{N}$, the harmonic conjugate of c_n with respect to p_n and p_{n+1} is $c_n^\perp \wedge (p_n \vee p_{n+1})$.

This formulation of Solèr's condition through harmonic conjugates is due to Holland [Hol95, Theorems 3.7 and 4.1]; Morash's angle-bisecting axiom [Mor73], which asks for bisectors as in the left half of Figure 4, may be used in its place (see Holland [Hol95, §4]).

Lemma 3.4. *Let Λ be an ortholattice that satisfies (OM), (A), (CP), (I), (D) and (S), and let E be a hermitian space over a division ring Δ and $p \mapsto R_p$ a bijection from the atoms of Λ onto the rays of E that Theorem 3.3 provides for Λ . After a rescaling of the form, the space E contains an infinite orthonormal sequence.*

Proof. Let p_n and c_n , for $n \in \mathbb{N}$, be atoms as in (S). Choose e_0 with $R_{p_0} = \Delta e_0$, and inductively choose e_{n+1} with $R_{p_{n+1}} = \Delta e_{n+1}$ and $R_{c_n} = \Delta(e_n + e_{n+1})$; this is possible because R_{c_n} is a ray of the plane spanned by e_n and e_{n+1} , distinct from Δe_n and Δe_{n+1} . By (1) and the computation that follows it, condition (S) gives $\langle e_n | e_n \rangle = \langle e_{n+1} | e_{n+1} \rangle$ for every $n \in \mathbb{N}$; call this common value ζ , which satisfies $\zeta^* = \zeta \neq 0$. The form $\langle x | y \rangle' = \langle x | y \rangle \zeta^{-1}$ is hermitian for the involution $\alpha \mapsto \zeta \alpha^* \zeta^{-1}$, has the same orthogonality relation, and satisfies $\langle e_n | e_n \rangle' = 1$ for every n . ☺

Definition 3.5. A *Solèr lattice* is an ortholattice that satisfies (C), (OM), (A), (CP), (I), (D) and (S). A *pre-Solèr lattice* is an ortholattice that satisfies (OM), (A), (CP), (I), (D) and (S), but not necessarily (C).

Theorem 3.6 (Piron, Amemiya and Araki, Solèr; in this form Holland [Hol95, Theorem 4.1]). *An ortholattice is a Solèr lattice if and only if it is isomorphic to $L(H)$ for some infinite-dimensional Hilbert space H over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Sketch. Let Λ be a Solèr lattice, with γ and E as in Theorem 3.3. Given $M \in \mathcal{C}(E)$, completeness provides the join a of the atoms p with $R_p \subseteq M$, and $\gamma(a)^\perp = \gamma(a^\perp) = M^\perp$, whence $\gamma(a) = M$; thus the embedding γ is an isomorphism onto $\mathcal{C}(E)$, which is therefore orthomodular, as is, by Proposition 2.8, the space E . By Lemma 3.4, the space E contains, after a rescaling of the form that changes neither $\mathcal{C}(E)$ nor its orthocomplementation, an infinite orthonormal sequence. Solèr’s theorem [Sol95] (see also Prestel [Pre95]) asserts that an orthomodular space containing an infinite orthonormal sequence is a Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$; the space E , with its form so rescaled, is therefore such a Hilbert space. Conversely, the lattice $L(H)$ is complete and satisfies the other conditions; for (S), take an orthonormal sequence $(e_n)_{n \in \mathbb{N}}$ in H , and let p_n be the ray of e_n and c_n the ray of $e_n + e_{n+1}$. ☺

Condition (S) cannot be omitted from Theorem 3.6. Keller [Kel80] constructed infinite-dimensional orthomodular hermitian spaces over division rings other than $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$; for such a space E , the lattice $\mathcal{C}(E)$ satisfies (C), (OM), (A), (CP), (I) and (D) (see Proposition 2.8 and Lemmas 2.7 and 2.6), yet, by the uniqueness clause of Theorem 3.3, it is isomorphic to no lattice $L(H)$ of a Hilbert space H over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

✂ **Historical note.** Von Staudt’s *Geometrie der Lage* of 1847 was the first treatment of projective geometry free of any metric, and his later calculus of “throws” outlined the algebraization of an axiomatic geometry. Hilbert showed in his *Grundlagen der Geometrie* of 1899 that Desargues’s theorem follows from the axioms of space but cannot be proved in the plane without the axioms of congruence, and he founded on it an algebra of segments that does not use those axioms [Hil99]. Veblen and Young gave axioms for projective geometry in 1908 [VY08]; in their book of 1910 they introduced coordinates through an algebra of points that descends from von Staudt’s algebra of throws [VY10]. Solèr, a student of Herbert Gross, had just begun work on Gross’s orthomodular problem when Gross died suddenly in 1989; she completed it under the informal guidance of Alexander Prestel, a friend of Gross’s at Konstanz, and her theorem appeared in 1995 [Sol95]. In a survey of the same year, Holland restated her condition through harmonic conjugates, constructed as in the book of Veblen and Young, in the form used in this section [Hol95].

4. WHAT A PRE-SOLÈR LATTICE IS MISSING (OPTIONAL)

A pre-Solèr lattice satisfies every condition in the definition of a Solèr lattice except, possibly, completeness. We show that the MacNeille completion of such a lattice supplies the joins and meets that the lattice may lack, and that the completion is a Solèr lattice exactly when it is orthomodular, which need not be the case. Neither completeness nor the orthomodularity of the completion can be expressed, among pre-Solèr lattices, by a sentence of two-valued $L_{\omega_1, \omega}$ about the lattice alone (see Proposition 4.5 below); this is why the vocabulary of Section 5 has, besides the Solèr sort, sorts for vectors and for sequences of vectors.

Throughout this section, we fix a pre-Solèr lattice Λ and a hermitian space E that Theorem 3.3 provides for Λ , and we identify Λ with its image $\gamma(\Lambda) \subseteq \mathcal{C}(E)$ under the embedding γ of that theorem.

Theorem 4.1. *Let Λ be a pre-Solèr lattice, let E be a hermitian space that Theorem 3.3 provides for Λ , and identify Λ with its image $\gamma(\Lambda)$ in $\mathcal{C}(E)$ under the embedding γ of that*

theorem. Then Λ is join-dense and meet-dense in $\mathcal{C}(E)$; consequently $\mathcal{C}(E)$, with $M \mapsto M^\perp$, is isomorphic as an ortholattice to the MacNeille completion of Λ .

Proof. Every $M \in \mathcal{C}(E)$ is the join in $\mathcal{C}(E)$ of the rays it contains, which belong to Λ by Theorem 3.3. If $M^\perp = \{0\}$, then M is E , which belongs to Λ ; otherwise M equals $M^{\perp\perp} = \bigcap \{y^\perp : 0 \neq y \in M^\perp\}$, a meet of hyperplanes, each of which belongs to Λ . Both assertions now follow from Proposition 1.5. 

The elements of the MacNeille completion that do not come from Λ are therefore orthogonally closed subspaces M such that M and $M^\perp$ are both infinite-dimensional, since Λ contains every finite-dimensional subspace and the orthogonal complement of each (see Theorem 3.3). Adjoining them need not produce a lattice isomorphic to some $L(H)$, because the completion may fail to be orthomodular (see Example 4.3); the next theorem shows that orthomodularity of the completion is exactly what is needed.

Theorem 4.2. *Let Λ be a pre-Solèr lattice, and let E be a hermitian space that Theorem 3.3 provides for Λ . Then the following are equivalent:*

- (1) *the MacNeille completion of Λ is orthomodular;*
- (2) *the space E is orthomodular;*
- (3) *the lattice Λ is isomorphic to a join-dense and meet-dense sub-ortholattice of $L(H)$ for some infinite-dimensional Hilbert space H over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

When these conditions hold, the space E , with its form suitably rescaled, is a Hilbert space as in (3). Moreover, the lattice Λ is isomorphic to $L(H)$ for such an H if and only if Λ is complete.

Proof. Conditions (1) and (2) are equivalent by Theorem 4.1 and Proposition 2.8. If (2) holds, then, by Lemma 3.4 and Solèr’s theorem, the space E , with its form rescaled, is a Hilbert space; rescaling changes neither $\mathcal{C}(E)$ nor its orthocomplementation, and (3) follows by Theorem 4.1. If (3) holds, then, by Proposition 1.5, the lattice $L(H)$, which is orthomodular, is isomorphic, together with its orthocomplementation, to the MacNeille completion of Λ ; this is (1). For the last assertion, a complete pre-Solèr lattice is a Solèr lattice, and therefore, by Theorem 3.6, isomorphic to $L(H)$ for an infinite-dimensional Hilbert space H over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$; conversely, every lattice $L(H)$ is complete. 

Example 4.3. Let E_0 be the space of finitely supported complex sequences $x = (x_n)_{n \in \mathbb{N}}$, with the inner product $\langle x | y \rangle = \sum_n x_n \overline{y_n}$ of the space ℓ^2 of square-summable sequences, and let Λ_0 be the set of finite-dimensional subspaces of E_0 and their orthogonal complements in E_0 . Then Λ_0 is a sub-ortholattice of $\mathcal{C}(E_0)$: its elements are orthogonally closed (the finite-dimensional ones by Lemma 2.6); for finite-dimensional F and G , the joins $F \vee G = F + G$, $F^\perp \vee G^\perp = (F \cap G)^\perp$ and $F \vee G^\perp = (F^\perp \cap G)^\perp$ belong to Λ_0 , whence, by the De Morgan law, so do the meets. The lattice Λ_0 is orthomodular, because every $M \in \Lambda_0$ satisfies $M + M^\perp = E_0$ by Lemma 2.6, and the argument of Proposition 2.8 applies. Since it is a sub-ortholattice of $\mathcal{C}(E_0)$ that contains every ray of E_0 , it inherits (A), (CP) and (I) from $\mathcal{C}(E_0)$, in which (CP) holds by Lemma 2.7; conditions (D) and (S) hold with $p_n = \mathbb{C}e_n$ and $c_n = \mathbb{C}(e_n + e_{n+1})$, where $(e_n)_{n \in \mathbb{N}}$ is the sequence of standard unit vectors. Consequently Λ_0 is a pre-Solèr lattice, and the space E_0 , with $R_p = p$ for every atom p of Λ_0 , is a hermitian space that Theorem 3.3 provides for it; the map γ is then the inclusion of Λ_0 in $\mathcal{C}(E_0)$. The MacNeille completion of Λ_0 , which is $\mathcal{C}(E_0)$ by Theorem 4.1, is not orthomodular: by the Amemiya–Araki theorem [AA66], a complex inner product space whose lattice of orthogonally closed

subspaces is orthomodular is complete, which E_0 is not. Hence Λ_0 is an orthomodular lattice whose MacNeille completion is not orthomodular; Harding [Har91] studies such lattices.

Orthomodularity of the completion can be stated through the atoms of Λ alone, without reference to the completion or to the space E ; it is in this form that Proposition 4.5 below shows that it cannot be expressed by a sentence of two-valued $L_{\omega_1, \omega}$ about the lattice. For a set $\mathcal{A}$ of atoms, let $\mathcal{A}^\perp$ be the set of atoms orthogonal to every element of $\mathcal{A}$, and call $\mathcal{A}$ *orthoclosed* when $\mathcal{A} = \mathcal{A}^{\perp\perp}$. Consider the following condition, named (MOM) for MacNeille orthomodularity:

(MOM) for every orthoclosed set $\mathcal{A}$ of atoms and every atom $p \notin \mathcal{A}$, some atom $q \in \mathcal{A}^\perp$ lies in $(\mathcal{A} \cup \{p\})^{\perp\perp}$.

Theorem 4.4. *The MacNeille completion of a pre-Solèr lattice is orthomodular if and only if the lattice satisfies (MOM).*

Proof. The map $\mathcal{A} \mapsto \{0\} \cup \bigcup \{R_p : p \in \mathcal{A}\}$, where R_p is the ray of E that corresponds to the atom p under Theorem 3.3, is a bijection from the orthoclosed sets of atoms onto $\mathcal{C}(E)$, carrying $\mathcal{A}^\perp$ to the orthogonal complement of the image of $\mathcal{A}$. If M is the image of $\mathcal{A}$ and $R_p = \Delta x$, where Δ is the division ring of scalars of E , then by Lemma 2.7 the set $(\mathcal{A} \cup \{p\})^{\perp\perp}$ corresponds to $M + \Delta x$. Condition (MOM) therefore says that $(M + \Delta x) \cap M^\perp \neq \{0\}$ whenever $M \in \mathcal{C}(E)$ and $x \notin M$. If $y = v + \alpha x$ is nonzero, with $v \in M$ and $y \in M^\perp$, then the scalar α is nonzero, since $y \notin M$, whence the vector $x = \alpha^{-1}(y - v)$ lies in $M + M^\perp$; conversely, if $x = v + y$ with $v \in M$ and $y \in M^\perp$, then $y = x - v$ is a nonzero element of $(M + \Delta x) \cap M^\perp$. Thus (MOM) is equivalent to the orthomodularity of E , and the theorem follows from Theorem 4.2. 

When $\mathcal{A}$ or $\mathcal{A}^\perp$ corresponds to a finite-dimensional subspace, the requirement in (MOM) holds automatically, by Lemma 2.6; the content of (MOM) lies in the orthoclosed sets $\mathcal{A}$ for which both $\mathcal{A}$ and $\mathcal{A}^\perp$ correspond to infinite-dimensional subspaces.

Proposition 4.5. *Neither (C) nor (MOM) is equivalent, among pre-Solèr lattices, to a sentence of two-valued $L_{\omega_1, \omega}$ in the vocabulary of ortholattices expanded by constants for atoms p_n and c_n , for $n \in \mathbb{N}$, that witness (S).*

Proof. Suppose that a sentence φ of $L_{\omega_1, \omega}$ were equivalent to (MOM) among pre-Solèr lattices. The axioms of ortholattices and the conditions (OM), (A), (CP), (I), (D) and (S) are expressed by countably many sentences of $L_{\omega_1, \omega}$ in this vocabulary; together with φ , they hold in $L(\ell^2)$. By the downward Löwenheim–Skolem theorem for a countable fragment, they hold in a countable structure Λ' , which is a pre-Solèr lattice satisfying (MOM). By Theorems 4.4 and 4.2, the lattice Λ' contains every ray of an infinite-dimensional Hilbert space, and is therefore uncountable, a contradiction. The same argument applies to (C), using Theorem 3.6. 

 **Historical note.** MacNeille introduced the completion by cuts in 1937 [Mac37], generalizing the cuts by which Dedekind had constructed the real numbers from the rationals in 1872 [Ded72]. Banaschewski and Schmidt characterized it in 1956 by join-density and meet-density [Ban56, Sch56]. In 1969 Adams solved negatively Problem 36 of Birkhoff’s *Lattice Theory*, in its third edition of 1967, by giving an orthocomplemented modular lattice whose completion by cuts is not orthomodular [Ada69]. His lattice consists of the finite-dimensional subspaces of an incomplete complex inner product space and their orthogonal complements, as in Example 4.3, and his argument also rests on the Amemiya–Araki theorem. Harding

gave in 1991 a construction of orthomodular lattices whose completions by cuts are not orthomodular that avoids that theorem: following a construction that Kalmbach introduced in 1977, it glues together the Boolean algebras generated by the chains of a lattice [Har91]. Harding noted that whether every orthomodular lattice can be embedded in a complete one was still open.

5. THE LOGIC AND THE VOCABULARY

Completeness, the one condition in the definition of a Solèr lattice that a pre-Solèr lattice need not satisfy, cannot be expressed by sentences of $L_{\omega_1, \omega}$, two-valued or real-valued, that speak of the lattice alone; the obstacle is the downward Löwenheim–Skolem theorem. For every countable fragment that contains sentences expressing the axioms of ortholattices and the other conditions in the definition of a Solèr lattice, a Solèr lattice, expanded by constants for atoms that witness Solèr’s condition (S), has a countable elementary substructure, in which every sentence of the fragment takes the value that it takes in the Solèr lattice. This substructure is a pre-Solèr lattice, yet it is not complete, since a complete one would be a Solèr lattice, which is isomorphic, by Theorem 3.6, to the lattice of closed subspaces of an infinite-dimensional Hilbert space and is therefore uncountable (for the details in the two-valued case, see the optional Proposition 4.5). Axioms about the lattice alone cannot, then, demand the joins that a pre-Solèr lattice may lack, but a Hilbert space placed beside the lattice can supply them. In this section we recall the features of the logic that we use and present a vocabulary in which the lattice stands beside a Hilbert space and a space of sequences of vectors, the three being linked by function symbols for orthogonal projection, for the passage between vectors and atoms, for the passage between vectors and sequences, and for the closed span of a sequence.

5.1. Real-valued $L_{\omega_1, \omega}$. We use the real-valued infinitary logic of Aubone, Dueñez and Iovino [ADI26], which extends both Eagle’s infinitary $[0, 1]$ -valued logic [Eag14] and the logic of Keisler’s model theory for real-valued structures [Kei23]. Of this logic we use only the following features:

- ♠ The vocabulary is countable and many-sorted. No metric is given a priori, and function symbols need not be continuous.
- ♠ Each predicate symbol P comes with a nonnegative real number r_P , its *a priori bound*, and takes its values in $[-r_P, r_P]$. In this paper the a priori bound of every predicate symbol is a positive integer. The vocabulary contains a nullary predicate symbol **1**, the *logical unity constant*, with a priori bound 1 and value 1 in every structure.
- ♠ Formulas are built from the atomic formulas by the connectives $s \cdot \varphi$, for $s \in \mathbb{Q}$, and $\varphi \vee \psi$, interpreted as multiplication by s and as the maximum, and by the connectives $\sup_x$, one for each variable x , interpreted as the supremum over the sort of x . They are, moreover, closed under countable joins $\bigvee_n \varphi_n$, or countable suprema, of families $(\varphi_n)_{n \in \mathbb{N}}$ of formulas whose free variables lie in one finite set and whose values lie, in every structure, in a common interval $[-r, r]$, for a positive rational number r . The number r is part of the syntax of the join and serves as its a priori bound; the join is interpreted as the supremum of the values of its members, which may themselves contain joins. In the logic of Aubone, Dueñez and Iovino, a join is formed only when the a priori bounds r_{φ_n} of its members have a finite supremum, which is then its a priori bound; the explicit number r admits joins, such as $\lceil \varphi \rceil$ in (2) below, whose members take their values in $[0, 1]$ although their a priori bounds grow without bound. In this paper we also allow continuous connectives $f(\varphi_1, \dots, \varphi_\ell)$,

for continuous functions $f: \mathbb{R}^\ell \rightarrow \mathbb{R}$, which we need for sums and products. The continuous connectives are taken from a fixed countable set, which contains those occurring in the theories we consider; with all continuous functions as connectives, the first-order formulas over a countable vocabulary would already be uncountable, and no fragment would be countable. Every formula φ receives an a priori bound r_φ , by an induction given in Section 10, and takes its values in the closed interval $I_\varphi = [-r_\varphi, r_\varphi]$ in every structure. Following Aubone, Dueñez and Iovino [ADI26], we call a *fragment* a set of formulas that contains the first-order formulas and is closed under subformulas, under the connectives $s \cdot \varphi$, $\varphi \vee \psi$ and $\sup_x$, and under substitution of terms. For a countable fragment $\mathcal{F}$, let C be a set of new constants, countably many of each sort, let $\mathcal{F}_C$ be the fragment that $\mathcal{F}$ generates in the vocabulary enlarged by C , and let $S_C^\mathcal{F}$ be the set of sentences of $\mathcal{F}_C$; an $S_C^\mathcal{F}$ -Henkin type is an element of the product of the intervals I_φ , over the sentences φ of $S_C^\mathcal{F}$, that satisfies the *Henkin conditions*: the constant $\mathbf{1}$ receives the value 1, the value of $s \cdot \varphi$ is s times that of φ , the value of $\varphi \vee \psi$ is the maximum of the values of φ and ψ , the value of a countable join is the supremum of the values of its members, and the value of $\sup_x \varphi(x)$ is the supremum of the values of the sentences $\varphi(t)$, for closed terms t of the sort of x . For formulas built with continuous connectives, we add the condition that the value of $f(\varphi_1, \dots, \varphi_\ell)$ is the value of f at the values of $\varphi_1, \dots, \varphi_\ell$. With the topology induced by the product topology, the $S_C^\mathcal{F}$ -Henkin types form the *space of Henkin types* $\mathcal{H}_C^\mathcal{F}$, a G_δ subset of the product; we return to it in Appendix A.

- ♠ We write 0 for $0 \cdot \mathbf{1}$, $-\varphi$ for $(-1) \cdot \varphi$, $\varphi \wedge \psi$ for $-(\neg\varphi \vee \neg\psi)$, $\inf_x \varphi$ for $-\sup_x -\varphi$, and $\bigwedge_n \varphi_n$ for $-\bigvee_n -\varphi_n$, a countable infimum. These are interpreted as the constant value 0, the negative, the minimum, the infimum over the sort of x , and the infimum of the values.
- ♠ There is no equality symbol. As in Keisler's model theory for real-valued structures [Kei23], a structure is *reduced* when any two elements that agree in every atomic formula with parameters are equal.

The statements of the logic are not formulas but conditions, which compare the value of a sentence with a rational number. For a sentence φ and a rational number r , a *condition* is an expression $\varphi > r$ or $\varphi < r$; a structure $\mathfrak{M}$ satisfies $\varphi > r$ when the value $\varphi^\mathfrak{M}$ of φ in $\mathfrak{M}$ is greater than r , and $\varphi < r$ when it is less than r . A *theory* is a set of conditions; a *model* of a theory is a reduced structure that satisfies every condition in it. Aubone, Dueñez and Iovino do not require models to be reduced; the difference is immaterial, since identifying the elements of a structure that agree in every atomic formula with parameters yields a reduced structure in which every sentence keeps its value (see the end of the proof of Theorem 10.3). Non-strict inequalities are recovered as countable theories: the *closed condition* $\varphi \geq r$ denotes the theory $\{\varphi > r - \frac{1}{n} : n \geq 1\}$, the closed condition $\varphi \leq r$ denotes $\{\varphi < r + \frac{1}{n} : n \geq 1\}$, and the closed condition $\varphi = r$ denotes the union of the two. An *axiom* is a closed condition on a formula $\varphi(\bar{x})$, possibly without free variables, required of all values of $\bar{x}$: the axiom $\varphi(\bar{x}) \leq r$ is the closed condition $\sup_{\bar{x}} \varphi(\bar{x}) \leq r$, the axiom $\varphi(\bar{x}) \geq r$ is $\inf_{\bar{x}} \varphi(\bar{x}) \geq r$, and the axiom $\varphi(\bar{x}) = r$ is the union of the two. For formulas φ and ψ , we write $\varphi \leq \psi$ and $\varphi = \psi$ for the axioms $\varphi - \psi \leq 0$ and $\varphi - \psi = 0$, which are closed conditions when φ and ψ are sentences.

As in the work of Aubone, Dueñez and Iovino [ADI26], we think of 1 as true and of -1 as false. A formula is *two-valued* in a structure if it takes only the values 1 and -1 there. For two-valued formulas, negation, disjunction and conjunction are expressed by $-\varphi$, $\varphi \vee \psi$ and $\varphi \wedge \psi$, the existential and the universal quantifier by $\sup_x$ and $\inf_x$, and countable disjunction

and conjunction by $\bigvee_n$ and $\bigwedge_n$. A two-valued formula θ is asserted by the axiom $\theta \geq 1$; the implication from θ to θ' is the two-valued formula $-\theta \vee \theta'$. Real quantities, such as inner products, norms and the measure, keep their numerical values, every statement about them being a closed condition or an axiom with a rational threshold; a measure or an assignment that takes only the values 0 and 1 is always described as such, never as two-valued. For the measure μ , a predicate on the Solèr sort that we introduce in Section 5.2, the axioms are written with μ itself, which they confine to $[0, 1]$, whereas the truth value of a proposition a , an element of the Solèr sort, is $2\mu(a) - 1$.

An implication whose hypothesis is a closed condition on a quantity requires a formula that records whether the quantity is positive. For a formula φ , let

$$\lceil \varphi \rceil = \bigvee_{n \geq 1} (((n \cdot \varphi) \wedge \mathbf{1}) \vee 0), \quad (2)$$

a countable join with a priori bound 1, whose members take their values in $[0, 1]$ in every structure, as the definition of a join requires. The formula $\lceil \varphi \rceil$ takes the value 0 where $\varphi \leq 0$ and the value 1 where $\varphi > 0$; its values are thus 0 and 1, not -1 and 1, a fact on which form (iii) below depends. Let θ be a formula that is two-valued in the structures under consideration, and let ψ be a formula with $|\psi| \leq c$ in each of them, for a positive rational number c . Three forms of implication then occur among the axioms:

- (i) The axiom $\theta \vee \lceil \varphi \rceil \geq 1$ says that $\varphi \leq 0$ implies θ .
- (ii) The axiom $\theta \wedge \psi \leq 0$ says that θ implies $\psi \leq 0$.
- (iii) The axiom $\psi \leq c \lceil \varphi \rceil$ says that $\varphi \leq 0$ implies $\psi \leq 0$.

In each case, if the hypothesis fails, the axiom holds whatever the value of the conclusion.

5.2. The vocabulary. The vocabulary $\mathcal{L}$ has three sorts. Two of them, the Hilbert sort and the sequence sort, are *vector sorts*, intended to be the closed unit balls of Hilbert spaces; working in unit balls keeps every predicate bounded.

- ♠ The *Hilbert sort* $\mathbf{H}$ has, for each pair λ, λ' of elements of the field $\mathbb{Q}(i)$ of complex numbers with rational real and imaginary parts such that $|\lambda| + |\lambda'| \leq 1$, a binary function $(x, y) \mapsto \lambda x + \lambda' y$; the predicates $\text{Re}\langle x | y \rangle$, $\text{Im}\langle x | y \rangle$ and $\|x\|$, with a priori bound 1; and constants e_n , for $n \in \mathbb{N}$. We write 0 for the closed term $0e_0 + 0e_0$, and $\langle x | y \rangle$ for the pair of predicates $\text{Re}\langle x | y \rangle$ and $\text{Im}\langle x | y \rangle$.
- ♠ The *sequence sort* $\mathbf{K}$ has the same functions and predicates as $\mathbf{H}$, and one constant ε ; in this sort, we write 0 for the closed term $0\varepsilon + 0\varepsilon$.
- ♠ The *Solèr sort* $\mathbf{L}$ has the constants 0 and 1, the binary functions $\wedge$ and $\vee$, the unary function $^\perp$, and a predicate $e(a, b)$ with a priori bound 1, the *equality predicate*.
- ♠ Function symbols connecting the sorts:

$$\begin{aligned} \pi: \mathbf{L} \times \mathbf{H} &\rightarrow \mathbf{H}, & \text{at}: \mathbf{H} &\rightarrow \mathbf{L}, & \text{rep}: \mathbf{L} &\rightarrow \mathbf{H}, \\ \iota_n: \mathbf{H} &\rightarrow \mathbf{K}, & \kappa_n: \mathbf{K} &\rightarrow \mathbf{H} \ (n \in \mathbb{N}), & \text{span}: \mathbf{K} &\rightarrow \mathbf{L}. \end{aligned}$$

- ♠ A predicate μ on $\mathbf{L}$, the *measure*, with a priori bound 2. The bound is wider than 1 so that the inequality $\mu(a) \leq 1$, like $\mu(a) \geq 0$, is imposed by the axiom (M1) of Section 6 below, and not by the vocabulary.

The symbol 0 thus names a formula, a closed term of each vector sort and a constant of the Solèr sort, while the symbols $\wedge$ and $\vee$ name both connectives and functions of the Solèr sort; the arguments to which they are applied always show which is meant. For vector terms t_1, t_2 of the same sort, the equation $t_1 \doteq t_2$ abbreviates $\|\frac{1}{2}t_1 - \frac{1}{2}t_2\| \leq 0$; for terms t_1, t_2 of the Solèr sort, the equation $t_1 = t_2$ abbreviates $e(t_1, t_2) \geq 1$. An equation that stands as an axiom in

Section 6 is an axiom in the sense of Section 5, required of all values of its free variables. A vector x *belongs* to an element a of the Solèr sort, or is a *member* of a , when $\pi(a, x) \doteq x$, that is, when $\|\frac{1}{2}\pi(a, x) - \frac{1}{2}x\| = 0$.

The intended meanings are as follows. Each element a of the Solèr sort names a closed subspace of the Hilbert space whose closed unit ball is the Hilbert sort, and the function $\pi(a, \cdot)$ is the orthogonal projection onto that subspace; the members of a are therefore the vectors of the unit ball that lie in it. The predicate e is equality, with the value 1 at equal elements and -1 at distinct ones. Vectors and elements of the Solèr sort are linked by at , which sends a nonzero vector x to the element whose members are the multiples of x in the unit ball, and by rep , which sends a nonzero element a to a unit vector that belongs to a . The constants e_n , for $n \in \mathbb{N}$, form an orthonormal basis of the Hilbert space. The elements of the sequence sort are the square-summable sequences of vectors of that space whose ℓ^2 norm is at most 1; they are linked to the Hilbert sort by ι_n , which sends a vector to the sequence that has it in position n and 0 elsewhere, and by κ_n , which reads the entry in position n . The constant ε is the sequence $(2^{-n-1}e_n)_{n \in \mathbb{N}}$, and the function span sends a sequence s to the element whose members are the vectors of the unit ball in the closed span of the entries of s .

The axiom of Theorem B, which excludes the points of a Borel set from the range of the measure, needs a two-valued formula that says whether a value of the measure lies in that set. For a Borel set $B \subseteq \mathbb{R}$, we write $\beta_B(a)$ for any formula of real-valued $L_{\omega_1, \omega}$, with the single free variable a of the Solèr sort, whose value at every element a of every structure for $\mathcal{L}$ is 1 if $\mu(a) \in B$ and -1 otherwise.

Lemma 5.1. *For every Borel set $B \subseteq \mathbb{R}$, there is a formula $\beta_B(a)$ as just described.*

Proof. Let $\mathcal{S}$ be the class of sets $B \subseteq \mathbb{R}$ for which such a formula exists. For rational numbers $q \leq q'$, the distance from a real number t to the interval $[q, q']$ is $\text{dist}(t, [q, q']) = \max(0, q - t, t - q')$, a continuous function of t and hence a continuous connective. The formulas $(\mathbf{1} - n \cdot \text{dist}(\mu(a), [q, q'])) \vee -\mathbf{1}$, for $n \geq 1$, take their values in $[-1, 1]$, and their countable infimum

$$\bigwedge_{n \geq 1} ((\mathbf{1} - n \cdot \text{dist}(\mu(a), [q, q'])) \vee -\mathbf{1})$$

takes the value 1 when $\mu(a) \in [q, q']$ and -1 otherwise. Thus $\mathcal{S}$ contains the closed intervals with rational endpoints. If $B \in \mathcal{S}$, then $-\beta_B(a)$ is a formula for $\mathbb{R} \setminus B$. If $B_n \in \mathcal{S}$ for every $n \in \mathbb{N}$, then the formulas $\beta_{B_n}(a)$ take their values in $[-1, 1]$, and their countable join is a formula for $\bigcup_n B_n$. Hence $\mathcal{S}$ is a σ -algebra. Every open subset of $\mathbb{R}$ is a countable union of closed intervals with rational endpoints, so $\mathcal{S}$ contains every Borel set. 

In the formula $\beta_B(a)$ that the proof constructs, the countable joins and infima are nested as in the construction of B from closed intervals with rational endpoints; the depth of this nesting is a countable ordinal, unbounded below ω_1 as B varies, whence the axioms of Theorem B involve formulas whose joins are nested to arbitrarily large countable depth, although none of them contains a quantifier.

Definition 5.2. A structure $\mathfrak{M}$ for $\mathcal{L}$, or for a vocabulary obtained from $\mathcal{L}$ by removing the predicate μ , the sequence sort together with the symbols that involve it, or both, is *standard* when its Hilbert sort is the closed unit ball of a complex Hilbert space H , its sequence sort, if it has one, is the closed unit ball of a complex Hilbert space K , and the functions $\lambda x + \lambda' y$ and the predicates $\text{Re}\langle x \mid y \rangle$, $\text{Im}\langle x \mid y \rangle$ and $\|x\|$ are the restrictions of the operations, the inner products and the norms of these spaces.

Besides T , we define in Section 6 below two theories contained in it: the theory T_0 , in the vocabulary without the measure and the sequence sort, and the theory T_1 , in the vocabulary without the measure alone. A model of one of the theories T_0 , T_1 and T is called *nonstandard* when it is not standard. The word has nothing to do with the nonstandard models of arithmetic or of nonstandard analysis: every predicate takes real values, and no element is infinitely large or infinitesimal. In Lemma 6.3 below, we show that the Hilbert sort of every model of T_0 , and hence of every model of T_1 or of T , can be identified with a subset of the closed unit ball of a complex Hilbert space, a subset whose $\mathbb{Q}(i)$ -linear span is dense in that space, the functions $\lambda x + \lambda' y$ and the predicates being the restrictions of the operations, the inner product and the norm of that space. A model of T_0 is standard if and only if that subset is the whole ball; a nonstandard model of T_0 therefore lacks some of the vectors of the ball and contains none that the ball lacks. In a nonstandard model of T_1 or of T , the sequence sort, which Lemma 6.3 does not concern, may fail to be the closed unit ball of a complex Hilbert space, whether or not the Hilbert sort is the whole ball.

Standardness is a hypothesis of some of our theorems, not a condition imposed by axioms; indeed, no countable theory in one of the vocabularies of Definition 5.2 that contains T_0 and has a model has only standard models. Every standard model of T_0 has an uncountable Hilbert sort, the closed unit ball of an infinite-dimensional Hilbert space (see Proposition 7.1 below), while the downward Löwenheim–Skolem theorem gives every such theory a model whose sorts are countable. In particular, the theories T_0 , T_1 and T , which have standard models by Theorem A(2) and by taking reducts, have nonstandard models whose sorts are countable. The countable models of T nevertheless fall under Theorem C, whereas the conclusions of Theorem A(1) and Theorem B fail in them (see Remark 11.2).

✂ **Historical note.** Carol Karp, a student of Henkin’s, wrote her dissertation of 1959 on languages with expressions of infinite length and published a book on them in 1964 [Kar64]. The model existence theorem for $L_{\omega_1, \omega}$, which builds a model from a consistency property in the manner of Henkin’s completeness proof, is due to Makkai, who proved it in 1969 [Mak69]; in his book of 1971, Keisler proves the theorems of that logic through consistency properties [Kei71]. Model theory with truth values in the unit interval goes back to Chang’s work of 1958 and 1961 on the infinite-valued logic of Łukasiewicz [Cha58, Cha61]; the continuous model theory of Chang and Keisler of 1966 took its truth values in a compact Hausdorff space [CK66]. Henson introduced the logic of positive bounded formulas in 1976 to study a Banach space through its nonstandard hull [Hen76], and in 2008 Ben Yaacov, Berenstein, Henson and Usvyatsov set out the model theory of metric structures in continuous logic [BYBH08]. In 2009 Ben Yaacov and Iovino introduced an infinitary continuous logic for model-theoretic forcing [BYI09], and in 2014 Eagle proved the omitting types theorem for infinitary $[0, 1]$ -valued logic [Eag14]. In Keisler’s model theory for real-valued structures, of 2023 [Kei23], no metric is given a priori; the logic of this paper, whose conditions are strict inequalities, is the infinitary extension of the logic of that theory due to Aubone, Dueñez and Iovino [ADI26].

6. THE AXIOMS

In this section we state the axioms of the theory T of Theorem A and of two smaller theories contained in it, the first in the vocabulary without the measure and the sequence sort, the second in the vocabulary without the measure alone. Interpreted in a standard model, the axioms say that the constants e_n form an orthonormal basis; that each element of the Solèr sort determines an orthogonal projection acting on the Hilbert sort, and that these

projections are ordered and combined as the closed subspaces onto which they project; that atoms correspond to unit vectors up to a phase, that is, up to a complex factor of modulus one; that the function span sends each sequence of the sequence sort to an element of the Solèr sort whose members are the vectors of the unit ball lying in the closed span of the entries of that sequence; and that the measure μ is σ -additive, with values in $[0, 1]$ and the value 1 at the greatest element of the Solèr sort. They fall into eight groups: one for each of the three sorts, namely (H), (OL) and (K), the last of which also governs the functions ι_n and κ_n ; four for the remaining symbols that connect the sorts, namely (P) and (Sub) for the projection π and its relation to the lattice operations, (At) for at and rep, and (Sp) for span; and one, (M), for the measure.

Free variables are universally quantified: the variables x, y, z range over $\mathbf{H}$, the variables s, t over $\mathbf{K}$, and the variables a, b, c over $\mathbf{L}$. The letters λ, λ' , which range over the elements of $\mathbb{Q}(i)$ with $|\lambda| + |\lambda'| \leq 1$, and the letters n, m, k, N , which range over $\mathbb{N}$, are not variables of the logic: an axiom in which they occur free stands for countably many axioms, one for each choice of their values, whereas a bound occurrence indexes a finite sum, a countable join or a countable infimum. Each axiom is understood in the sense of Section 5, as a closed condition required of all values of its free variables; the equations written with $\doteq$ between vector terms and with $=$ between terms of the Solèr sort are the abbreviations introduced in Section 5.2. An implication between two-valued statements is asserted, as in Section 5, by requiring the disjunction of its conclusion with the negation of its hypothesis to be at least 1; an implication that involves a real quantity takes one of the forms (i)–(iii) that follow (2), of which (i) and (iii) use the formula $\lceil \varphi \rceil$. Some axioms are followed by a reading in words, in italics, which describes what the axiom says in standard models.

(H) The Hilbert sort.

- (H1) *Inner product space.* For any two terms built from variables by the functions $\lambda x + \lambda' y$ that denote the same $\mathbb{Q}(i)$ -linear combination of the variables, the equation between them; the closed conditions $\langle \lambda x + \lambda' y \mid z \rangle = \lambda \langle x \mid z \rangle + \lambda' \langle y \mid z \rangle$ and $\langle y \mid x \rangle = \overline{\langle x \mid y \rangle}$, each written as two real closed conditions; and $\|x\| = \sqrt{\max(0, \operatorname{Re} \langle x \mid x \rangle)}$.
- (H2) *Separability.* $\langle e_n \mid e_m \rangle = \delta_{nm}$, where δ_{nm} is the Kronecker delta, and

$$\|x\|^2 = \bigvee_N \left(\mathbf{1} \wedge \sum_{n < N} |\langle x \mid e_n \rangle|^2 \right).$$

The constants e_n are orthonormal and satisfy Parseval's identity.

(OL) The Solèr sort.

- (OL1) The equations of ortholattices:

$$\begin{aligned} a \wedge b &= b \wedge a, & a \wedge (b \wedge c) &= (a \wedge b) \wedge c, & a \wedge (a \vee b) &= a, \\ a \vee b &= b \vee a, & a \vee (b \vee c) &= (a \vee b) \vee c, & a \vee (a \wedge b) &= a, \\ a \wedge 0 &= 0, & a \vee 1 &= 1, & a^{\perp \perp} &= a, & (a \wedge b)^{\perp} &= a^{\perp} \vee b^{\perp}, & a \wedge a^{\perp} &= 0. \end{aligned}$$

(OL2) *The equality predicate.*

$$\begin{aligned}
& e(a, b) \vee -e(a, b) \geq 1, \quad e(a, a) \geq 1, \\
& -e(a, b) \vee e(b, a) \geq 1, \quad -e(a, b) \vee -e(b, c) \vee e(a, c) \geq 1; \\
& -e(a, b) \vee e(a \wedge c, b \wedge c) \geq 1, \quad -e(a, b) \vee e(a \vee c, b \vee c) \geq 1, \\
& -e(a, b) \vee e(a^\perp, b^\perp) \geq 1; \\
& e(a, b) \wedge \left\| \frac{1}{2}\pi(a, x) - \frac{1}{2}\pi(b, x) \right\| \leq 0, \quad e(a, b) \wedge \left\| \frac{1}{2}\text{rep}(a) - \frac{1}{2}\text{rep}(b) \right\| \leq 0, \\
& e(a, b) \wedge |\mu(a) - \mu(b)| \leq 0.
\end{aligned}$$

The predicate e is a two-valued equivalence relation, respected by the lattice operations, the projections, the function rep and the measure.

(P) **Each element of the Solèr sort determines a projection.**

- (P1) $\pi(a, \lambda x + \lambda' y) \doteq \lambda \pi(a, x) + \lambda' \pi(a, y)$.
(P2) $\pi(a, \pi(a, x)) \doteq \pi(a, x)$.
(P3) $\langle \pi(a, x) \mid y \rangle = \langle x \mid \pi(a, y) \rangle$.
(P4) *Extensionality.* $e(a, b) \vee \left\lceil \sup_x \left\| \frac{1}{2}\pi(a, x) - \frac{1}{2}\pi(b, x) \right\| \right\rceil \geq 1$.
Two elements with the same projection are equal.

(Sub) **The order and the operations of the Solèr sort are those of subspaces.**

- (Sub0) $\pi(0, x) \doteq 0$ and $\pi(1, x) \doteq x$.
Only the zero vector belongs to 0; every vector belongs to 1.
(Sub1)

$$\begin{aligned}
& e(a \wedge b, a) \wedge \left\| \frac{1}{2}\pi(b, \pi(a, x)) - \frac{1}{2}\pi(a, x) \right\| \leq 0, \\
& e(a \wedge b, a) \vee \left\lceil \sup_x \left\| \frac{1}{2}\pi(b, \pi(a, x)) - \frac{1}{2}\pi(a, x) \right\| \right\rceil \geq 1.
\end{aligned}$$

The relation $a \leq b$ holds if and only if every member of a is a member of b .

- (Sub2) $\bigwedge_N \bigvee_{n \geq N} \left\| \frac{1}{2}\pi(a \wedge b, x) - \frac{1}{2}(P_a P_b)^n x \right\| \leq 0$, where $(P_a P_b)^n x$ is the term obtained by applying $\pi(b, \cdot)$ and then $\pi(a, \cdot)$, n times.

The members of $a \wedge b$ are the common members of a and b .

- (Sub3) $\bigwedge_N \bigvee_{n \geq N} \left\| \frac{1}{4}\pi(a \vee b, x) - \frac{1}{4}x + \frac{1}{4}(P_{a^\perp} P_{b^\perp})^n x \right\| \leq 0$, where $(P_{a^\perp} P_{b^\perp})^n x$ is the term of

(Sub2) for $a^\perp$ and $b^\perp$, and the vector inside the norm is denoted by a term built from the functions $\lambda x + \lambda' y$.

The closed subspace spanned by the members of $a \vee b$ is the closure of the sum of the closed subspaces spanned by the members of a and by those of b .

- (Sub4) $\frac{1}{2}\pi(a^\perp, x) \doteq \frac{1}{2}x - \frac{1}{2}\pi(a, x)$.

The members of $a^\perp$ are the vectors orthogonal to all members of a .

Axiom (At3) below needs a formula that recognizes atoms. Let $\text{nonatom}(a)$ be the formula $\sup_b (-e(b, 0) \wedge -e(b, a) \wedge e(a \wedge b, b))$. By (OL2), it is two-valued, with the value 1 if some b lies strictly between 0 and a , and -1 otherwise. Thus an element a with $e(a, 0) = -1$ is an atom exactly when $\text{nonatom}(a) = -1$.

(At) Atoms and unit vectors.

(At1) $\pi(\text{at}(x), x) \doteq x$, and $|\langle \pi(\text{at}(x), y) \mid \pi(\text{at}(x), z) \rangle|^2 = \|\pi(\text{at}(x), y)\|^2 \|\pi(\text{at}(x), z)\|^2$.
Every vector x belongs to $\text{at}(x)$, and any two members of $\text{at}(x)$ are linearly dependent.

(At2) $\pi(a, \text{rep}(a)) \doteq \text{rep}(a)$, and $-e(a, 0) \wedge \|\text{rep}(a)\| - 1 \leq 0$.
The vector $\text{rep}(a)$ belongs to a , and it is a unit vector when $a \neq 0$.

(At3) $e(a, 0) \vee \text{nonatom}(a) \vee e(\text{at}(\text{rep}(a)), a) \geq 1$.
Every atom a equals $\text{at}(\text{rep}(a))$.

(At4) $e(\text{at}(x), \text{at}(y)) \wedge (\|x\| \|y\| - |\langle x \mid y \rangle|) \leq 0$, and

$$e(\text{at}(x), \text{at}(y)) \vee \left[\|x\| \|y\| - |\langle x \mid y \rangle| + |1 - \|x\|| + |1 - \|y\|| \right] \geq 1.$$

Two unit vectors determine the same atom if and only if one is a multiple of the other by a complex number of modulus one.

(At5) $\text{at}(0) = 0$, and $e(\text{at}(x), \text{at}(y)) \vee \left[\|\frac{1}{2}x - \frac{1}{2}y\| \right] \geq 1$.

(K) The sequence sort.

(K1) The axioms (H1), written for **K**.

(K2) $\iota_n(\lambda x + \lambda' y) \doteq \lambda \iota_n x + \lambda' \iota_n y$ and $\kappa_n(\lambda s + \lambda' t) \doteq \lambda \kappa_n s + \lambda' \kappa_n t$; $\langle \iota_n x \mid \iota_m y \rangle = \delta_{nm} \langle x \mid y \rangle$;
 and $\langle \kappa_n s \mid x \rangle = \langle s \mid \iota_n x \rangle$.

(K3) $\|s\|^2 = \bigvee_N \left(\mathbf{1} \wedge \sum_{n < N} \|\kappa_n s\|^2 \right)$.

The norm of a sequence is the ℓ^2 norm of its entries, whence, by the linearity of the maps κ_n in (K2), a sequence is determined by its entries.

(K4) $\kappa_n \varepsilon \doteq 2^{-n-1} e_n$.

(Sp) Spans.

(Sp1) $\pi(\text{span}(s), \kappa_n s) \doteq \kappa_n s$.
Every entry of s belongs to $\text{span}(s)$.

(Sp2) $\|\pi(\text{span}(s), x)\|^2 \leq \left[\bigvee_N \left(\mathbf{1} \wedge \sum_{n < N} |\langle x \mid \kappa_n s \rangle|^2 \right) \right]$.

A vector orthogonal to every entry of s is orthogonal to every member of $\text{span}(s)$.

(Sp3) $e(\text{span}(s), \text{span}(t)) \vee \left[\|\frac{1}{2}s - \frac{1}{2}t\| \right] \geq 1$.

The axiom of σ -additivity concerns sequences whose entries are mutually orthogonal; it therefore needs a quantity that measures how far the entries of a sequence are from being mutually orthogonal. Let $\text{od}(s)$, the *orthogonality defect* of s , be the formula $\bigvee_N (\mathbf{1} \wedge \sum_{n, m < N, n \neq m} |\langle \kappa_n s \mid \kappa_m s \rangle|)$. It takes values in $[0, 1]$, and it vanishes exactly when the entries $\kappa_n s$ of s are mutually orthogonal.

(M) The measure.

(M1) $\mu(a) \geq 0$ and $\mu(a) \leq 1$.

(M2) $\mu(1) = 1$.

(M3) σ -additivity.

$$\left| \mu(\text{span}(s)) - \bigvee_N \left(\left((2 \cdot \mathbf{1}) \wedge \sum_{n < N} \mu(\text{at}(\kappa_n s)) \right) \vee 0 \right) \right| \leq 2 \lceil \text{od}(s) \rceil.$$

If the entries of s are mutually orthogonal, the value of μ at $\text{span}(s)$ is the sum of its values at the elements $\text{at}(\kappa_n s)$.

Since μ takes its values in $[0, 1]$ in every structure that satisfies (M1), the left side of (M3) is at most 2 in such a structure; there, axiom (M3) is an implication of the form (iii), with that left side in place of ψ , the bound 2 in place of the constant of (iii), and $\text{od}(s)$ in place of φ .

Definition 6.1. The theory T_0 consists of the axioms (H), (OL), (P), (Sub) and (At), without the clause on μ in (OL2), in the vocabulary obtained from $\mathcal{L}$ by removing the sequence sort with its constant ε , the functions ι_n , κ_n and span , and the predicate μ . The theory T_1 consists of T_0 together with (K) and (Sp), in the vocabulary $\mathcal{L}$ without μ ; the theory T consists of T_1 together with (M) and the clause on μ in (OL2).

Lemma 6.2. *In a model of T_0 , two elements a, b of the Solèr sort are equal if and only if $e(a, b) = 1$, and two vectors x, y are equal if and only if $\|\frac{1}{2}x - \frac{1}{2}y\| = 0$; the same holds in the sequence sort of a model of T_1 .*

Proof. Since models are reduced, it suffices to show that two elements that the predicate e , or the norm of their half-difference, does not distinguish agree in every atomic formula; this rests on the Cauchy–Schwarz inequality, hence on the positivity of the form, which we prove first. By (H1) and (H2), the form given by the pair of predicates $\langle x \mid y \rangle$ is positive semidefinite on the Hilbert sort, that is, $\text{Re}\langle x \mid x \rangle \geq 0$ for every element x of that sort. Indeed, suppose that $\text{Re}\langle x \mid x \rangle < 0$ for some x ; then (H1) yields $\|x\| = 0$, and (H2) gives $\langle x \mid e_n \rangle = 0$ for every n . For the vector $\frac{1}{2}x + \frac{1}{2}e_0$, axiom (H2) then gives $\|\frac{1}{2}x + \frac{1}{2}e_0\|^2 = \frac{1}{4}$, whereas (H1) gives $\|\frac{1}{2}x + \frac{1}{2}e_0\|^2 = \max(0, \frac{1}{4} \text{Re}\langle x \mid x \rangle + \frac{1}{4}) < \frac{1}{4}$, which is absurd. In a model of T_1 , an argument of the same kind, which uses (K1)–(K3), the positivity just proved and the vector $\frac{1}{2}s + \frac{1}{2}\iota_0 e_0$, shows that the form is positive semidefinite on the sequence sort as well; the Cauchy–Schwarz inequality therefore holds for the predicates of each vector sort. With it, axioms (H1), (OL2), (P1)–(P3) and (At5), together with (K1), (K2) and (Sp3) in a model of T_1 , show that every atomic formula takes the same values at a and b when $e(a, b) = 1$, and at x and y when $\|\frac{1}{2}x - \frac{1}{2}y\| = 0$; since models are reduced, the elements in question are equal. The converse implications hold by (OL2) and (H1). ☺

Lemma 6.3. *Let $\mathfrak{M}$ be a model of T_0 . There is a complex Hilbert space $\bar{E}$ such that the Hilbert sort of $\mathfrak{M}$ can be identified with a subset Y of the closed unit ball of $\bar{E}$ whose $\mathbb{Q}(i)$ -linear span E is dense, the functions $\lambda x + \lambda' y$ and the predicates being the restrictions of the operations, the inner product and the norm of $\bar{E}$. For every element a of the Solèr sort, the map $\pi(a, \cdot)$ extends uniquely to an orthogonal projection $\bar{P}_a$ of $\bar{E}$. If $\mathfrak{M}$ is standard, then Y is the whole closed unit ball of $\bar{E}$.*

Proof. The idea is to recover a vector space from the Hilbert sort by formal multiples. Each pair (k, x) , with $k \geq 1$ an integer and x an element of the Hilbert sort, stands for kx ; we put $(k, x) \sim (l, y)$ when $\|\frac{1}{2l}x - \frac{1}{2k}y\| = 0$, and define the sum by $(k, x) + (l, y) = (2kl, \frac{1}{2l}x + \frac{1}{2k}y)$, the product by an element λ of $\mathbb{Q}(i)$ by $\lambda(k, x) = (km, \frac{\lambda}{m}x)$, where m is a positive integer with $m \geq |\lambda|$, and the form by $\langle (k, x) \mid (l, y) \rangle = kl \langle x \mid y \rangle$. Axiom (H1), together with the positivity of the form established in the proof of Lemma 6.2, makes the quotient a

$\mathbb{Q}(i)$ -vector space E with a positive definite hermitian form, and, by Lemma 6.2, the map that sends x to the class of $(1, x)$ is injective; we let $\bar{E}$ be the completion of E , and Y the image of the Hilbert sort, whose $\mathbb{Q}(i)$ -linear span is E . By (P1)–(P3), the map $\pi(a, \cdot)$ extends to a $\mathbb{Q}(i)$ -linear idempotent map Q_a on E , symmetric with respect to the form; hence $\|Q_a x\|^2 = \langle x \mid Q_a x \rangle \leq \|x\| \|Q_a x\|$, the map Q_a is contractive, and it extends by continuity to a self-adjoint idempotent $\bar{P}_a$ of $\bar{E}$. In a standard model, the Hilbert sort is the closed unit ball of a Hilbert space H , and the map that sends the class of (k, x) to kx identifies E with H , since every vector of H is a positive integer multiple of a vector of that ball; as H is complete, it follows that $\bar{E} = H$, and that Y is the whole closed unit ball of $\bar{E}$. 

 **Historical note.** In the continuous logic set out by Ben Yaacov, Berenstein, Henson and Usvyatsov in 2008, the unit ball of a Banach space carries the maps $(x, y) \mapsto \alpha x + \beta y$ with $|\alpha| + |\beta| \leq 1$ as functions, and a Hilbert space carries in addition its inner product as a predicate [BYBH08]; the Hilbert sort follows this practice. Von Neumann’s alternating projection theorem, on which axioms (Sub2) and (Sub3) rest, comes from lecture notes that he distributed in mimeographed form in 1933. The notes were printed in 1950 as the second volume of his *Functional Operators* [vN50].

7. STANDARD MODELS OF T_0 : PRE-SOLÈR LATTICES

In this section we study the standard models of T_0 , the theory whose vocabulary has neither the sequence sort nor the measure. We show that the Solèr sort of such a model is a pre-Solèr lattice, an ortholattice that satisfies every condition in the definition of a Solèr lattice except, possibly, completeness. We then show that the map sending each element of the Solèr sort to the closed subspace spanned by its members embeds the sort, as a join-dense and meet-dense sub-ortholattice in the order-theoretic sense of Section 1, in the lattice of closed subspaces of the Hilbert space whose closed unit ball is the Hilbert sort. Completeness cannot be obtained at this level, since a standard model of T_0 may have an incomplete Solèr sort (see Example 7.6); Gleason’s theorem, by contrast, already holds there, in the sense that the σ -additive states on the Solèr sort, defined below, are given by density operators.

Throughout this section, we fix a standard model $\mathfrak{M}$ of T_0 . We write H for the Hilbert space whose closed unit ball is the Hilbert sort of $\mathfrak{M}$ (see Definition 5.2), and $\mathbf{L}^{\mathfrak{M}}$ for the Solèr sort of $\mathfrak{M}$. For each element a of $\mathbf{L}^{\mathfrak{M}}$, we write P_a for the projection $\bar{P}_a$ of Lemma 6.3, which extends $\pi(a, \cdot)$, and S_a for its range; this range is the closed subspace of H spanned by the members of a , the vectors x of the unit ball with $\pi(a, x) = x$.

Proposition 7.1. *The space H is a separable infinite-dimensional complex Hilbert space, and the constants e_n of the Hilbert sort, for $n \in \mathbb{N}$, form an orthonormal basis of H .*

Proof. By (H2), the vectors e_n are orthonormal, and Parseval’s identity puts every vector of the unit ball, and hence every vector of H , in their closed span. Thus $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of H , which is therefore separable and infinite-dimensional; it is complex by Definition 5.2. 

We next show that the map $a \mapsto S_a$ respects the order and the operations of the Solèr sort. Axioms (Sub2) and (Sub3) describe meets and joins through von Neumann’s alternating projection theorem, which reaches the projection onto the intersection of two closed subspaces by projecting alternately onto each.

Theorem 7.2 (von Neumann’s alternating projection theorem [vN49, Lemma 22]). *Let U and V be closed subspaces of a Hilbert space, and let P_U , P_V and $P_{U \cap V}$ be the orthogonal*

projections onto U , V and $U \cap V$. Then, for every vector x of the space, the sequence $((P_U P_V)^n x)_{n \in \mathbb{N}}$ converges in norm to $P_{U \cap V} x$.

Proposition 7.3. *The map $\tau: a \mapsto S_a$ is an embedding of ortholattices from $\mathbf{L}^{\mathfrak{M}}$ into the lattice $L(H)$ of closed subspaces of H ; that is, it is injective and satisfies*

$$S_0 = \{0\}, \quad S_1 = H, \quad S_{a \wedge b} = S_a \cap S_b, \quad S_{a \vee b} = \overline{S_a + S_b}, \quad S_{a^\perp} = S_a^\perp.$$

Moreover, the relation $a \leq b$ holds if and only if $S_a \subseteq S_b$.

Proof. By (OL1) and Lemma 6.2, the Solèr sort $\mathbf{L}^{\mathfrak{M}}$ is an ortholattice. Axiom (Sub0) gives the first two equalities, and axiom (Sub4) gives $P_{a^\perp} = \text{Id} - P_a$, whose range is $S_a^\perp$. For meets and joins we combine the axioms with Theorem 7.2: axiom (Sub2) shows that $P_{a \wedge b} = P_{S_a \cap S_b}$, and axiom (Sub3) that $P_{a \vee b} = \text{Id} - P_{S_a^\perp \cap S_b^\perp}$, whose range is $(S_a^\perp \cap S_b^\perp)^\perp = \overline{S_a + S_b}$. Injectivity follows from extensionality: if $S_a = S_b$, then the projections P_a and P_b coincide, and axiom (P4) gives $a = b$. Finally, the order is read off from axiom (Sub1), which shows that $a \wedge b = a$ exactly when $P_b P_a = P_a$, that is, when $S_a \subseteq S_b$. $\color{violet}{\smileyemote}$

Proposition 7.4. *For every nonzero x in the unit ball, the subspace $S_{\text{at}(x)}$ is $\mathbb{C}x$. The atoms of $\mathbf{L}^{\mathfrak{M}}$ are exactly the elements $\text{at}(x)$ with $x \neq 0$, and the map $x \mapsto \text{at}(x)$ induces a bijection from the unit sphere of H , modulo multiplication by complex numbers of modulus one, onto the set of atoms of $\mathbf{L}^{\mathfrak{M}}$, whose inverse sends an atom a to the class of $\text{rep}(a)$.*

Proof. By (At1), the vector x lies in $S_{\text{at}(x)}$, and any two vectors of $S_{\text{at}(x)}$ attain equality in the Cauchy-Schwarz inequality, hence are linearly dependent; therefore $S_{\text{at}(x)} = \mathbb{C}x$. Since this ray is an atom of $L(H)$ and the map τ is an order embedding, the element $\text{at}(x)$ is an atom of $\mathbf{L}^{\mathfrak{M}}$. If a is an atom of $\mathbf{L}^{\mathfrak{M}}$, then axiom (At2) shows that $\text{rep}(a)$ is a unit vector, and axiom (At3) gives $a = \text{at}(\text{rep}(a))$. For unit vectors x and y , axiom (At4) gives $\text{at}(x) = \text{at}(y)$ if and only if $|\langle x | y \rangle| = 1$, that is, if and only if y is a multiple of x by a complex number of modulus one. In particular, for a unit vector x , the unit vector $\text{rep}(\text{at}(x))$, which lies in $S_{\text{at}(x)} = \mathbb{C}x$ by (At2), is such a multiple of x , and the inverse of the bijection is as stated. $\color{violet}{\smileyemote}$

Theorem 7.5. *Let $\mathfrak{M}$ be a standard model of T_0 , let H be the Hilbert space whose closed unit ball is its Hilbert sort, and, for each element a of its Solèr sort $\mathbf{L}^{\mathfrak{M}}$, let S_a be the closed subspace of H spanned by the members of a . Then $\mathbf{L}^{\mathfrak{M}}$ is a pre-Solèr lattice, and the image of the map $\tau: a \mapsto S_a$ is join-dense and meet-dense in the lattice $L(H)$ of closed subspaces of H , which is therefore isomorphic to the MacNeille completion of $\mathbf{L}^{\mathfrak{M}}$. The image of τ contains every finite-dimensional subspace of H and the orthogonal complement of each, and the map τ sends every join that exists in $\mathbf{L}^{\mathfrak{M}}$ to the corresponding join in $L(H)$.*

Proof. We begin with the finite-dimensional subspaces and then verify, condition by condition, that $\mathbf{L}^{\mathfrak{M}}$ is a pre-Solèr lattice. The zero subspace is S_0 ; if F is a nonzero finite-dimensional subspace of H with a basis $x_1, \dots, x_k$ of unit vectors, then, by Propositions 7.3 and 7.4, the map τ sends $\text{at}(x_1) \vee \dots \vee \text{at}(x_k)$ to the closure of F , which is F itself, and $(\text{at}(x_1) \vee \dots \vee \text{at}(x_k))^\perp$ to $F^\perp$. Condition (OM) holds because $\tau(\mathbf{L}^{\mathfrak{M}})$ is a sub-ortholattice of the orthomodular lattice $L(H)$. For (A), let a be a nonzero element of $\mathbf{L}^{\mathfrak{M}}$; the atoms below a are the elements $\text{at}(x)$ with $x \in S_a$ and $0 < \|x\| \leq 1$, and any upper bound b of them in $\mathbf{L}^{\mathfrak{M}}$ satisfies $S_b \supseteq S_a$, hence $b \geq a$, which makes a their join. For (CP), let a be an element of $\mathbf{L}^{\mathfrak{M}}$ and $\text{at}(x)$ an atom not below a ; then the subspace S_a does not contain x , and, by Proposition 7.3 and Lemma 2.7, the subspace $\tau(a \vee \text{at}(x))$ is $S_a + \mathbb{C}x$, in which S_a has codimension one; since τ is an order embedding, no element lies strictly between a and $a \vee \text{at}(x)$. Condition (I) holds because the join of two distinct atoms corresponds to a

plane, which contains infinitely many rays, all in $\tau(\mathbf{L}^{\mathfrak{M}})$. The atoms $\text{at}(e_n)$ give (D). For (S), take $p_n = \text{at}(e_n)$ and $c_n = \text{at}(\frac{1}{2}(e_n + e_{n+1}))$; since τ is injective and preserves the lattice operations and the orthocomplementation, it suffices to verify the condition for the images of these atoms, the rays of e_n and of $e_n + e_{n+1}$, where it follows from (1) and the computation following it, because $\|e_n\| = \|e_{n+1}\|$.

Join-density and meet-density of the image of τ follow from two facts about each closed subspace S of H : it is the join in $L(H)$ of its rays, and it equals $\bigcap \{y^\perp : y \in S^\perp, 0 < \|y\| \leq 1\}$, where each $y^\perp = S_{\text{at}(y)^\perp}$ lies in the image of τ . By Proposition 1.5, the lattice $L(H)$ is therefore isomorphic to the MacNeille completion of $\mathbf{L}^{\mathfrak{M}}$. Finally, let $A \subseteq \mathbf{L}^{\mathfrak{M}}$ have a join c in $\mathbf{L}^{\mathfrak{M}}$, and let W be the closed span of $\bigcup_{a \in A} S_a$; clearly $W \subseteq S_c$. If y is a vector of $S_c \cap W^\perp$ with $0 < \|y\| \leq 1$, then every $a \in A$ lies below $\text{at}(y)^\perp$, hence so does c ; the vector y , which lies in S_c , is then orthogonal to itself, which is absurd. Since $S_c = W + (S_c \cap W^\perp)$ by the projection theorem, it follows that $S_c = W$. 

Example 7.6. Let L_{fc} (for “finite dimension or finite codimension”) be the set of finite-dimensional subspaces of the space ℓ^2 of square-summable complex sequences and their orthogonal complements; by the argument of Example 4.3, it is a sub-ortholattice of $L(\ell^2)$. Take the closed unit ball of ℓ^2 as Hilbert sort, the standard unit vectors as the constants e_n , and L_{fc} , with its lattice operations and orthocomplementation, as Solèr sort. Interpret e as equality, with $e(S, S') = 1$ when $S = S'$ and $e(S, S') = -1$ otherwise. Put $\pi(S, x) = P_S x$ and $\text{at}(x) = \mathbb{C}x$, let $\text{rep}(S)$ be a unit vector chosen in S for each $S \neq \{0\}$, and let $\text{rep}(\{0\})$ be 0. Axioms (Sub2) and (Sub3) hold by Theorem 7.2, and the other axioms of T_0 by construction; the result is therefore a standard model of T_0 . Its Solèr sort L_{fc} is not complete. Indeed, an upper bound in L_{fc} of the rays $\mathbb{C}e_{2n}$, for $n \in \mathbb{N}$, is infinite-dimensional, hence of the form $G^\perp$ for a finite-dimensional subspace G orthogonal to every e_{2n} . The closed span of the vectors e_{2n+1} is infinite-dimensional, whereas G is finite-dimensional; that span therefore contains a unit vector y orthogonal to G , and the subspace $(G + \mathbb{C}y)^\perp$, which contains every e_{2n} but not y , is an upper bound of the same rays that lies strictly below $G^\perp$. The rays $\mathbb{C}e_{2n}$ therefore have no join in L_{fc} .

Gleason’s theorem, in the form we need, concerns two notions: density operators, by which quantum mechanics represents the states of a system, and σ -additive states on an ortholattice. A *density operator* on H is a positive operator ρ whose trace $\text{Tr } \rho = \sum_k \langle \rho u_k \mid u_k \rangle$, computed in any orthonormal basis $(u_k)_{k \in \mathbb{N}}$ of H , equals 1. A σ -*additive state* on an ortholattice Λ is a map $\nu : \Lambda \rightarrow [0, 1]$ with $\nu(1) = 1$ such that $\nu(\bigvee_{n \in J} a_n) = \sum_{n \in J} \nu(a_n)$ for every family $(a_n)_{n \in J}$ of mutually orthogonal elements of Λ , indexed by a set $J \subseteq \mathbb{N}$, whose join exists in Λ .

Theorem 7.7 (Gleason’s theorem for T_0). *Let $\mathfrak{M}$ be a standard model of T_0 , let H be the Hilbert space whose closed unit ball is its Hilbert sort, and, for each element a of its Solèr sort $\mathbf{L}^{\mathfrak{M}}$, let P_a be the orthogonal projection onto the closed subspace spanned by the members of a . The σ -additive states on $\mathbf{L}^{\mathfrak{M}}$ are exactly the maps $a \mapsto \text{Tr}(\rho P_a)$, where ρ ranges over the density operators on H , and each such state determines its density operator.*

Proof. That the maps $a \mapsto \text{Tr}(\rho P_a)$ are σ -additive states rests on Proposition 7.3 and Theorem 7.5, by which a join that exists in $\mathbf{L}^{\mathfrak{M}}$ corresponds to the closure of the sum of the corresponding subspaces of H ; the converse is Gleason’s theorem, applied to the function that assigns to each unit vector x the value of the state at the atom $\text{at}(x)$, whose sum along every orthonormal basis of H is 1. Let ρ be a density operator on H . If the elements a_n , for n in a set $J \subseteq \mathbb{N}$, are mutually orthogonal and have a join c in $\mathbf{L}^{\mathfrak{M}}$, then, by Proposition 7.3

and Theorem 7.5, the subspace S_c is the closure of the sum of the mutually orthogonal subspaces S_{a_n} , whence $P_c x = \sum_{n \in J} P_{a_n} x$ for every $x \in H$ and $\text{Tr}(\rho P_c) = \sum_{n \in J} \text{Tr}(\rho P_{a_n})$; moreover, the map $a \mapsto \text{Tr}(\rho P_a)$ takes the value $\text{Tr} \rho = 1$ at 1, and its values lie in $[0, 1]$. Conversely, let ν be a σ -additive state on $\mathbf{L}^{\mathfrak{M}}$; additivity on the orthogonal pair $1, 0$, whose join is 1, gives $\nu(0) = 0$. For a unit vector x , put $f(x) = \nu(\text{at}(x))$. For $a \neq 0$, let $\mathcal{U}_a$ be an orthonormal basis of S_a , which is countable, as H is separable; the atoms $\text{at}(u)$, for $u \in \mathcal{U}_a$, are mutually orthogonal, and their join in $\mathbf{L}^{\mathfrak{M}}$ is a , because a is an upper bound of them and every upper bound b of them satisfies $S_b \supseteq S_a$; hence $\nu(a) = \sum_{u \in \mathcal{U}_a} f(u)$. Taking $a = 1$ shows that f is a nonnegative *frame function* of weight 1: its sum over every orthonormal basis of H is 1. Since H is separable of dimension at least three, Gleason’s theorem [Gle57] gives a unique density operator ρ with $f(x) = \langle \rho x \mid x \rangle$ for every unit vector x , whence $\nu(a) = \sum_{u \in \mathcal{U}_a} \langle \rho u \mid u \rangle = \text{Tr}(\rho P_a)$ for every $a \neq 0$; the same equality holds for $a = 0$, since $\nu(0) = 0$. 

 **Historical note.** Gleason proved his theorem in 1957 to answer a question of Mackey’s: whether the Born rule is the only way to assign probabilities to the closed subspaces of a Hilbert space [Mac57, Gle57]. The Born rule gives a closed subspace, in the state described by a density operator ρ , the probability $\text{Tr}(\rho P)$, where P is the orthogonal projection onto that subspace. Frame functions are the heart of Gleason’s proof: he shows first, by means of spherical harmonics, that the continuous frame functions on the unit sphere of $\mathbb{R}^3$ come from quadratic forms; then, in the hardest step, that every nonnegative frame function on that sphere is continuous; and finally he passes from $\mathbb{R}^3$ to real and complex separable Hilbert spaces of dimension at least three. Gleason noted that the restriction on the dimension is essential: in dimension two the theorem fails. Cooke, Keane and Moran gave in 1985 an elementary proof, longer than Gleason’s but with fewer prerequisites [CKM85].

8. STANDARD MODELS OF T_1 , PROOFS OF THEOREMS A AND B, AND DEFINABLE SETS OF VALUES

The axioms of T that mention neither the sequence sort nor the measure, which form the theory T_0 of Section 7, constrain only the elements that the Solèr sort contains; none of them compels it to contain a given closed subspace of infinite dimension and infinite codimension, as the lattice L_{fc} of Example 7.6 shows. The sequence sort supplies what is missing: every closed subspace of a separable Hilbert space is the closed span of a square-summable sequence of vectors, and the function span sends such a sequence to an element of the Solèr sort whose members span that subspace. In this section we show accordingly that, in a standard model of T_1 , the part of T that does not mention the measure, every closed subspace of the Hilbert space whose closed unit ball is the Hilbert sort is spanned by the members of an element of the Solèr sort, and we prove Theorems A and B; in Section 8.1 we identify the sets of values that formulas define in the standard models.

Throughout this section, we fix a standard model $\mathfrak{M}$ of T_1 ; we write H and K for the Hilbert spaces whose closed unit balls are its Hilbert sort and its sequence sort, respectively, and $\mathbf{L}^{\mathfrak{M}}$ for its Solèr sort. Since T_1 contains T_0 , the reduct of $\mathfrak{M}$ to the vocabulary of T_0 is a standard model of T_0 , to which the results of Section 7 apply; as there, for each element a of $\mathbf{L}^{\mathfrak{M}}$ we write P_a for the orthogonal projection of H that extends $\pi(a, \cdot)$, S_a for its range, the closed subspace spanned by the members of a , and τ for the map $a \mapsto S_a$.

Lemma 8.1. *For each $n \in \mathbb{N}$, the map κ_n from the sequence sort to the Hilbert sort, which in the intended interpretation reads the entry in position n of a sequence (see Section 5.2),*

extends to a bounded linear map from K to H , and the map $s \mapsto (\kappa_n s)_{n \in \mathbb{N}}$ is a linear isometry of K onto the space $\ell^2(\mathbb{N}, H)$ of sequences $(x_n)_{n \in \mathbb{N}}$ in H with $\sum_n \|x_n\|^2 < \infty$.

Proof. By (K2), each map κ_n is linear on the unit ball of K , and by (K3) it satisfies $\|\kappa_n s\| \leq \|s\|$; it therefore extends to a bounded linear map from K to H , and (K3) shows that the map $s \mapsto (\kappa_n s)_{n \in \mathbb{N}}$ is isometric. By (K2), we also have $\kappa_m \iota_n x = \delta_{mn} x$, and the maps ι_n , which in the intended interpretation place a vector in position n of a sequence, extend to linear isometries from H into K with mutually orthogonal ranges. Given $(x_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, H)$, the partial sums $\sum_{n < N} \iota_n x_n$ therefore form a Cauchy sequence in K ; since K is complete, they converge to some s , and $\kappa_n s = x_n$ by the continuity of κ_n . 

Lemma 8.2. *For every s in the unit ball of K , the subspace $S_{\text{span}(s)}$ is the closed span of the entries $\kappa_n s$.*

Proof. Let W be that closed span. By (Sp1), the subspace $S_{\text{span}(s)}$ contains every entry of s ; hence $W \subseteq S_{\text{span}(s)}$. If $x \perp W$, axiom (Sp2) gives $P_{\text{span}(s)} x = 0$; hence $S_{\text{span}(s)} \subseteq W^{\perp\perp} = W$. 

Theorem 8.3. *Let $\mathfrak{M}$ be a standard model of T_1 , let H be the Hilbert space whose closed unit ball is its Hilbert sort, and, for each element a of its Solèr sort, let S_a be the closed subspace of H spanned by the members of a . Then H is a separable infinite-dimensional complex Hilbert space, and the map $\tau: a \mapsto S_a$ is an isomorphism of ortholattices from the Solèr sort of $\mathfrak{M}$ onto the lattice $L(H)$ of closed subspaces of H ; in particular, that Solèr sort is a Solèr lattice. Conversely, for every separable infinite-dimensional complex Hilbert space H' , there is a standard model of T_1 whose Hilbert sort is the closed unit ball of H' and whose Solèr sort is $L(H')$.*

Proof. By Proposition 7.1, the space H is separable and infinite-dimensional. By Proposition 7.3, the map τ is an embedding; it remains to be shown that it is onto. Let S be a closed subspace of H , and let $(u_n)_{n \in \mathbb{N}}$ enumerate an orthonormal basis of S , which is countable by Proposition 7.1, followed by zeros if the basis is finite. By Lemma 8.1 there is s in the unit ball of K with $\kappa_n s = 2^{-n-1} u_n$ for all n , since $\sum_n 4^{-n-1} < 1$, and the subspace $S_{\text{span}(s)}$ is S by Lemma 8.2.

For the converse, we build the model sort by sort. The Hilbert sort is the closed unit ball of H' , and the constants e_n are interpreted as an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of H' . The Solèr sort is $L(H')$, with e the equality predicate, $\pi(S, x) = P_S x$, where P_S denotes the orthogonal projection onto S , and $\text{at}(x) = \mathbb{C}x$; for each $S \neq \{0\}$ we let $\text{rep}(S)$ be a unit vector chosen in S , and we let $\text{rep}(\{0\})$ be 0. The sequence sort is the closed unit ball of $\ell^2(\mathbb{N}, H')$, with $\iota_n x$ the sequence with x in position n and 0 elsewhere, $\kappa_n s$ the entry of s in position n , and $\varepsilon = (2^{-n-1} e_n)_{n \in \mathbb{N}}$. Finally, the function span sends each element s of the sequence sort to the closed span of its entries. Axioms (Sub2) and (Sub3) hold by Theorem 7.2, and the others by construction. 

Proof of Theorem A. Let $\mathfrak{M}$ be a standard model of T . Its reduct to the vocabulary of T_1 is a standard model of T_1 , to which the notation fixed at the beginning of this section applies; by Theorem 8.3, the space H is a separable infinite-dimensional complex Hilbert space, and the map τ is an isomorphism of $\mathbf{L}^{\mathfrak{M}}$ onto $L(H)$. The idea is that axiom (M3), applied to a sequence whose entries are multiples of the vectors of an orthonormal basis of a closed subspace, expresses the measure of that subspace as the sum of the measures of the atoms of

those vectors (Step 2); this makes μ a σ -additive state, to which Gleason's theorem applies (Step 3).

Step 1: $\mu(0) = 0$. Recall that the formula od of Section 6 vanishes exactly at the sequences whose entries are mutually orthogonal. It therefore vanishes at the zero element s of the sequence sort, whose entries, all 0, are mutually orthogonal; moreover, the subspace $S_{\text{span}(s)}$ is $\{0\}$ by Lemma 8.2, whence $\text{span}(s) = 0$ by the injectivity of τ . Since these entries are 0, axiom (At5) gives $\text{at}(\kappa_n s) = \text{at}(0) = 0$; axiom (M3) then says that $\mu(0)$ is the supremum over N of $\max(0, \min(2, N\mu(0)))$. If $\mu(0)$ were positive, this supremum would be 2, contradicting (M1).

Step 2: sums along orthonormal bases. Let a be an element of $\mathbf{L}^{\mathfrak{M}}$, and let $(u_k)_{k \in \mathbb{N}}$ enumerate an orthonormal basis of S_a , followed by zeros if it is finite. By Lemma 8.1 there is s in the unit ball of K with $\kappa_k s = 2^{-k-1} u_k$. Its entries are mutually orthogonal, whence $\text{od}(s) = 0$; by Lemma 8.2 and the injectivity of τ , the element $\text{span}(s)$ is a . For $u_k \neq 0$, the subspaces $S_{\text{at}(2^{-k-1} u_k)}$ and $S_{\text{at}(u_k)}$ are both $\mathbb{C} u_k$ by Proposition 7.4, and the injectivity of τ gives $\text{at}(\kappa_k s) = \text{at}(u_k)$; for $u_k = 0$, the element $\text{at}(\kappa_k s)$ is 0, whose measure is 0 by Step 1. Axiom (M3) and $\mu(a) \leq 1$ give

$$\mu(a) = \sum_k \mu(\text{at}(u_k)). \quad (*)$$

Step 3: application of Gleason's theorem. We show that μ is a σ -additive state on $\mathbf{L}^{\mathfrak{M}}$, in order to apply Theorem 7.7. Its values lie in $[0, 1]$ by (M1), and $\mu(1) = 1$ by (M2). Let $(a_n)_{n \in J}$, for a set $J \subseteq \mathbb{N}$, be a family of mutually orthogonal elements of $\mathbf{L}^{\mathfrak{M}}$, and let c be its join, which exists because the map τ is onto $L(H)$. By Proposition 7.3 and Theorem 7.5, the subspaces S_{a_n} are mutually orthogonal, and the closure of their sum is S_c ; orthonormal bases of the subspaces S_{a_n} therefore form together an orthonormal basis of S_c . Applied to c , the identity $(*)$ expresses $\mu(c)$ as the sum of the values $\mu(\text{at}(u))$ over the vectors u of that basis; applied to each a_n , it expresses $\mu(a_n)$ as the sum of the same values over the vectors that come from S_{a_n} . Since these values are nonnegative by (M1), the first sum may be grouped by n , whence $\mu(c) = \sum_{n \in J} \mu(a_n)$. The reduct of $\mathfrak{M}$ to the vocabulary of T_0 is a standard model of T_0 with the same Hilbert sort and the same Solèr sort, to which Theorem 7.7 applies: there is a unique density operator ρ on H with $\mu(a) = \text{Tr}(\rho P_a)$ for every $a \in \mathbf{L}^{\mathfrak{M}}$.

The converse. Given a separable infinite-dimensional complex Hilbert space H' and a density operator ρ on H' , expand a standard model of T_1 built for H' as in the proof of Theorem 8.3 by $\mu(a) = \text{Tr}(\rho P_a)$, where P_a is the orthogonal projection onto the closed subspace a of H' . The clause on μ in (OL2) holds because e is the equality predicate. Axioms (M1) and (M2) hold because $\text{Tr}(\rho P) \in [0, 1]$ for every orthogonal projection P and $\text{Tr} \rho = 1$. For (M3), let s be an element of the sequence sort, and suppose first that its entries $x_n = \kappa_n s$, for $n \in \mathbb{N}$, are mutually orthogonal. The element $\text{span}(s)$ is then their closed span, and, with $\hat{x}_n = x_n / \|x_n\|$ for $x_n \neq 0$, we have $\text{Tr}(\rho P_{\text{span}(s)}) = \sum_{x_n \neq 0} \langle \rho \hat{x}_n \mid \hat{x}_n \rangle = \sum_n \mu(\text{at}(x_n)) \leq 1$; the left side of (M3) is therefore 0. If the entries are not mutually orthogonal, the right side of (M3) is 2, while the left side is at most 2. 😊

The proof of Theorem B rests on the fact that, for a density operator ρ on a separable infinite-dimensional complex Hilbert space, the map $S \mapsto \text{Tr}(\rho P_S)$ on the closed subspaces of that space takes every value in $[0, 1]$, unlike a measure with only the values 0 and 1.

Lemma 8.4. *For every separable infinite-dimensional complex Hilbert space H and every density operator ρ on H , the set $\{\text{Tr}(\rho P_S) : S \in L(H)\}$, where P_S is the orthogonal projection onto S , is the interval $[0, 1]$.*

Proof. The values 0 and 1 are attained at $\{0\}$ and H , respectively. Fix α with $0 < \alpha < 1$. We rotate one vector of a subspace spanned by eigenvectors of ρ continuously, from a subspace whose value under $S \mapsto \text{Tr}(\rho P_S)$ is at least α to one whose value is smaller, and apply the intermediate value theorem. Since ρ is a positive compact operator of trace 1, there are orthonormal eigenvectors $v_1, v_2, \dots$ (finitely or infinitely many) whose eigenvalues $\eta_1 \geq \eta_2 \geq \dots > 0$ are the nonzero eigenvalues of ρ , repeated according to multiplicity, with $\sum_i \eta_i = 1$. Write $\eta_{\leq k} = \eta_1 + \dots + \eta_k$, with $\eta_{\leq 0} = 0$, and choose an integer $k \geq 0$ with $\eta_{\leq k} < \alpha \leq \eta_{\leq k+1}$. On the unit vectors of $W = \{v_1, \dots, v_{k+1}\}^\perp$, the quantity $\langle \rho x \mid x \rangle$ has infimum 0: either W meets the kernel of ρ in a nonzero vector, or the eigenvalues η_i with $i > k+1$ tend to 0. Choose a unit vector $u \in W$ with $\langle \rho u \mid u \rangle < \alpha - \eta_{\leq k}$. For $0 \leq \vartheta \leq 1$, let w_ϑ be the unit vector $\sqrt{1-\vartheta} v_{k+1} + \sqrt{\vartheta} u$, and let S_ϑ be the subspace spanned by $v_1, \dots, v_k$ and w_ϑ . Since $\rho v_{k+1} = \eta_{k+1} v_{k+1}$ and the vector u is orthogonal to $v_1, \dots, v_{k+1}$, the cross terms vanish, and

$$\text{Tr}(\rho P_{S_\vartheta}) = \eta_{\leq k} + (1-\vartheta)\eta_{k+1} + \vartheta\langle \rho u \mid u \rangle,$$

a continuous function of ϑ that equals $\eta_{\leq k+1} \geq \alpha$ at $\vartheta = 0$ and is smaller than α at $\vartheta = 1$. By the intermediate value theorem, it takes the value α . 

Proof of Theorem B. Let $\mathfrak{M}$ be a standard model of T , and let H be the Hilbert space whose closed unit ball is its Hilbert sort. By Theorem A, there is a density operator ρ on H with $\mu(a) = \text{Tr}(\rho P_a)$ for every element a of the Solèr sort, and the map $a \mapsto S_a$ is onto $L(H)$; by Lemma 8.4, the range of μ is $[0, 1]$. Since B meets $[0, 1]$, some element a of the Solèr sort satisfies $\mu(a) \in B$, and the value of $\beta_B(a)$ at that element is 1. Hence the axiom $\beta_B(a) \leq -1$ of Theorem B fails in $\mathfrak{M}$. 

 **Historical note.** Sierpiński proved in 1922 that a measure without atoms takes, on the measurable subsets of a set, every value between 0 and the measure of that set [Sie22]. A $\{0, 1\}$ -valued measure, which is what an assignment of truth values amounts to, is the opposite extreme: the whole space is an atom. The measures $S \mapsto \text{Tr}(\rho P_S)$ that density operators ρ define on the closed subspaces of a separable infinite-dimensional Hilbert space, by contrast, take every value in $[0, 1]$ (see Lemma 8.4), even when ρ is a pure state, the orthogonal projection onto a ray, whose classical counterpart would be a point mass.

8.1. Definable sets of values. The formulas $\beta_B(a)$ constructed in the proof of Lemma 5.1 use the connectives of the logic but none of its quantifiers. With quantifiers over the Solèr sort, formulas define more sets of values in the standard models, because the measure takes every value in $[0, 1]$ there; the sets they define are exactly the σ -projective subsets of $[0, 1]$ (see Theorem 8.6), and the conclusion of Theorem B extends to all of them.

Aguilera, Müller and Schlicht [AMS21] study the σ -projective sets, under this name, as subsets of the finite powers of $\mathbb{N}^\mathbb{N}$. Let Γ assign to each Polish space X a class $\Gamma(X)$ of subsets of X , the least such assignment for which $\Gamma(X)$ contains the Borel subsets of X and is closed under complements in X and countable unions, and for which, whenever $A \in \Gamma(X \times \mathbb{N}^\mathbb{N})$, the set $\{x \in X : (x, z) \in A \text{ for some } z \in \mathbb{N}^\mathbb{N}\}$ belongs to $\Gamma(X)$. A subset of a Polish space X is σ -projective when it belongs to $\Gamma(X)$. On the finite powers of $\mathbb{N}^\mathbb{N}$, these are the sets of Aguilera, Müller and Schlicht: their class is closed under the operations that define Γ , and, conversely, the sets $A \subseteq X$ such that $g^{-1}[A]$ belongs to their class for every k and every Borel map $g: (\mathbb{N}^\mathbb{N})^k \rightarrow X$ include the Borel sets and are closed under complements, countable unions and projections along $\mathbb{N}^\mathbb{N}$. Every projective set [Kec95, Section 37] is σ -projective, and so is every countable union of projective sets. The σ -projective sets are closed under preimages by Borel maps between Polish spaces: the sets $A \subseteq X$ such that $g^{-1}[A]$ is

σ -projective for every Borel map g from a Polish space into X include the Borel sets and are closed under complements, countable unions and projections along $\mathbb{N}^{\mathbb{N}}$, since $(g \times \text{id})^{-1}$ commutes with the projection; they therefore include every σ -projective set. They are also closed under projections along every nonempty Polish space Y : if $f: \mathbb{N}^{\mathbb{N}} \rightarrow Y$ is a continuous surjection [Kec95, Theorem 7.9] and $A \subseteq X \times Y$, the projection of A to X is the projection along $\mathbb{N}^{\mathbb{N}}$ of the preimage of A under $\text{id} \times f$. A function f from a Polish space to $\mathbb{R}$ is Γ -measurable when the set $\{f > r\}$ is σ -projective for every real number r ; the set $\{f \geq r\}$, the intersection of the sets $\{f > r - \frac{1}{n}\}$, is then σ -projective as well.

Definition 8.5. A set $D \subseteq [0, 1]$ is *definable over T* if there is a formula $\gamma(a)$ of real-valued $L_{\omega_1, \omega}$, with the single free variable a of the Solèr sort, whose value, in every standard model of T and at every element a of its Solèr sort, is 1 if $\mu(a) \in D$ and -1 otherwise; we say that γ *defines* D . Since (M1) confines μ to $[0, 1]$ in every model of T , a formula whose value is 1 exactly when $\mu(a)$ lies in a set $E \subseteq \mathbb{R}$ defines $E \cap [0, 1]$; only subsets of $[0, 1]$ need be considered.

Theorem 8.6. *A subset of $[0, 1]$ is definable over T if and only if it is σ -projective. Consequently, if $D \subseteq [0, 1]$ is a nonempty σ -projective set and $\gamma(a)$ defines D , then the theory T together with the axiom $\gamma(a) \leq -1$, which says that no value of μ lies in D , has no standard models.*

The class of σ -projective subsets of $[0, 1]$ is strictly larger than that of projective ones. For each $n \geq 1$, let $C_n \subseteq [0, 1]$ be a Σ_{n+1}^1 set that is not Σ_n^1 , which exists by the hierarchy theorem for projective sets [Kec95, Section 37], and let h_n be a homeomorphism of $[0, 1]$ onto $[2^{-n}, 2^{-n+1}]$. The set $C = \bigcup_n h_n[C_n \cap (0, 1)]$ is a countable union of projective sets, and it contains none of the points 2^{-m} . If C were Σ_k^1 , then, for $n > k$, the set $h_n^{-1}[C]$, which equals $C_n \cap (0, 1)$, would be Σ_k^1 , and so would C_n , which differs from it by at most two points; hence C_n would be Σ_n^1 , which is false. If C were Π_k^1 , it would be Σ_{k+1}^1 , which is excluded in the same way. Hence C is not projective, and by Theorem 8.6 it is definable over T .

Proof of Theorem 8.6. Every σ -projective subset of $[0, 1]$ is definable. For $m \geq 1$, a formula $\gamma(a_1, \dots, a_m)$, with free variables $a_1, \dots, a_m$ of the Solèr sort, *strongly defines* a set $A \subseteq [0, 1]^m$ if it is two-valued in every structure for $\mathcal{L}$ and, in every standard model of T , takes the value 1 at $(a_1, \dots, a_m)$ exactly when $(\mu(a_1), \dots, \mu(a_m)) \in A$. Let $\mathcal{A}$ be the class of sets $A \subseteq [0, 1]^m$, for $m \geq 1$, that some formula strongly defines. We show that $\mathcal{A}$ contains every σ -projective subset of every cube $[0, 1]^m$; a formula that strongly defines a set $D \subseteq [0, 1]$ defines it in the sense of Definition 8.5.

First, $\mathcal{A}$ contains the Borel sets. The proof of Lemma 5.1, with the closed boxes $[q_1, q'_1] \times \dots \times [q_m, q'_m]$, for rational numbers $q_i \leq q'_i$, in place of the closed intervals with rational endpoints, and with the distance to such a box written as the maximum

$$\text{dist}(\mu(a_1), [q_1, q'_1]) \vee \dots \vee \text{dist}(\mu(a_m), [q_m, q'_m])$$

of the distances of the coordinates, by means of the connectives $\text{dist}(\cdot, [q, q'])$ of that proof, gives, for every Borel set $B \subseteq \mathbb{R}^m$, a formula $\beta_B(a_1, \dots, a_m)$ whose value at every tuple of elements of the Solèr sort of every structure for $\mathcal{L}$ is 1 if $(\mu(a_1), \dots, \mu(a_m)) \in B$ and -1 otherwise; it strongly defines $B \cap [0, 1]^m$. Next, if γ strongly defines $A \subseteq [0, 1]^m$, then $-\gamma$ strongly defines $[0, 1]^m \setminus A$; and if γ_n strongly defines $A_n \subseteq [0, 1]^m$ for every $n \in \mathbb{N}$, then the join $\bigvee_n \gamma_n$, whose members take their values in $[-1, 1]$, strongly defines $\bigcup_n A_n$. Finally, if $\gamma(a_1, \dots, a_m, b)$ strongly defines $A \subseteq [0, 1]^{m+1}$, then $\sup_b \gamma(a_1, \dots, a_m, b)$ is two-valued in every structure, since the Solèr sort contains the constant 0 and a supremum of values in $\{-1, 1\}$ over a nonempty set is attained; and in a standard model of T it takes the value

1 at $(a_1, \dots, a_m)$ exactly when $(\mu(a_1), \dots, \mu(a_m), \mu(b)) \in A$ for some element b , that is, by Theorem B, exactly when $(\mu(a_1), \dots, \mu(a_m), t) \in A$ for some $t \in [0, 1]$. Thus $\mathcal{A}$ contains the Borel subsets of the cubes $[0, 1]^m$ and is closed under complements, countable unions and projections along $[0, 1]$.

It remains to see that these operations produce every σ -projective subset of every cube $[0, 1]^m$. Let Γ_0 be the least class of subsets of the cubes $[0, 1]^m$, for $m \geq 1$, that contains their Borel subsets and is closed under complements in the cube, countable unions and projections from $[0, 1]^{m+1}$ to $[0, 1]^m$; we have shown that $\Gamma_0 \subseteq \mathcal{A}$. Fix a Borel isomorphism θ of $[0, 1]$ onto $\mathbb{N}^{\mathbb{N}}$ [Kec95, Theorem 15.6], and let $\Gamma'(X)$, for a Polish space X , be the class of sets $A \subseteq X$ such that $g^{-1}[A] \in \Gamma_0$ for every $m \geq 1$ and every Borel map $g: [0, 1]^m \rightarrow X$. Each class $\Gamma'(X)$ contains the Borel subsets of X and is closed under complements in X and countable unions. If $A \in \Gamma'(X \times \mathbb{N}^{\mathbb{N}})$ and $g: [0, 1]^m \rightarrow X$ is Borel, then, since θ is onto $\mathbb{N}^{\mathbb{N}}$,

$$g^{-1}[\{x : \exists z \in \mathbb{N}^{\mathbb{N}} (x, z) \in A\}] = \{u \in [0, 1]^m : \exists t \in [0, 1] (u, t) \in (g \times \theta)^{-1}[A]\},$$

the projection along $[0, 1]$ of a set of Γ_0 , because $g \times \theta$ is a Borel map on $[0, 1]^{m+1}$; it belongs to Γ_0 . Hence Γ' is closed under projections along $\mathbb{N}^{\mathbb{N}}$, and $\Gamma(X) \subseteq \Gamma'(X)$ for every Polish space X . For a σ -projective set $A \subseteq [0, 1]^m$, taking for g the identity shows that $A \in \Gamma_0$, and therefore $A \in \mathcal{A}$.

Every definable subset of $[0, 1]$ is σ -projective. Let $\gamma(a)$ define a set $D \subseteq [0, 1]$. Let $H = \ell^2(\mathbb{N})$, let v and w be orthogonal unit vectors of H , and let ρ be the projection onto $\mathbb{C}v$, a density operator. By Theorem A(2) and its proof, there is a standard model $\mathfrak{M}$ of T whose Hilbert sort is the closed unit ball B_H of H , whose sequence sort is the closed unit ball B_K of $K = \ell^2(\mathbb{N}, H)$, whose Solèr sort is $L(H)$, with the symbols interpreted as in that proof, the constants e_n being the standard unit vectors of $\ell^2(\mathbb{N})$, and with $\mu(S) = \text{Tr}(\rho P_S)$; in that proof, $\text{rep}(S)$ may be any unit vector of S when $S \neq \{0\}$, and we let it be $P_S e_n / \|P_S e_n\|$ for the least n with $P_S e_n \neq 0$. We identify $L(H)$ with the set of orthogonal projections on H , with the strong operator topology. For a dense sequence $(x_k)_{k \in \mathbb{N}}$ in B_H , the map $Q \mapsto (Qx_k)_{k \in \mathbb{N}}$ is a homeomorphism of the set of operators of norm at most 1, with the strong operator topology, onto its image in the Polish space $B_H^{\mathbb{N}}$, since for operators of norm at most 1 convergence at the vectors x_k implies convergence at every vector. The image is closed: if $(y_k)_{k \in \mathbb{N}}$ is the limit of sequences $(Q_j x_k)_{k \in \mathbb{N}}$ in the image, then $\|\sum_k c_k y_k\| \leq \|\sum_k c_k x_k\|$ for every finitely supported family of complex numbers c_k , so that $x_k \mapsto y_k$ extends to an operator of norm at most 1. The projections form a closed subset, since a strong limit Q of projections satisfies $\langle Qx \mid x \rangle = \|Qx\|^2 = \langle Q^* Qx \mid x \rangle$ for every x , whence $Q = Q^* Q$ by polarization, and so $Q = Q^* = Q^2$. Thus $L(H)$ is Polish, and so are B_H and B_K with their norm topologies.

Every symbol of $\mathcal{L}$ is interpreted in $\mathfrak{M}$ by a Borel function on a product of these spaces. The vector operations, the inner products, the norms, ι_n , κ_n , the constants and the orthocomplement $Q \mapsto I - Q$ are continuous; so is $(Q, x) \mapsto Qx$, the interpretation of π , since $\|Qx - Q_0 x_0\| \leq \|x - x_0\| + \|(Q - Q_0)x_0\|$; and so is $\mu(Q) = \text{Tr}(\rho Q) = \langle Qv \mid v \rangle$. The meet $Q \wedge Q'$ is the strong limit of $(QQ'Q)^n$ as $n \rightarrow \infty$, since $(QQ'Q)^n = (QQ')^n Q$ and by Theorem 7.2; it is a pointwise limit of continuous maps, hence Borel, and $Q \vee Q' = I - ((I - Q) \wedge (I - Q'))$. The predicate e is 1 on the diagonal, a closed set, and -1 off it. The map at sends $x \neq 0$ to the projection $y \mapsto \langle y \mid x \rangle x / \|x\|^2$, continuously, and 0 to 0. For rep , the set of projections Q with $Qe_k = 0$ for $k < n$ and $Qe_n \neq 0$ is Borel, and on it $\text{rep}(Q) = Qe_n / \|Qe_n\|$ is continuous. For span , let $R_N(s)$ be the projection onto the span of the entries $\kappa_0 s, \dots, \kappa_{N-1} s$ of s . On each of the finitely many sets on which the set of indices $i < N$ such that $\kappa_i s$ is not in the span of $\kappa_0 s, \dots, \kappa_{i-1} s$ is constant, the map R_N is continuous, being given by Gram–Schmidt orthonormalization of the corresponding entries; these sets are Borel, since the rank of the

Gram matrix of $(\kappa_0 s, \dots, \kappa_i s)$ is a Borel function of s , the set where it is at least r being open. The projections $R_N(s)$ increase with N to the projection $\text{span}(s)$ onto the closed span of the entries of s , strongly, and span is Borel. It follows by induction on terms that every term is interpreted by a Borel function.

We show, by induction on formulas, that for every formula $\varphi(\bar{x})$ the function $\bar{x} \mapsto \varphi^{\mathfrak{M}}(\bar{x})$, on the product of the Polish spaces that interpret the sorts of $\bar{x}$, is Γ -measurable. Atomic formulas are Borel functions of terms, hence Borel. For a continuous connective $f(\varphi_1, \dots, \varphi_\ell)$, the set $\{f(\varphi_1, \dots, \varphi_\ell) > r\}$ is the preimage under $(\varphi_1, \dots, \varphi_\ell)$ of an open subset of $\mathbb{R}^\ell$, a countable union of open boxes with rational corners, and the preimage of each box is a finite intersection of sets $\{\varphi_i > q\}$ and complements of sets $\{\varphi_i \geq q'\}$; the connectives $s \cdot \varphi$ and $\varphi \vee \psi$ are of this kind. For a countable join, $\{\bigvee_n \varphi_n > r\} = \bigcup_n \{\varphi_n > r\}$. For a supremum over a variable y , the set $\{\bar{x} : \sup_y \varphi(\bar{x}, y) > r\}$ is the projection of $\{(\bar{x}, y) : \varphi(\bar{x}, y) > r\}$ along the Polish space that interprets the sort of y . In each case the set is σ -projective.

Now $\gamma^{\mathfrak{M}}$ is Γ -measurable, and $\{Q \in L(H) : \gamma^{\mathfrak{M}}(Q) > 0\}$ is σ -projective. For $t \in [0, 1]$, let $\xi(t) = \text{at}(\sqrt{t}v + \sqrt{1-t}w)$, the projection onto the line spanned by the unit vector $\sqrt{t}v + \sqrt{1-t}w$; the map ξ is continuous, and $\mu(\xi(t)) = |\langle v \mid \sqrt{t}v + \sqrt{1-t}w \rangle|^2 = t$. Since $\gamma^{\mathfrak{M}}$ takes the value 1 exactly at the elements Q with $\mu(Q) \in D$,

$$D = \{t \in [0, 1] : \gamma^{\mathfrak{M}}(\xi(t)) > 0\} = \xi^{-1}[\{\gamma^{\mathfrak{M}} > 0\}],$$

a continuous preimage of a σ -projective set, hence σ -projective.

The consequence follows as in the proof of Theorem B, with γ in place of β_B : in a standard model, some element a has $\mu(a) \in D$, and $\gamma(a)$ takes the value 1 there. 

The *quantifier depth* $\text{qd}(\varphi)$ of a formula φ , a countable ordinal, is defined by induction: $\text{qd}(\varphi) = 0$ for atomic φ ; the depth of $f(\varphi_1, \dots, \varphi_\ell)$, of $s \cdot \varphi$ and of $\varphi \vee \psi$ is the largest depth of the formulas to which the connective is applied; $\text{qd}(\bigvee_n \varphi_n) = \sup_n \text{qd}(\varphi_n)$; and $\text{qd}(\sup_x \varphi) = \text{qd}(\varphi) + 1$. In the model $\mathfrak{M}$ of the proof, the induction on formulas given there shows that, for every $k < \omega$ and every formula φ with $\text{qd}(\varphi) \leq k$, each set $\{\varphi^{\mathfrak{M}} > r\}$ is Δ_{k+1}^1 . For $k = 0$, the formula φ contains no supremum over a variable, and these sets are Borel. For $k \geq 1$, if $\text{qd}(\sup_y \varphi) \leq k$, then $\text{qd}(\varphi) \leq k - 1$, and the set $\{(\sup_y \varphi)^{\mathfrak{M}} > r\}$ is the projection of a Δ_k^1 set, hence Σ_k^1 and so Δ_{k+1}^1 ; the other cases use only countable unions, countable intersections and complements, under which Δ_{k+1}^1 is closed. If a formula γ with $\text{qd}(\gamma) \leq k$ defines D , then $D = \xi^{-1}[\{\gamma^{\mathfrak{M}} > 0\}]$ is therefore Δ_{k+1}^1 . Since the set C above is not projective, every formula that defines it has quantifier depth at least ω ; since the subformula relation is well founded, such a formula has a subformula of depth at least ω whose proper subformulas all have finite depth, and that subformula is a countable join whose members have unbounded finite depths.

The definition asks for one formula that defines D in every standard model, whatever orthonormal basis interprets the constants e_n , whatever density operator gives the measure and whatever unit vectors interpret rep . The requirement cannot be weakened to a single standard model. Write $\langle \varphi \rangle$ for the two-valued formula $2[\varphi] - \mathbf{1}$, which takes the value 1 where $\varphi > 0$ and -1 elsewhere; write $z_i(b)$ for the pair of predicates $\langle \text{rep}(b) \mid e_i \rangle$; let $p(b)$ and $q(b)$ be the real and imaginary parts of $z_1(b)\overline{z_0(b)}$, written with the sum and product connectives; and let $\delta(a, b)$ be the formula $\beta_B(a, b)$ of the proof for the diagonal $B = \{(u, u) : u \in \mathbb{R}\}$. Let

$\gamma_0(a)$ be the formula

$$\sup_b \left(-e(b, 0) \wedge \text{nonatom}(b) \wedge e(b \wedge (\text{at}(e_0) \vee \text{at}(e_1)), b) \wedge \delta(a, b) \right. \\ \left. \wedge \langle p(b) \rangle \wedge -\langle |q(b)| \rangle \wedge \langle \text{Re}(\text{rep}(b) \mid e_0) \rangle \right).$$

Given $X \subseteq (0, 1)$, take the model of the proof of Theorem A(2) on $\ell^2(\mathbb{N})$, with the standard unit vectors as the constants e_n , with ρ the projection onto $\mathbb{C}e_0$, and with rep sending the atom spanned by $\sqrt{t}e_0 + \sqrt{1-t}e_1$, for $0 < t < 1$, to $\sqrt{t}e_0 + \sqrt{1-t}e_1$ if $t \in X$ and to its negative otherwise. For $0 < t < 1$, the only atom b below $\text{at}(e_0) \vee \text{at}(e_1)$ with $\mu(b) = t$, $p(b) > 0$ and $q(b) = 0$ is this one, and for $t \in \{0, 1\}$ there is none; hence $\gamma_0(a)$ takes the value 1 in that model exactly when $\mu(a) \in X$, and -1 otherwise. For the model $\mathfrak{M}$ of the proof, whose symbols are Borel, the second half of the proof shows that a set D such that some formula takes, at every element Q of the Solèr sort of $\mathfrak{M}$, the value 1 if $\mu(Q) \in D$ and -1 otherwise, is σ -projective, and hence definable over T .

In a model that is not standard, a formula that defines a set D over T need not take the value 1 exactly when $\mu(a) \in D$. By Remark 11.2, the theory T has a countable model $\mathfrak{N}$; the range of μ in $\mathfrak{N}$ is countable, and some $c \in [0, 1]$ is not a value of μ in $\mathfrak{N}$. By Theorem B, the formula $\sup_b \beta_{\mathbb{R} \times \{c\}}(a, b)$, with $\beta_{\mathbb{R} \times \{c\}}$ as in the proof of Theorem 8.6, defines $[0, 1]$ over T , yet it takes the value -1 everywhere in $\mathfrak{N}$. The formulas $\beta_B(a)$ constructed in the proof of Lemma 5.1 have no quantifiers, and in every structure they take the value 1 exactly when $\mu(a) \in B$; the cases of the remark on standard models in Section 10, which rest on facts about all models of T , depend on this, and for this reason that remark is stated for them, whereas Theorem 8.6 is a theorem about the standard models only.

9. THE PROOF OF THEOREM C

In this section we prove Theorem C: in no model of T , standard or not, does the measure μ take only the values 0 and 1. Since the model need not be standard, the conclusions of Theorem A are not available; besides the constants e_n , which are orthonormal by (H2), and the constant ε , whose entries are the vectors $2^{-n-1}e_n$ by (K4), the proof uses only the eighteen vectors of Table 1, which are built from four of the vectors e_n .

We argue by contradiction, and fix throughout this section a model $\mathfrak{M}$ of T in which μ takes only the values 0 and 1. Let $\bar{E}$ be a complex Hilbert space as in Lemma 6.3: the Hilbert sort of $\mathfrak{M}$ is identified with a subset Y of its closed unit ball whose linear span with coefficients in $\mathbb{Q}(i)$, the field of complex numbers with rational real and imaginary parts, is dense in $\bar{E}$. For each element a of the Solèr sort, let $\bar{P}_a$ be the orthogonal projection of $\bar{E}$ that extends $\pi(a, \cdot)$. By (H2), the vectors e_n form an orthonormal basis of $\bar{E}$: Parseval's identity puts every vector of Y , and hence every vector of $\bar{E}$, in their closed span. The proof rests on three facts that hold in every model of T .

Lemma 9.1. *Let $\mathfrak{M}$ be a model of T , standard or not, and let $\bar{E}$ and $\bar{P}_a$ be as in Lemma 6.3. Then the following hold:*

- (1) *For every nonzero x in the Hilbert sort and every $\lambda \in \mathbb{Q}(i)$ with $0 < |\lambda| \leq 1$, we have $\text{at}(\lambda x) = \text{at}(x)$, and the range of $\bar{P}_{\text{at}(x)}$ is $\mathbb{C}x$.*
- (2) *Let s be an element of the sequence sort, and let F be the closed span of its entries in $\bar{E}$. If every e_n lies in F or in $F^\perp$, then the range of $\bar{P}_{\text{span}(s)}$ is F .*
- (3) *The measure satisfies $\mu(0) = 0$, and $\text{span}(\varepsilon) = 1$.*

Proof. (1) By (At1) and continuity, the range of $\bar{P}_{\text{at}(x)}$ contains x , and any two of its vectors are linearly dependent; hence it is $\mathbb{C}x$, and likewise for λx . The two projections coincide, and axiom (P4) gives $\text{at}(\lambda x) = \text{at}(x)$.

(2) By (Sp1) and linearity, the projection $\bar{P}_{\text{span}(s)}$ fixes every entry of s ; hence its range contains F . If $e_n \in F^\perp$, then the vector e_n is orthogonal to every entry, and axiom (Sp2) gives $\bar{P}_{\text{span}(s)}e_n = 0$. The vectors e_n that lie in $F^\perp$ span a dense subspace of $F^\perp$, because the e_n form an orthonormal basis and each lies in F or in $F^\perp$; by continuity, the projection $\bar{P}_{\text{span}(s)}$ vanishes on $F^\perp$, and its range is F .

(3) For $s = 0$, whose entries are 0 by (K2), every vector of the Hilbert sort is orthogonal to the entries, and axiom (Sp2) gives $\bar{P}_{\text{span}(0)} = 0$, which is $\bar{P}_0$ by (Sub0); axiom (P4) then gives $\text{span}(0) = 0$, and, since $\text{od}(0) = 0$, the remainder of Step 1 in the proof of Theorem A, which uses only (At5), (M3) and (M1), gives $\mu(0) = 0$. For $s = \varepsilon$, the entries are $2^{-n-1}e_n$ by (K4); the space F of part (2) is then $\bar{E}$, and part (2) gives $\bar{P}_{\text{span}(\varepsilon)} = \text{Id}$, which is $\bar{P}_1$ by (Sub0), whence $\text{span}(\varepsilon) = 1$ by (P4). 

basis	vectors (coordinates with respect to an orthonormal basis g_0, g_1, g_2, g_3)			
1	(0, 0, 0, 1)	(0, 0, 1, 0)	(1, 1, 0, 0)	(1, -1, 0, 0)
2	(0, 0, 0, 1)	(0, 1, 0, 0)	(1, 0, 1, 0)	(1, 0, -1, 0)
3	(1, -1, 1, -1)	(1, -1, -1, 1)	(1, 1, 0, 0)	(0, 0, 1, 1)
4	(1, -1, 1, -1)	(1, 1, 1, 1)	(1, 0, -1, 0)	(0, 1, 0, -1)
5	(0, 0, 1, 0)	(0, 1, 0, 0)	(1, 0, 0, 1)	(1, 0, 0, -1)
6	(1, -1, -1, 1)	(1, 1, 1, 1)	(1, 0, 0, -1)	(0, 1, -1, 0)
7	(1, 1, -1, 1)	(1, 1, 1, -1)	(1, -1, 0, 0)	(0, 0, 1, 1)
8	(1, 1, -1, 1)	(-1, 1, 1, 1)	(1, 0, 1, 0)	(0, 1, 0, -1)
9	(1, 1, 1, -1)	(-1, 1, 1, 1)	(1, 0, 0, 1)	(0, 1, -1, 0)

TABLE 1. The configuration of Cabello, Estebaranz and García-Alcaine [CEGA96]: nine orthogonal bases of a four-dimensional space, formed by eighteen vectors, each of which occurs in exactly two bases. In the proof of Theorem C, the vectors $g_0, \dots, g_3$ are four consecutive vectors $e_{4j}, \dots, e_{4j+3}$, for some $j \in \mathbb{N}$, of the orthonormal basis $(e_n)_{n \in \mathbb{N}}$.

Proof of Theorem C. We first show that the basis $(e_n)_{n \in \mathbb{N}}$ contains exactly one vector whose atom has measure 1, and then apply a parity argument to the bases of Table 1 within the block of four basis vectors that contains it. The entries of ε are mutually orthogonal, whence $\text{od}(\varepsilon) = 0$, and part (1) of Lemma 9.1 gives $\text{at}(2^{-n-1}e_n) = \text{at}(e_n)$; by (M2), (K4), (M3) and Lemma 9.1(3), we obtain

$$1 = \mu(1) = \mu(\text{span}(\varepsilon)) = \sum_n \mu(\text{at}(e_n)).$$

As μ takes only the values 0 and 1, there is exactly one $n_0 \in \mathbb{N}$ with $\mu(\text{at}(e_{n_0})) = 1$. Let $j \in \mathbb{N}$ satisfy $4j \leq n_0 < 4j + 4$, put $g_i = e_{4j+i}$ for $i = 0, 1, 2, 3$, and let F be the span of $g_0, \dots, g_3$.

The basis $g_0, \dots, g_3$ of F contains exactly one vector, e_{n_0} , whose atom has measure 1; we show that every orthogonal basis of F whose vectors have coordinates in $\{0, 1, -1\}$ with

respect to $g_0, \dots, g_3$ has the same property. Let $\mathcal{B}$ be such a basis, consisting of vectors $v_0, \dots, v_3$. Each vector $\frac{1}{4}v_i$ has coordinates whose absolute values sum to at most 1, and it is therefore denoted by a term built from the constants e_n by the functions $\lambda x + \lambda' y$. Hence the term $\frac{1}{2}(\frac{1}{2}\iota_0(\frac{1}{4}v_0) + \frac{1}{2}\iota_1(\frac{1}{4}v_1)) + \frac{1}{2}(\frac{1}{2}\iota_2(\frac{1}{4}v_2) + \frac{1}{2}\iota_3(\frac{1}{4}v_3))$ denotes an element $s_{\mathcal{B}}$ of the sequence sort. By (K2), which gives $\kappa_m \iota_n x = \delta_{mn} x$, the entries $\kappa_n s_{\mathcal{B}}$ are $\frac{1}{16}v_n$ for $n < 4$ and 0 for $n \geq 4$; they are mutually orthogonal, whence $\text{od}(s_{\mathcal{B}}) = 0$. Each e_n lies in F or in $F^\perp$; by Lemma 9.1(2), the projection $\bar{P}_{\text{span}(s_{\mathcal{B}})}$ is therefore the projection onto F . For a vector v with coordinates in $\{0, 1, -1\}$, we write $\text{at}(v)$ for $\text{at}(\frac{1}{16}v)$; by Lemma 9.1(1), this element does not change when the sign of v is changed. By (P4), all the elements $\text{span}(s_{\mathcal{B}})$ are equal to one element a_F of the Solèr sort, and axiom (M3), Lemma 9.1(1) and $\mu(0) = 0$ give

$$\mu(a_F) = \sum_{v \in \mathcal{B}} \mu(\text{at}(v)).$$

For the basis $g_0, \dots, g_3$ this sum contains $\mu(\text{at}(e_{n_0})) = 1$ and three zeros; hence $\mu(a_F) = 1$, and every such basis $\mathcal{B}$ contains exactly one vector v with $\mu(\text{at}(v)) = 1$.

The nine bases of Table 1 are orthogonal bases of F of this kind, and their eighteen vectors span pairwise distinct rays; we conclude by counting in two ways the pairs $(\mathcal{B}, v)$ in which $\mathcal{B}$ is one of the nine bases, v is a vector of $\mathcal{B}$ and $\mu(\text{at}(v)) = 1$. By the preceding paragraph, each basis contains exactly one such vector, and there are therefore exactly nine pairs. On the other hand, each of the eighteen vectors occurs in exactly two of the bases; each vector v with $\mu(\text{at}(v)) = 1$ thus belongs to exactly two pairs, and the number of pairs is even. This contradiction proves the theorem. 

In physical terms, the Bell–Kochen–Specker theorem, Theorem C, excludes the hidden-variable theories of the kind that Kochen and Specker considered. Such a theory assigns to every experimentally verifiable proposition about a quantum system a definite truth value, true or false, which a measure with only the values 0 and 1 would record as 1 or 0, respectively. In a model of T , where the propositions are represented by the elements of the Solèr sort, such a measure cannot serve as the interpretation of μ : by Theorem C, the axioms of T , σ -additivity among them, exclude it. Together with axiom (M1), the theorem yields the following corollary.

Corollary 9.2 (No hidden variables). *In every model of T , standard or not, some element a of the Solèr sort satisfies $0 < \mu(a) < 1$.*

Proof. By (M1), the measure takes values in $[0, 1]$, and by Theorem C it does not take only the values 0 and 1. 

 **Historical note.** Specker announced in 1960 that the closed subspaces of a three-dimensional Hilbert space cannot be mapped into a Boolean algebra in a way that respects the operations on orthogonal subspaces, and said of the proof only that it rests on an elementary geometric argument [Spe60]. Bell derived from Gleason’s theorem that no hidden-variable theory of the kind described above is possible, in a paper written in 1964 and published in 1966 [Bel66], and in 1967 Kochen and Specker gave a finite proof with 117 vectors in $\mathbb{R}^3$ [KS67]. The configurations then shrank: Peres found configurations of 33 vectors in $\mathbb{R}^3$ and of 24 in $\mathbb{R}^4$ in 1991 [Per91], Kernaghan one of 20 vectors in $\mathbb{R}^4$ in 1994 [Ker94], and Cabello, Estebaranz and García-Alcaine the eighteen vectors of Table 1 in 1996 [CEGA96]. Peres conjectured that no smaller configuration exists, and Xu, Chen and Gühne proved in 2020 that no Kochen–Specker set, in any dimension, has fewer than eighteen vectors [XCG20, BCG⁺22]. The titles

of the papers of Kernaghan and of Cabello, Estebarez and García-Alcaine call the result the Bell–Kochen–Specker theorem, a name that records Bell’s share and that Theorem C keeps.

10. THE MODEL EXISTENCE GAME AND FINITE OBSTRUCTIONS

By Theorem C, the measure μ takes a value other than 0 and 1 in every model of the theory T of Theorem A, standard or not; in other words, the theory obtained from T by adjoining the axiom of Theorem C has no models. In first-order logic, compactness would turn this fact into a finite obstruction, a finite set of axioms that already has no models, just as finitely many vectors suffice in the Kochen–Specker theorem [KS67]. Real-valued $L_{\omega_1, \omega}$ has a model existence theorem [ADI26]: every Henkin type, an assignment of values to sentences that satisfies the Henkin conditions recalled in Section 5, is the type of a canonical structure for the enlarged vocabulary, one in which every element is the value of a closed term. In this section we state that theorem as a game, from which the finite obstructions arise. The proof of the omitting types theorem in the same work, an application of the Baire category theorem in the space of Henkin types, also has a counterpart in the game (see Appendix A).

The game follows Väänänen’s model existence game for two-valued logic [Vää11]. It is played by Olivia, who claims that a given countable theory has no models, and Josh, who claims that it has one. At each round Olivia makes a challenge, and Josh answers it by adding to a growing description of a structure (see Figure 5); Olivia wins as soon as Josh cannot answer without contradicting himself, and Josh wins if that never happens.

We show that the theory has no models exactly when Olivia has a winning strategy, and that every winning strategy of Olivia determines a well-founded tree of finite plays, at each leaf of which the commitments made along the play form a finite obstruction: a finite set of statements about the values of sentences that no structure satisfies. The tree branches infinitely only where Olivia challenges a countable supremum or infimum; once Josh’s answers to those challenges are fixed, every play ends within a bounded number of rounds. The first three subsections depend only on Section 5. In the last, we return to the theory of Theorem C and describe a winning strategy of Olivia, modeled on the proof of that theorem, under which every play in which Josh makes all the commitments that the parity argument of that proof requires ends at a position whose commitments concern a configuration of the Kochen–Specker type.

10.1. The setting. Until the last subsection, we work with a countable many-sorted vocabulary $\mathcal{L}$ for the logic of Section 5 and a nonempty countable theory T in $\mathcal{L}$; in the last subsection, the vocabulary and the theory are again those of Section 5.2 and of Theorem A. As usual, every sort of a structure is nonempty.

In the game, Josh commits the values of sentences to intervals, and the rules require, for each sentence, an interval fixed in advance that contains all its possible values. By the convention of Section 5, the a priori bound r_P of each predicate symbol P is a positive integer, and each countable supremum $\bigvee_n \varphi_n$ or infimum $\bigwedge_n \varphi_n$ comes, as part of its syntax, with a positive rational number r such that every φ_n takes its values in $[-r, r]$ in every structure. By induction on formulas, each formula φ receives a nonnegative rational number r_φ such that φ takes its values in the closed interval $I_\varphi = [-r_\varphi, r_\varphi]$ in every structure; this number does not change when terms are substituted for the free variables of φ . The bounds of $P(\bar{t})$, $s \cdot \varphi$, $\varphi \vee \psi$ and $\sup_x \varphi$ are those of Aubone, Dueñez and Iovino [ADI26], namely r_P , $|s|r_\varphi$, $\max(r_\varphi, r_\psi)$ and r_φ ; the bound of $\bigvee_n \varphi_n$ is the number r of its family, as in Section 5, and, moreover, $r_{\inf_x \varphi} = r_\varphi$ and $r_{\bigwedge_n \varphi_n} = r$. For any other continuous connective f , the number $r_{f(\varphi_1, \dots, \varphi_\ell)}$ is the least positive integer that is at least the maximum of $|f|$ on the box $I_{\varphi_1} \times \dots \times I_{\varphi_\ell}$. In the

game, $\inf_x$ and $\bigwedge_n$ are treated as primitive, with rules of their own, whereas $s \cdot \varphi$ and $\varphi \vee \psi$ are handled by the rule (Cn) of Definition 10.2 as the connectives $t \mapsto st$ and $(t, u) \mapsto \max(t, u)$. Finally, the logical unity constant $\mathbf{1}$, a nullary predicate symbol, takes only the value 1; for the definition of infeasible positions below, we write $R_1 = \{1\}$, and $R_P = [-r_P, r_P]$ for every other predicate symbol P .

The axioms enter the game through one of its rules, which is simplest to state when all axioms have the same form. As everywhere in the paper, an axiom of T is a closed condition required of all values of its free variables; by the conventions of Section 5, an axiom $\varphi(\bar{x}) \leq r$ is the closed condition $\sup_{\bar{x}} \varphi \leq r$, an axiom $\varphi(\bar{x}) \geq r$ is $\inf_{\bar{x}} \varphi \geq r$, and an axiom $\varphi(\bar{x}) = r$ is the union of the two, where $\bar{x}$ lists the free variables of φ . Every axiom of T may therefore be regarded as a closed condition $\psi \leq r$ or $\psi \geq r$ for a sentence ψ and a rational number r , an axiom $\varphi(\bar{x}) = r$ counting as two. Theories that contain strict conditions are discussed in Remark 10.6.

To name the elements of a structure under construction, we add to $\mathcal{L}$, for each sort $\mathbf{S}$, a set $\mathcal{W}_{\mathbf{S}} = \{w_{\mathbf{S},i} : i \in \mathbb{N}\}$ of new constants of sort $\mathbf{S}$, the *witnesses*; we write $\mathcal{W}$ for the union of the sets $\mathcal{W}_{\mathbf{S}}$, and $\mathcal{L}_{\mathcal{W}}$ for the vocabulary obtained by adding the witnesses to $\mathcal{L}$. We use the letters w and w' for witnesses. Constant symbols of $\mathcal{L}$ are treated as function symbols with no arguments, so that the rules below handle them together with the other function symbols. A sentence of $\mathcal{L}_{\mathcal{W}}$ is *basic* when it has the form $P(w_1, \dots, w_q)$, where P is a predicate symbol and $w_1, \dots, w_q$ are witnesses; the number q may be 0, and the sentence $\mathbf{1}$, in particular, is basic.

In the course of a play, Josh builds, in answer to Olivia's challenges, a partial description of a structure whose elements are named by witnesses. Such a description consists of commitments, which say that the value of a sentence lies in a given interval, and records, which say what a function symbol does to the elements named by given witnesses. The rules of the game (see Definition 10.2), apart from those that bring in the axioms and make the function symbols total, follow the inductive definition of the value of a sentence: a commitment on a sentence formed by a continuous connective, or by a countable supremum or infimum, is challenged through the sentences from which it is formed; a commitment on a supremum or an infimum over a sort, through witnesses; and a commitment on an atomic sentence, through records, which replace its terms by witnesses until a basic sentence remains.

Definition 10.1. A *commitment* is an expression $\varphi \in I$, where φ is a sentence of $\mathcal{L}_{\mathcal{W}}$ and $I = [\alpha, \beta]$ is an interval with rational endpoints $\alpha \leq \beta$; the interval may be a single point. A *record* is an expression $F(\bar{w}) \mapsto w'$, where F is a function symbol and $\bar{w}, w'$ are witnesses of the appropriate sorts. A *position* is a finite set $\mathfrak{p}$ of commitments and records that contains at most one record $F(\bar{w}) \mapsto w'$ for each function symbol F and each tuple $\bar{w}$. A witness *occurs* in $\mathfrak{p}$ when it occurs in a commitment or a record of $\mathfrak{p}$. A position $\mathfrak{p}$ is *infeasible* when there is a basic sentence $P(\bar{w})$ such that the intervals I for which $\mathfrak{p}$ contains the commitment $P(\bar{w}) \in I$, together with R_P , have empty intersection.

An infeasible position is contradictory in the most elementary way: it assigns to a basic sentence $P(\bar{w})$ finitely many rational intervals that have no common point in R_P (see Figure 7).

The rule for continuous connectives asks Josh not for the exact values of the arguments of a connective but for small cells of a grid that contain them, fine enough that the connective varies by at most a prescribed amount on each cell. Its statement requires three notions: dyadic grids, a modulus of uniform continuity for each connective, and widened intervals. For $m \in \mathbb{N}$, the *dyadic grid* $\mathcal{G}_m$ is the set of intervals $[i2^{-m}, (i+1)2^{-m}]$, for $i \in \mathbb{Z}$. Let ℓ be a positive integer, let $f: \mathbb{R}^{\ell} \rightarrow \mathbb{R}$ be a continuous connective, let $\varphi_1, \dots, \varphi_{\ell}$ be sentences, and

let $k \in \mathbb{N}$ be given. The function f is uniformly continuous on the compact box $I_{\varphi_1} \times \cdots \times I_{\varphi_\ell}$; we write $m(f, \bar{\varphi}, k)$ for the least $m \in \mathbb{N}$ such that $|f(\xi) - f(\xi')| \leq 2^{-k}$ for all points ξ, ξ' of the box with $|\xi_j - \xi'_j| \leq 2^{-m}$ for $j = 1, \dots, \ell$. For an interval $I = [\alpha, \beta]$ and $k \in \mathbb{N}$, we write $I^{(k)} = [\alpha - 2^{-k}, \beta + 2^{-k}]$.

Definition 10.2 (The model existence game). The game $G(T)$ is played by Olivia and Josh. It starts at the empty position; at each round, Olivia makes a *challenge*, and Josh *answers* it by adding to the current position $\mathfrak{p}$ commitments and records, if any, in the manner that the rule of the challenge prescribes. When Josh answers a challenge, the *first fresh witness* of a sort $\mathbf{S}$ is the witness $w_{\mathbf{S},i}$ with the least index i that occurs neither in $\mathfrak{p}$ nor in the challenge. In each rule other than (Ax) and (Tot), the commitment named by Olivia belongs to $\mathfrak{p}$, and its interval is written $I = [\alpha, \beta]$.

- (Ax): Olivia names an axiom $\psi \leq r$ or $\psi \geq r$ of T . Josh adds $\psi \in [\min(-r_\psi, r), r]$ or $\psi \in [r, \max(r, r_\psi)]$, respectively.
- (Tot): Olivia names a function symbol F and witnesses $\bar{w}$ of the appropriate sorts. If $\mathfrak{p}$ contains no record $F(\bar{w}) \mapsto w'$, Josh adds such a record, choosing for w' a witness of the sort of the values of F that occurs in $\mathfrak{p}$ or is the first fresh witness of that sort; otherwise he adds nothing.
- (Tm): Olivia names a commitment $P(\bar{t}) \in I$, where P is a predicate symbol and $\bar{t}$ is a tuple of terms, together with a term $F(\bar{w})$ that occurs in $\bar{t}$ and in which $\bar{w}$ is a tuple of witnesses (possibly empty). Josh first answers as in (Tot) for F and $\bar{w}$; then, with w' the witness such that the enlarged position contains the record $F(\bar{w}) \mapsto w'$, he adds $P(\bar{t}') \in I$, where $\bar{t}'$ is obtained from $\bar{t}$ by replacing every occurrence of $F(\bar{w})$ by w' .
- (Cn): Olivia names a commitment $f(\varphi_1, \dots, \varphi_\ell) \in I$, where f is a continuous connective, and a number $k \in \mathbb{N}$. With $m = m(f, \bar{\varphi}, k)$, Josh chooses intervals $D_1, \dots, D_\ell$ of $\mathcal{G}_m$ such that each D_j meets I_{φ_j} and f maps $(D_1 \cap I_{\varphi_1}) \times \cdots \times (D_\ell \cap I_{\varphi_\ell})$ into $I^{(k)}$, and adds the commitments $\varphi_j \in D_j \cap I_{\varphi_j}$ (see Figure 6).
- (sup $\leq$): Olivia names a commitment $\sup_x \varphi(x) \in I$, where x is a variable of sort $\mathbf{S}$, and a witness $w \in \mathcal{W}_{\mathbf{S}}$. Josh adds $\varphi(w) \in [-r_\varphi, \beta]$.
- (sup $\geq$): Olivia names a commitment $\sup_x \varphi(x) \in I$, where x is a variable of sort $\mathbf{S}$, and a number $k \in \mathbb{N}$. Josh chooses a witness $w \in \mathcal{W}_{\mathbf{S}}$ that occurs in $\mathfrak{p}$ or is the first fresh witness of sort $\mathbf{S}$, and adds $\varphi(w) \in [\alpha - 2^{-k}, r_\varphi]$.
- (inf $\geq$): Olivia names a commitment $\inf_x \varphi(x) \in I$, where x is a variable of sort $\mathbf{S}$, and a witness $w \in \mathcal{W}_{\mathbf{S}}$. Josh adds $\varphi(w) \in [\alpha, r_\varphi]$.
- (inf $\leq$): Olivia names a commitment $\inf_x \varphi(x) \in I$, where x is a variable of sort $\mathbf{S}$, and a number $k \in \mathbb{N}$. Josh chooses a witness w as in (sup $\geq$), and adds $\varphi(w) \in [-r_\varphi, \beta + 2^{-k}]$.
- ($\bigvee \leq$): Olivia names a commitment $\bigvee_n \varphi_n \in I$, where $(\varphi_n)_{n \in \mathbb{N}}$ is a family of formulas with common rational bound $r = r_{\bigvee_n \varphi_n}$, and a number $n \in \mathbb{N}$. Josh adds $\varphi_n \in [-r, \beta]$.
- ($\bigvee \geq$): Olivia names a commitment $\bigvee_n \varphi_n \in I$ as in the preceding rule, and a number $k \in \mathbb{N}$. Josh chooses a number $n \in \mathbb{N}$ and adds $\varphi_n \in [\alpha - 2^{-k}, r]$.
- ($\bigwedge \geq$): Olivia names a commitment $\bigwedge_n \varphi_n \in I$, where $(\varphi_n)_{n \in \mathbb{N}}$ is a family of formulas with common rational bound $r = r_{\bigwedge_n \varphi_n}$, and a number $n \in \mathbb{N}$. Josh adds $\varphi_n \in [\alpha, r]$.
- ($\bigwedge \leq$): Olivia names a commitment $\bigwedge_n \varphi_n \in I$ as in the preceding rule, and a number $k \in \mathbb{N}$. Josh chooses a number $n \in \mathbb{N}$ and adds $\varphi_n \in [-r, \beta + 2^{-k}]$.

An answer is *legal* when it obeys the rule and every interval that it adds is nonempty, that is, has its left endpoint at most its right endpoint. Josh *loses* at a round if he has no legal answer to Olivia's challenge, or if his answer makes the position infeasible. A *play* is a finite or infinite sequence of rounds played according to these rules; it ends at the round, if any,

at which Josh loses, a round that consists of Olivia's challenge alone when he has no legal answer to it. Olivia wins a play if Josh loses at some round; Josh wins a play if he never loses, in which case the play is infinite. A *strategy* for a player chooses that player's next move from the moves made so far; it is *winning* when that player wins every play in which the strategy is followed, whatever the opponent does.

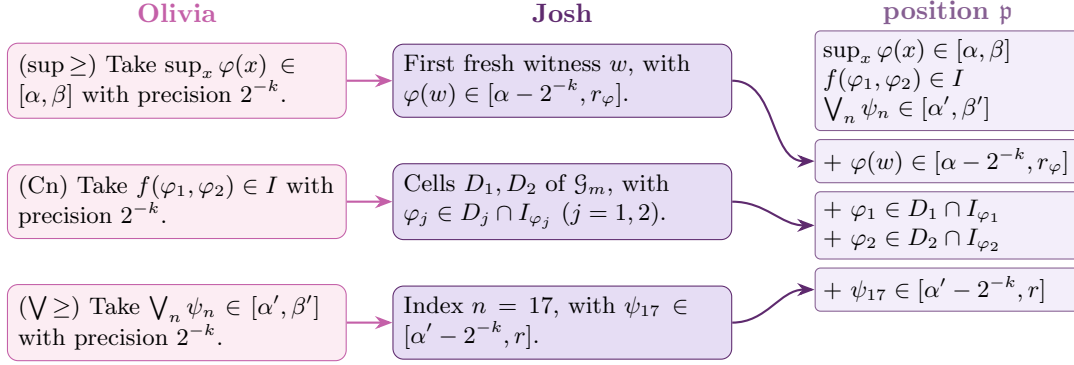


FIGURE 5. Three rounds of the game. Olivia's challenges are on the left and Josh's answers in the middle; the position, on the right, grows by one card at each answer. The first round concerns a supremum over a sort, the second a continuous connective and the third a countable supremum.

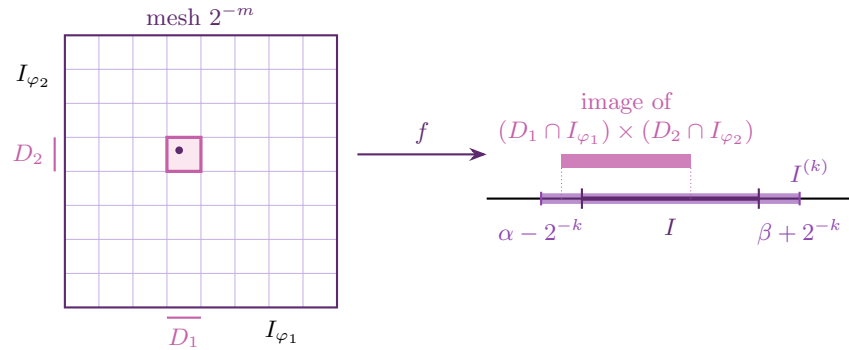


FIGURE 6. The rule (Cn) for a connective f of two variables. Josh must choose a cell $D_1 \times D_2$ of the dyadic grid of mesh 2^{-m} whose intersection with the box $I_{\varphi_1} \times I_{\varphi_2}$ is mapped by f into the widened interval $I^{(k)}$, where k is the number named by Olivia and $m = m(f, \bar{\varphi}, k)$ is the modulus of uniform continuity of f on that box defined before Definition 10.2. A cell that contains the point (the dot) whose coordinates are the values of φ_1 and φ_2 in a structure satisfying the commitment $f(\varphi_1, \varphi_2) \in I$ always qualifies, by uniform continuity, although its image may leave I itself.

Josh's answers are of three kinds, according to the number of choices that the rules leave him. The rules (Ax), (sup ≤), (inf ≥), (∨ ≤) and (∧ ≥) leave him no choice: the answer is determined by the challenge. Under (Tot), (Tm), (Cn), (sup ≥) and (inf ≤), he has finitely many choices, since only finitely many witnesses occur in a position and only finitely many intervals of $\mathcal{G}_m$ meet a bounded interval. Only the rules (∨ ≥) and (∧ ≤), which concern

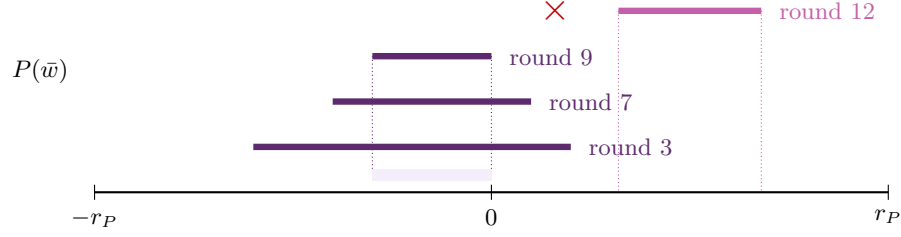


FIGURE 7. An infeasible position: at different rounds, Josh has committed the basic sentence $P(\bar{w})$ to intervals that have no common point.

countable suprema and infima, leave him a choice among infinitely many answers. We call the challenges made under these two rules *countable challenges*, and Josh's answers to them *countable answers*.

10.2. The model existence theorem as a game. To extract a model from a play that Josh survives, we need Olivia to have made, sooner or later, every challenge open to her. A challenge is *available* at a round when the rules allow Olivia to make it there, that is, when the commitment that it names, if any, belongs to the position at the start of that round; since positions only grow, a challenge that is available at a round remains available at every later round. Since a position is finite and the vocabulary $\mathcal{L}_{\mathcal{W}}$ and the theory T are countable, only countably many challenges are available at each round; there is at least one, since T is nonempty. Moreover, every sentence that occurs in a play is an atomic sentence of $\mathcal{L}_{\mathcal{W}}$ or is obtained from a subformula of an axiom of T by substituting closed terms of $\mathcal{L}_{\mathcal{W}}$ for its free variables, whence the challenges that can occur in plays form a countable set. A play is *fair* if every challenge that is available at some round is made by Olivia at that round or at a later one. Olivia has a *fair strategy*, one under which every infinite play is fair, whatever Josh does; we call the following one her *bookkeeping*. She fixes a bijection $i \mapsto (j(i), l(i))$ from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$ with $j(i) \leq i$ for every $i \in \mathbb{N}$, and at each round she enumerates, with repetitions if necessary, the challenges available at that round. At round i she makes the $l(i)$ -th challenge of her enumeration for round $j(i)$, which is still available because $j(i) \leq i$.

Theorem 10.3 (The model existence theorem as a game). *Let T be a nonempty countable theory of real-valued $L_{\omega_1, \omega}$ in a countable vocabulary, whose axioms are closed conditions $\psi \leq r$ or $\psi \geq r$ for sentences ψ and rational numbers r , and let $G(T)$ be the game of Definition 10.2. The following conditions are equivalent:*

- (1) *the theory T has a model;*
- (2) *Josh has a winning strategy in $G(T)$;*
- (3) *Olivia has no winning strategy in $G(T)$.*

Proof. (1) implies (2). Let $\mathfrak{N}$ be a model of T ; Josh answers every challenge truthfully in $\mathfrak{N}$. To do so, he keeps, along the play, a map h from the witnesses occurring in the current position to $\mathfrak{N}$ such that every commitment of the position holds in $\mathfrak{N}$ when each witness w is interpreted as $h(w)$, and $F^{\mathfrak{N}}(h(\bar{w})) = h(w')$ for every record $F(\bar{w}) \mapsto w'$ of the position. When a challenge names a witness that does not yet occur, Josh first extends h to it arbitrarily, which is possible because the sorts of $\mathfrak{N}$ are nonempty. The commitment that he adds under (Ax) holds because $\mathfrak{N}$ is a model of T . Under (Tot) and (Tm), if the position contains no record $F(\bar{w}) \mapsto w'$, Josh adds one, with w' the first fresh witness, and

puts $h(w') = F^{\mathfrak{M}}(h(\bar{w}))$; under (Tm), whether the record is new or not, the value of $P(\bar{t})$ in $\mathfrak{N}$ equals that of $P(\bar{t})$. To a challenge (Cn), Josh answers with cells $D_j \in \mathcal{G}_m$ that contain the values ξ_j of the sentences φ_j in $\mathfrak{N}$; every point of $(D_1 \cap I_{\varphi_1}) \times \cdots \times (D_\ell \cap I_{\varphi_\ell})$ then has coordinates within 2^{-m} of those of $(\xi_1, \dots, \xi_\ell)$, and the choice of m ensures that f maps it to within 2^{-k} of $f(\xi_1, \dots, \xi_\ell)$, which lies in I . The commitment added under $(\sup \leq)$ holds because the value of $\varphi(w)$ is at most the supremum, hence at most β ; under $(\sup \geq)$, Josh takes the first fresh witness w and, since the supremum is at least α , lets $h(w)$ be an element of $\mathfrak{N}$ at which φ takes a value at least $\alpha - 2^{-k}$. The remaining rules are handled in the same way; under $(\bigvee \geq)$, for instance, Josh names a number n for which the value of φ_n is at least $\alpha - 2^{-k}$. Every answer is legal, since the true values lie in the added intervals, and no position becomes infeasible, since the value in $\mathfrak{N}$ of a basic sentence $P(\bar{w})$ lies in every interval committed for it and in R_P . Thus Josh never loses.

(2) *implies* (1). Let Josh play a winning strategy against a fair strategy of Olivia. The resulting play is infinite and fair; let $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots$ be its positions, indexed by $i \in \mathbb{N}$, and $\mathfrak{p}_\infty$ their union. We define a structure $\mathfrak{M}$ for $\mathcal{L}_{\mathcal{W}}$ whose elements are the witnesses: the universe of sort $\mathbf{S}$ is $\mathcal{W}_{\mathbf{S}}$, and each witness is interpreted as itself. For a function symbol F and witnesses $\bar{w}$, Olivia makes the challenge (Tot) for F and $\bar{w}$ at some round, because the play is fair; the union $\mathfrak{p}_\infty$ therefore contains a record $F(\bar{w}) \mapsto w'$, which is unique because every $\mathfrak{p}_i$ is a position, and we put $F^{\mathfrak{M}}(\bar{w}) = w'$. For a basic sentence $P(\bar{w})$, the intervals I for which $\mathfrak{p}_\infty$ contains $P(\bar{w}) \in I$, together with R_P , are compact, and every finite subfamily has a common point because no position $\mathfrak{p}_i$ is infeasible; by the finite intersection property, the whole family has a common point, and we let $P^{\mathfrak{M}}(\bar{w})$ be one. As a point of R_P , this value respects the a priori bound of P , and it equals 1 when P is $\mathbf{1}$. Since no continuity is required of the interpretations $F^{\mathfrak{M}}$ and $P^{\mathfrak{M}}$ (see Section 5), these choices complete the definition of $\mathfrak{M}$.

We claim that every commitment $\varphi \in I$ of $\mathfrak{p}_\infty$ holds in $\mathfrak{M}$, and we prove it by induction on the construction of φ and, for atomic sentences, on the number of occurrences of function symbols. For a basic sentence, the claim holds by the choice of $P^{\mathfrak{M}}$. A non-basic atomic sentence $P(\bar{t})$ contains a term $F(\bar{w})$ whose arguments $\bar{w}$ are witnesses (possibly none, when F is a constant symbol); by fairness, Olivia made the challenge (Tm) for the commitment and this term, and Josh added $P(\bar{t}) \in I$, which holds by the induction hypothesis; the sentences $P(\bar{t})$ and $P(\bar{t})$ have the same value in $\mathfrak{M}$, because the witness w' that replaces $F(\bar{w})$ is $F^{\mathfrak{M}}(\bar{w})$. For $f(\varphi_1, \dots, \varphi_\ell) \in I$, fairness gives, for every $k \in \mathbb{N}$, intervals $D_1, \dots, D_\ell$ such that $\mathfrak{p}_\infty$ contains the commitments $\varphi_j \in D_j \cap I_{\varphi_j}$ and f maps $(D_1 \cap I_{\varphi_1}) \times \cdots \times (D_\ell \cap I_{\varphi_\ell})$ into $I^{(k)}$; by the induction hypothesis, the value of $f(\varphi_1, \dots, \varphi_\ell)$ in $\mathfrak{M}$ lies in $I^{(k)}$ for every k , hence in I . For $\sup_x \varphi(x) \in [\alpha, \beta]$, fairness puts in $\mathfrak{p}_\infty$ a commitment $\varphi(w) \in [-r_\varphi, \beta]$ for every witness w of the sort of x , that is, for every element of $\mathfrak{M}$ of that sort, and, for every $k \in \mathbb{N}$, a commitment $\varphi(w) \in [\alpha - 2^{-k}, r_\varphi]$ for some witness w ; by the induction hypothesis, the supremum lies in $[\alpha, \beta]$. The remaining cases are similar. Finally, every axiom $\psi \leq r$ of T was named in a challenge (Ax), to which Josh answered with the commitment $\psi \in [\min(-r_\psi, r), r]$; by the claim, the structure $\mathfrak{M}$ satisfies $\psi \leq r$, and the axioms $\psi \geq r$ are treated in the same way.

It remains to pass to a reduced structure. For witnesses w_1, w_2 of the same sort, let $w_1 \sim w_2$ when $\varphi(w_1, \bar{w})$ and $\varphi(w_2, \bar{w})$ have the same value in $\mathfrak{M}$ for every atomic formula $\varphi(x, \bar{y})$ of $\mathcal{L}$ and every tuple $\bar{w}$ of witnesses. If $w_1 \sim w_2$, then the witnesses $F^{\mathfrak{M}}(w_1, \bar{w}')$ and $F^{\mathfrak{M}}(w_2, \bar{w}')$ are equivalent for every function symbol F and every tuple $\bar{w}'$ of witnesses, since $\varphi(F(x, \bar{y}'), \bar{y})$ is again an atomic formula; the same holds at every argument place of F , and

by transitivity, the relation $\sim$ is a congruence for every function symbol, whether or not it is continuous. The quotient $\mathfrak{M}/\sim$ is reduced, and the quotient map is surjective and preserves the value of every atomic formula; by induction on formulas, it preserves the value of every formula. Hence the reduct of $\mathfrak{M}/\sim$ to $\mathcal{L}$ is a model of T .

(2) and (3) are equivalent. Regard $G(T)$ as an infinite game in which, after Josh has lost, the play continues with arbitrary moves. Olivia wins exactly when Josh loses at some finite round; in the language of descriptive set theory, the set of her wins is *open* in the space of infinite sequences of moves, whose basic open sets consist of the sequences that extend a given finite one. The Gale–Stewart theorem [GS53] says that an infinite game of perfect information in which the set of wins of one player is open is *determined*: one of the two players has a winning strategy. Since the two players cannot both have winning strategies, (2) and (3) are equivalent. 😊



Remark 10.4 (Baire category and the Banach–Mazur game). Olivia’s bookkeeping can be read as a Baire category argument in the space of plays, and as a winning strategy in a Banach–Mazur game on that space; we set out both readings in Appendix A, beside the Baire category argument of Aubone, Dueñez and Iovino [ADI26] in the space of Henkin types.

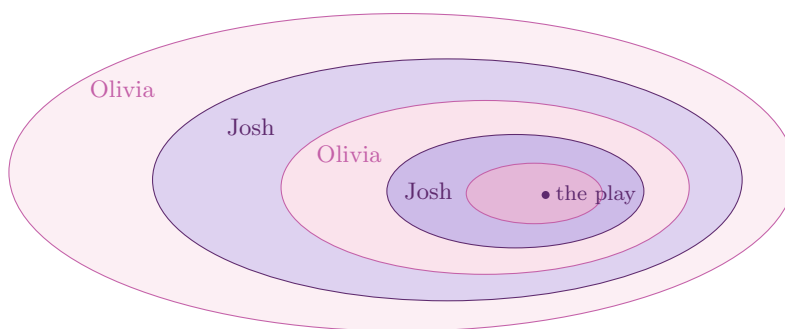


FIGURE 8. A play of the game as a sequence of nested basic open sets. Olivia (pink) and Josh (purple) move alternately, and each move, like a move of the Banach–Mazur game, shrinks the set of infinite plays that extend the current finite play; for an infinite play, the intersection of the nested sets consists of that play alone.

10.3. Finite obstructions. A winning strategy for Olivia is a certificate that a theory has no models, and its shape shows where finite obstructions suffice. Let σ be a strategy for Olivia. The *tree of σ* , denoted $\mathcal{P}_\sigma$, is the set of finite plays in which Olivia follows σ and Josh has not lost before the last round, ordered by extension. A finite play of $\mathcal{P}_\sigma$ in which Josh has not lost has as children its extensions by Olivia’s next challenge and a legal answer of Josh, and, when Josh has no legal answer to that challenge, its extension by the challenge alone. A *leaf* of $\mathcal{P}_\sigma$ is a finite play with no children, that is, one at which Josh has lost. The strategy σ is winning exactly when $\mathcal{P}_\sigma$ has no infinite branch, that is, when $\mathcal{P}_\sigma$ is *well-founded*. The *rank* of a well-founded tree is defined by recursion: a leaf has rank 0, a node that is not a leaf has as rank the least ordinal greater than the ranks of all its children, and the rank of the tree is the rank of its root. The tree $\mathcal{P}_\sigma$ is countably branching, hence countable; when it is well-founded, its rank is therefore a countable ordinal, and the rank is finite exactly when the plays in which Olivia follows σ have bounded length.

In Corollary 10.5 below, we show that $\mathcal{P}_\sigma$ branches infinitely only at nodes where σ makes a countable challenge; the following notions keep track of Josh's answers to such challenges. The *countable record* of a node of $\mathcal{P}_\sigma$ is the finite sequence of the numbers n that Josh names in his countable answers along the branch from the root to the node, in the order in which he names them. The *skeleton* of σ is the set of countable records of nodes of $\mathcal{P}_\sigma$, ordered by extension; it is a tree of finite sequences of natural numbers.

Corollary 10.5 (Finite obstructions). *Let T be a nonempty countable theory of real-valued $L_{\omega_1, \omega}$ in a countable vocabulary, whose axioms are closed conditions $\psi \leq r$ or $\psi \geq r$ for sentences ψ and rational numbers r , and suppose that T has no models. Then Olivia has a winning strategy in $G(T)$, and every winning strategy σ of Olivia has the following properties:*

- (1) *The tree $\mathcal{P}_\sigma$ is well-founded: every play in which Olivia follows σ ends after finitely many rounds, at a leaf.*
- (2) *At every leaf, the last position $\mathbf{p}$ is a finite set of commitments and records, involving finitely many axioms of T and finitely many witnesses, that is satisfied in no structure under any interpretation of the witnesses. Either $\mathbf{p}$ is infeasible, in which case a basic sentence $P(\bar{w})$ receives finitely many rational intervals with no common point in R_P , or Josh has no legal answer to Olivia's last challenge, which names a commitment of $\mathbf{p}$ that holds in no structure.*
- (3) *A node of $\mathcal{P}_\sigma$ has infinitely many children only if σ makes a countable challenge at it; every other node has finitely many children.*
- (4) *If every node of $\mathcal{P}_\sigma$ has finitely many children, as is the case when σ makes no countable challenge, and in particular when no axiom of T contains a countable supremum or infimum, then the tree $\mathcal{P}_\sigma$ is finite. In that case there is a number N_σ such that every play in which Olivia follows σ ends within N_σ rounds, and some finite set of axioms of T has no models.*
- (5) *The skeleton of σ is well-founded, and for every element $\vec{n}$ of the skeleton there is a number $N_\sigma(\vec{n})$ such that every play in which Olivia follows σ and whose countable record is $\vec{n}$ ends within $N_\sigma(\vec{n})$ rounds.*

Proof. By Theorem 10.3, Olivia has a winning strategy; let σ be any winning strategy of Olivia. Part (1) holds because σ is winning. For part (2), recall that a position contains finitely many commitments and records; the argument for “(1) implies (2)” in the proof of Theorem 10.3 shows that a position satisfied in some structure, under some interpretation of the witnesses, is not infeasible and allows Josh a legal answer to every challenge. The same argument shows that Josh has a legal answer to a challenge whenever the commitment that it names holds in some structure under some interpretation of its witnesses; under (Ax), (Tot) and (Tm), he always has one. Part (3) follows from the count of Josh's possible answers in the paragraph after Definition 10.2.

Parts (4) and (5) rest on König's lemma: an infinite tree in which every node has finitely many children has an infinite branch. For part (4), the tree $\mathcal{P}_\sigma$ is well-founded and, by hypothesis, finitely branching; hence it is finite, and the length of its longest branch gives N_σ . The hypothesis holds, by part (3), when σ makes no countable challenge, and in particular when no axiom of T contains a countable supremum or infimum, since every sentence that occurs in a play is atomic or is an instance of a subformula of an axiom. It remains to be shown that some finite set of axioms of T has no models. Let T_σ be the set of axioms named in challenges (Ax) at nodes of $\mathcal{P}_\sigma$, together with one axiom of T if there are none. In $G(T_\sigma)$, every challenge that σ makes at a node of $\mathcal{P}_\sigma$ is allowed by the rules, and Josh has the same

answers as in $G(T)$; the strategy σ therefore wins $G(T_\sigma)$, and the finite theory T_σ has no models by Theorem 10.3.

For part (5), let $\vec{n}$ be a finite sequence of natural numbers, and let $\mathcal{P}_\sigma^{\vec{n}}$ be the set of nodes of $\mathcal{P}_\sigma$ whose countable record is an initial segment of $\vec{n}$; it is a subtree of $\mathcal{P}_\sigma$, and every play in which Olivia follows σ and whose countable record is $\vec{n}$ runs inside it. In $\mathcal{P}_\sigma^{\vec{n}}$, a node at which σ makes a countable challenge has at most one child. If Josh has no legal answer to the challenge, its only child is the extension by the challenge alone. Otherwise, its only child is the extension by the countable answer whose number is the next entry of $\vec{n}$, unless the countable record of the node is $\vec{n}$ itself, in which case it has no child in $\mathcal{P}_\sigma^{\vec{n}}$. By part (3), every other node of $\mathcal{P}_\sigma^{\vec{n}}$ has finitely many children. Since $\mathcal{P}_\sigma^{\vec{n}}$ is well-founded, it is finite by König's lemma, and the length of its longest branch gives $N_\sigma(\vec{n})$. Finally, the skeleton is well-founded. If it had an infinite branch $\vec{n}_1 \subsetneq \vec{n}_2 \subsetneq \dots$, the set of nodes of $\mathcal{P}_\sigma$ whose countable record is an initial segment of some $\vec{n}_i$ would form a finitely branching subtree of $\mathcal{P}_\sigma$ containing nodes of every countable record $\vec{n}_i$, hence infinite; by König's lemma it would have an infinite branch, contradicting part (1). 

For countable theories of the finitary part of the logic, part (4) of Corollary 10.5 yields the compactness theorem: a countable theory of closed conditions without countable suprema and infima that has no models has a finite set of axioms that has no models. Part (3) locates the infinitary character of the obstruction: only at the countable challenges, where Josh chooses an index n among infinitely many, can the tree branch infinitely. By part (5), nothing else stands in the way of a uniform bound: the skeleton, a well-founded countably branching tree, records the infinitary shape of the obstruction, and once Josh's countable answers are fixed, the plays have bounded length (see Figures 9 and 10). For a theory with countable suprema and infima, the single finite obstruction of the compact case thus gives way to a well-founded tree of finite obstructions, one at each leaf.



Remark 10.6 (Strict conditions). The conditions of the logic are strict inequalities, and a closed condition is a countable theory of strict ones (see Section 5). For a theory that contains strict conditions, the game has one more rule.

(Str): Olivia names a condition $\varphi < r$ of T , for a sentence φ . Josh chooses a rational number $\varrho < r$ and adds $\varphi \in [-r_\varphi, \varrho]$. For a condition $\varphi > r$ of T , Josh chooses a rational number $\varrho > r$ and adds $\varphi \in [\varrho, r_\varphi]$.

With one more case for this rule, the proof of Theorem 10.3 applies to theories that consist of closed conditions as above and of strict conditions on sentences: a truthful Josh chooses ϱ between the value of φ and r , and in the structure built from a fair play the commitment $\varphi \in [-r_\varphi, \varrho]$ gives $\varphi \leq \varrho < r$. Under (Str), however, Josh chooses among infinitely many answers, and the challenges (Str) are accordingly counted as countable challenges in Corollary 10.5, with Josh's rational answers coded by natural numbers in the countable record; parts (1), (3) and (5) of the corollary then hold. Part (2) holds once the condition named in the last challenge is added to $\mathbf{p}$, since Josh has no legal answer to (Str) for $\varphi < r$ exactly when $r \leq -r_\varphi$, and none for $\varphi > r$ exactly when $r \geq r_\varphi$; in either case, the condition named is itself unsatisfiable. With the same counting, part (4) holds, except for its clause on theories without countable suprema and infima: a challenge (Str) with a legal answer has infinitely many children, and a theory without countable suprema and infima may have no models although each finite set of its axioms has one. For a nullary predicate symbol P with $r_P > 0$, the theory $\{P > 0\} \cup \{P < \frac{1}{n} : n \geq 1\}$ is such a theory. Olivia wins by naming first $P > 0$, to which Josh answers with some rational $\varrho > 0$ and the commitment $P \in [\varrho, r_P]$, and then a condition $P < \frac{1}{n}$ with $\frac{1}{n} < \varrho$; the tree of this strategy branches infinitely at its root. The

contrast between the two kinds of conditions has a topological counterpart in the space of Henkin types of Aubone, Dueñez and Iovino [ADI26], which we describe in Appendix A.

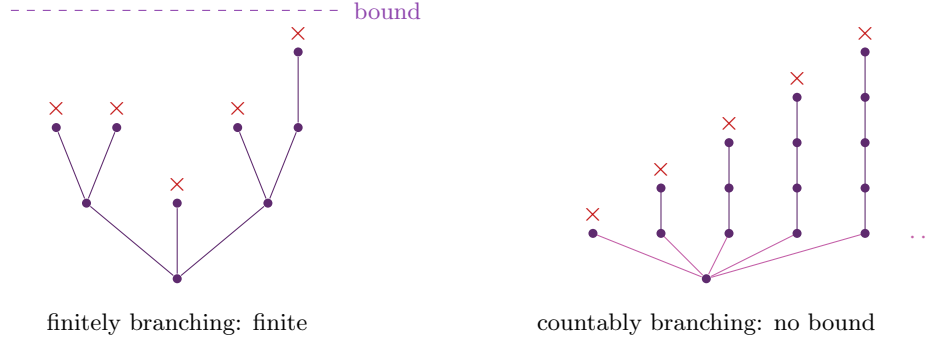


FIGURE 9. Trees grow upward from their roots. Left: a finitely branching tree without infinite branches is finite, by König’s lemma, and its branches have bounded length. Right: a countably branching tree without infinite branches may have branches of every finite length, as in this fan with one branch of each length.

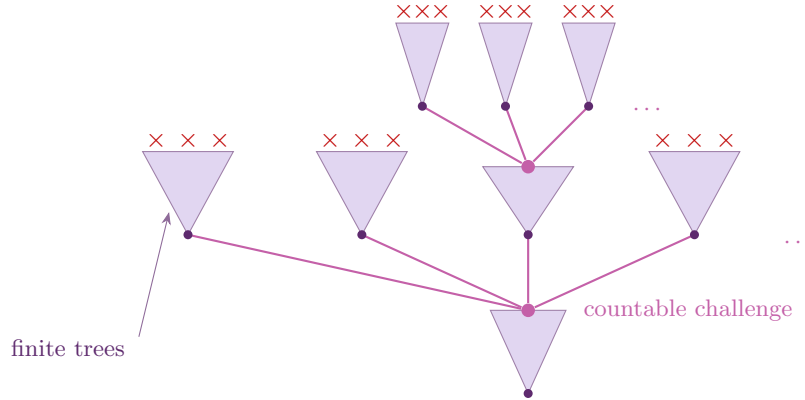


FIGURE 10. The tree of a winning strategy for Olivia, growing upward from its root. At each countable challenge (pink dot) to which Josh has a legal answer, the tree branches infinitely (pink fan), one branch for each number that he may name; the sequences of numbers that he names along the branches form the skeleton. The parts of the tree between countable challenges are finite (shaded), and at each of their leaves, marked $\times$, the commitments made along the play that ends there form a finite obstruction.

10.4. The Bell–Kochen–Specker obstruction. We now return to the theory T of Theorem A, with its measure μ , and to the axiom of Theorem C, which says that μ takes only the values 0 and 1. In the normal form described in the first subsection, that axiom becomes $\sup_a (\mu(a) \wedge (1 - \mu(a))) \leq 0$. By Theorem C, the theory obtained by adding the resulting axiom to T has no models, and the conclusions of Corollary 10.5 hold for it. We show that the tree of every winning strategy of Olivia for this theory branches infinitely, and we describe

a winning strategy, modeled on the proof of Theorem C, under which every play in which Josh makes all the commitments that the parity argument of that proof requires ends at a position whose commitments concern a configuration of the Kochen–Specker type.

Proposition 10.7. *Let T^{01} be the theory T of Theorem A together with the axiom*

$$\sup_a (\mu(a) \wedge (1 - \mu(a))) \leq 0.$$

- (1) *Olivia has a winning strategy in $G(T^{01})$; every play in which she follows it ends after finitely many rounds, at a position that is satisfied in no structure.*
- (2) *Every finite set of axioms of T^{01} has a model.*
- (3) *The skeleton of every winning strategy of Olivia in $G(T^{01})$ has more than one element: in some play in which Olivia follows such a strategy, Josh answers a countable challenge.*

Proof. Part (1) follows from parts (1) and (2) of Corollary 10.5, since T^{01} has no models by Theorem C.

For part (2), let a finite set of axioms of T^{01} be given, and let $Z \subseteq \mathbb{N}$ be the finite set of numbers n for which the instance $\kappa_n \varepsilon \doteq 2^{-n-1} e_n$ of axiom (K4) belongs to it. In the proof of Theorem C, it is the instances of (K4), taken together, that make the entries of ε span the whole space, and σ -additivity along ε then gives some atom the measure 1. The given finite set contains only finitely many of these instances; in a structure whose sequences are all finitely supported, axiom (M3) involves only finite-dimensional subspaces, on which a measure that vanishes on every finite-dimensional subspace satisfies it. Such a structure is the following. Its Hilbert sort is the closed unit ball of the space ℓ^2 of square-summable complex sequences; its Solèr sort is the lattice L_{fc} of Example 7.6, with the interpretations of e , π , at , rep and the constants e_n given there; its sequence sort consists of the finitely supported sequences in the closed unit ball of the space $\ell^2(\mathbb{N}, \ell^2)$ of square-summable sequences in ℓ^2 , with ι_n and κ_n as in the proof of Theorem 8.3; and the function span sends each element s of the sequence sort to the span of its entries, which is finite-dimensional. The measure μ vanishes on the finite-dimensional subspaces and equals 1 on their orthogonal complements; the constant ε is the finitely supported sequence whose entry of index n is $2^{-n-1} e_n$ for $n \in Z$ and 0 otherwise, and its norm is less than 1. The axioms of T_0 hold as in Example 7.6, and the axioms (K1)–(K3) and (Sp) as in the proof of Theorem 8.3, since the span of finitely many vectors is closed; of the instances of (K4), the only axiom in which ε occurs, exactly those with $n \in Z$ hold. The clause on μ in (OL2) holds because e is interpreted as equality; axioms (M1), (M2) and the axiom $\sup_a (\mu(a) \wedge (1 - \mu(a))) \leq 0$ hold because μ takes only the values 0 and 1, with $\mu(1) = 1$. For (M3), let s be an element of the sequence sort. If its entries are mutually orthogonal, the element $\text{span}(s)$ and the elements $\text{at}(\kappa_n s)$ are finite-dimensional subspaces, and both sides of (M3) vanish; otherwise the right side is 2, and the left side is at most 2. The structure is therefore a model of the given finite set of axioms.

For part (3), suppose that the skeleton of a winning strategy σ of Olivia had only one element, the empty sequence. Then Josh would give no countable answer at any node of $\mathcal{P}_\sigma$: a node at which σ makes a countable challenge would have only one child, the extension by the challenge alone, and every other node would have finitely many children by Corollary 10.5(3). By Corollary 10.5(4), the theory T^{01} would then have a finite set of axioms without models, contradicting part (2). 

Proof of Theorem D. In the model existence game, Olivia is the player who claims that the theory has no models, and the tree of plays of a strategy σ of hers is the tree $\mathcal{P}_\sigma$. Let T'

be a countable theory that has no models, with its axioms put in the normal form of the first subsection. The theory T' is then nonempty, since the empty theory has models, and the assertions of Theorem D about T' are those of Theorem 10.3 and Corollary 10.5, with T' in place of T . The theory of Theorem C, written in this normal form, is the theory T^{01} of Proposition 10.7; by part (2) of that proposition, every finite set of its axioms has a model, whose measure takes only the values 0 and 1, and by part (3), Josh gives a countable answer at some node of the tree of every winning strategy of Olivia. Since the interval that Josh adds under the rules $(\bigvee \geq)$ and $(\bigwedge \leq)$ does not depend on the number he names, every number is a legal answer at that node, and the tree branches infinitely there. 

We describe informally a winning strategy for Olivia that follows the proof of Theorem C (see Figures 11 and 12). It has two phases: the first finds a basis vector whose atom has measure close to 1, and the second runs the parity argument of Theorem C in the block of four basis vectors that contains it. Recall that the constant ε names the sequence $(2^{-n-1}e_n)_{n \in \mathbb{N}}$ built from the constants e_n , which form an orthonormal basis, and that the function at sends a nonzero vector x to an element of the Solèr sort whose members are multiples of x (see Lemma 9.1(1)). Olivia begins with the instance of axiom (M3) at ε ; by challenging it, together with instances of the axioms on which Lemmas 6.3 and 9.1 and the proof of Theorem C rest, among them (H1), (H2), (OL2), (P1)–(P4), (Sub0), (At1), (At5), (K2), (K4), (Sp1), (Sp2), (M1) and (M2), she obliges Josh to commit the sum of the values $\mu(\text{at}(\kappa_n \varepsilon))$ to an interval close to 1. A countable challenge $(\bigvee \geq)$ to the join over N of the truncated partial sums $\sum_{n < N} \mu(\text{at}(\kappa_n \varepsilon))$ in (M3) forces Josh to name a number N for which the finite sum $\sum_{n < N} \mu(\text{at}(\kappa_n \varepsilon))$ is close to 1. Challenges to the axiom of T^{01} at witnesses that name the elements $\text{at}(\kappa_n \varepsilon)$, for $n < N$, followed by challenges (Cn) and by challenges to (M1), make him commit each value $\mu(\text{at}(\kappa_n \varepsilon))$ to an interval close to 0 or close to 1. Since Olivia chooses the precision of these challenges after Josh has named N , and can take it finer than $1/(2N)$, the finite sum forces at least one $n_0 < N$ for which $\mu(\text{at}(\kappa_{n_0} \varepsilon))$ is close to 1. Through (K4) and the axioms behind Lemma 9.1(1), the element $\text{at}(\kappa_{n_0} \varepsilon)$ is identified with $\text{at}(e_{n_0})$. Further countable challenges arise, here and in the second phase, from the countable suprema in (H2), in the formula od that precedes (M3) and in the uses of the formula $[\varphi]$ of (2) in (P4), (At5), (Sp2) and (M3).

In the second phase, Olivia turns to the block of four basis vectors $e_{4j}, \dots, e_{4j+3}$, where $j \in \mathbb{N}$ satisfies $4j \leq n_0 < 4j + 4$, and challenges the instances of the axioms that the proof of Theorem C uses for the nine bases of Table 1 and for the block itself; as in that proof, we write $\text{at}(v)$ for $\text{at}(\frac{1}{16}v)$ when v is one of the eighteen vectors of the table. Josh's answers commit each of the eighteen values $\mu(\text{at}(v))$ to an interval close to 0 or close to 1, and each of the nine sums over a basis to an interval close to 1. Whatever finitely many choices Josh makes, the parity argument in the proof of Theorem C shows that these commitments are incompatible. In that proof, each of the nine bases $\mathcal{B}$, with vectors $v_0, \dots, v_3$, gives an element $s_{\mathcal{B}}$ of the sequence sort whose entries are $\frac{1}{16}v_0, \dots, \frac{1}{16}v_3$, followed by zeros; the instances of (K2) and (At5) identify each element $\text{at}(\kappa_i s_{\mathcal{B}})$, for $i < 4$, with $\text{at}(v_i)$, the instances of (P4) identify the nine elements $\text{span}(s_{\mathcal{B}})$ with one another, and the clause on μ in (OL2) carries the values of μ along these identifications. Through these instances, Olivia can then reduce the incompatibility to a basic sentence that receives intervals with no common point. Thus every play in which Josh makes all the commitments that the parity argument requires ends at a position whose commitments concern a finite configuration of the Kochen–Specker type inside one block of four basis vectors, the block that Josh's answer N bounds and his finitely many choices of intervals select.

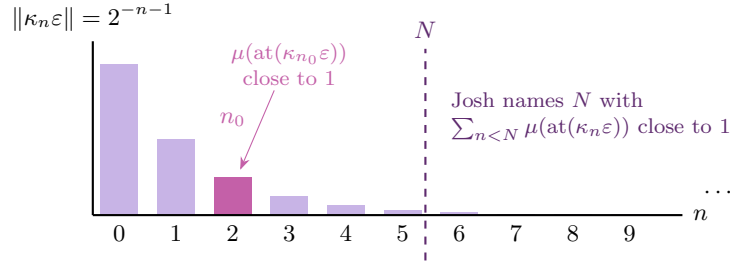


FIGURE 11. The decisive countable answer. The entries of ε have norms 2^{-n-1} ; Josh names N with $\sum_{n < N} \mu(\text{at}(\kappa_n \varepsilon))$ close to 1, and the axiom of T^{01} , together with (M1), then forces some $n_0 < N$ with value close to 1.

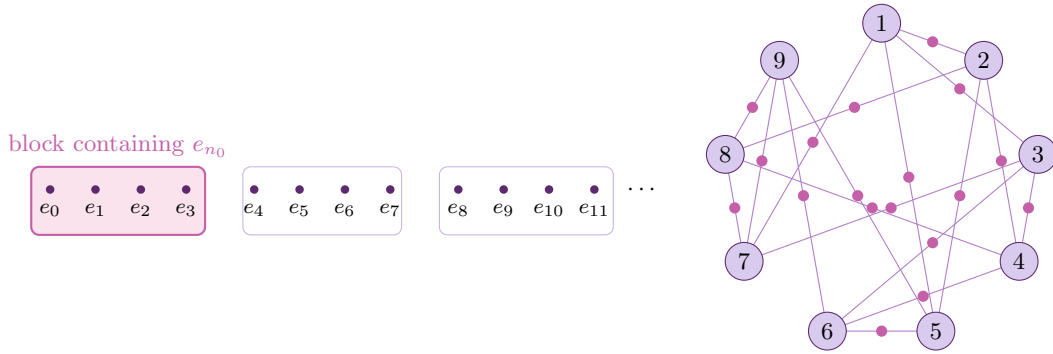


FIGURE 12. The finite obstruction at a leaf. Left: the basis vectors e_n grouped in blocks of four, with the block that contains e_{n_0} highlighted (here $n_0 = 2$, as in Figure 11). Right: the nine bases of Table 1 (numbered circles) and their eighteen rays (dots), each ray drawn on the segment that joins the two bases containing it. Since the element a_F of the proof of Theorem C, which stands for the span of the block, has measure 1, a $\{0, 1\}$ -valued measure would give the value 1 to exactly one ray in each basis; the pairs formed by a basis and its ray of value 1 would then number nine when counted basis by basis, and an even number when counted ray by ray, since each ray lies in exactly two bases.

By Proposition 10.7(3), Josh gives countable answers in some plays of every winning strategy of Olivia in $G(T^{01})$, and by Corollary 10.5(5), the plays of such a strategy have bounded length once those answers are fixed. Whether the length can be bounded independently of them remains open.

Question 10.8. Does Olivia have a winning strategy in $G(T^{01})$ whose tree has finite rank, that is, such that every play in which she follows it ends within a fixed number of rounds?



Remark 10.9 (Standard models). The game $G(T)$ concerns all models of T , standard or not, whereas Theorem B concerns standard models only, and standardness is not expressed by axioms. For a Borel set B that meets $[0, 1]$, let T_B be the theory T together with the axiom $\sup_a \beta_B(a) \leq -1$, the normal form of the axiom of Theorem B, where $\beta_B(a)$ is a formula, given by Lemma 5.1, whose value at a is 1 when $\mu(a) \in B$ and -1 otherwise. By Theorem 10.3, Olivia has a winning strategy in $G(T_B)$ exactly when T_B has no models at all, standard or not. This is the case when B contains 0 or 1, by Lemma 9.1(3) and (M2), and

when B contains $(0, 1)$, by Corollary 9.2; for the other Borel sets B that meet $[0, 1]$, we do not know whether T_B has a model.

 **Historical note.** The Banach–Mazur game began as Problem 43 of the Scottish Book, posed by Mazur; on 4 August 1935 Banach wrote in the book that Mazur’s conjecture is true, but his proof was never published, and Oxtoby published one in 1957 [Mau81, Oxt57]. Gale and Stewart proved in 1953 that open and closed games are determined [GS53]; Dénes König had published his lemma on finitely branching trees in 1927 [Kön27]. The witnesses of the game descend from the new constants of Henkin’s completeness proof of 1949 [Hen49]. For Väänänen there are essentially three kinds of games in logic, the semantic game, the model existence game and the Ehrenfeucht–Fraïssé game; his book of 2011 is built on them [Vää11].

11. REMARKS

In these remarks we explain why the vocabulary links vectors to elements of the Solèr sort through projections, why standardness cannot be dropped from part (1) of Theorem A or from Theorem B, and why axiom (K4) cannot be dropped from Theorem C. We then describe the changes needed for real and quaternionic scalars, and we record the logical form of the axioms and the redundancies among them.

 **Remark 11.1 (Projections rather than membership).** The vocabulary links a vector and an element of the Solèr sort through the projection π rather than through a membership relation: a vector x belongs to an element a of the Solèr sort exactly when $\|\frac{1}{2}x - \frac{1}{2}\pi(a, x)\| = 0$. The choice matters for joins: with a membership relation, the members of $a \vee b$ would naturally be described existentially, as the vectors x with $x = y + z$ for some member y of a and some member z of b , and in the logic an existential statement of this kind is expressed by the vanishing of an infimum, here the infimum of a scaled norm of $x - y - z$ over such y and z . In a standard model whose Hilbert sort is the closed unit ball of a Hilbert space H , this infimum vanishes exactly at the sums $y + z$ of such members, since these sums form a closed set: bounded sequences in the closed subspaces S_a and S_b spanned by the members of a and of b have weakly convergent subsequences. When $S_a + S_b$ is not closed, this set therefore misses the vectors of the unit ball that lie in the join $\overline{S_a + S_b}$ of S_a and S_b in $L(H)$ but not in $S_a + S_b$. Axiom (Sub3) reaches the join through the projections instead, by von Neumann’s alternating projection theorem. The functions at and rep provide the bijection between the rays of H , which correspond to the unit vectors up to a phase, and the atoms of the Solèr sort (see Proposition 7.4). Completeness of the Solèr sort is not an axiom: in standard models it is a consequence of the axioms of T_1 (see Theorem 8.3).

 **Remark 11.2 (Nonstandard models).** Standardness cannot be dropped from Theorem B or from part (1) of Theorem A: by the downward Löwenheim–Skolem theorem, the theory T has countable models, in which the range of μ is countable and the Solèr sort, being countable, is not isomorphic to the lattice of closed subspaces of an infinite-dimensional Hilbert space.

In a nonstandard model of T_0 , even the conclusion of Gleason’s theorem for T_0 (see Theorem 7.7) can fail. Let E_0 be the space of finitely supported complex sequences with the inner product of ℓ^2 , and let Λ_0 be the lattice of finite-dimensional subspaces of E_0 and their orthogonal complements in E_0 (see Example 4.3). Take the closed unit ball of E_0 as Hilbert sort, the standard unit vectors as the constants e_n , for $n \in \mathbb{N}$, and Λ_0 as Solèr sort, with e interpreted as equality, $\pi(a, x)$ the orthogonal projection of x onto a , $\text{at}(x) = \mathbb{C}x$, $\text{rep}(a)$ a unit vector chosen in a for each $a \neq \{0\}$, and $\text{rep}(\{0\}) = 0$. The resulting structure is a model of T_0 that is not standard. Every maximal orthonormal family of E_0 is countable and has join 1 in Λ_0 ; a σ -additive state on Λ_0 would therefore yield a frame function on

E_0 , a function on the unit vectors whose sum over every maximal orthonormal family is 1 and whose restriction to the unit vectors of every finite-dimensional subspace has the same sum over all the orthonormal bases of that subspace. By a theorem of Dvurečenskij [Dvu95, Theorem 3.1] (compare Hamhalter and Pták [HP87]), an inner product space that carries such a function is complete, which E_0 is not. The lattice Λ_0 therefore has no σ -additive states. On the other hand, every density operator ρ on ℓ^2 defines a map $a \mapsto \text{Tr}(\rho P_a)$ on Λ_0 , with P_a the projection of ℓ^2 onto the closure of a , and the conclusion of Theorem 7.7, which identifies the σ -additive states with these maps, therefore fails for this nonstandard model.

The axioms of Theorem B exclude, one Borel set, or more generally one σ -projective set (see Theorem 8.6), at a time, every range other than $[0, 1]$ in standard models; no single sentence, however, says that the range of μ is disconnected. Indeed, suppose that a sentence ψ of real-valued $L_{\omega_1, \omega}$ in the vocabulary of T took the value 1 exactly in the models of T in which the range of μ is disconnected. Let $\mathfrak{M}$ be a standard model of T , which exists by Theorem A(2); the range of μ in $\mathfrak{M}$ is $[0, 1]$ by Theorem B, and so the value of ψ in $\mathfrak{M}$ is not 1. By the downward Löwenheim–Skolem theorem, the model $\mathfrak{M}$ has a countable elementary substructure $\mathfrak{N}$ for a countable fragment containing T and ψ ; the value of ψ in $\mathfrak{N}$ is also not 1, yet the range of μ in $\mathfrak{N}$ is countable and contains $\mu(0) = 0$ and $\mu(1) = 1$, and it is therefore disconnected, a contradiction.

We leave open whether, in a nonstandard model of T , the measure μ can fail to be of the form $a \mapsto \text{Tr}(\rho \bar{P}_a)$ for a density operator ρ on a complex Hilbert space $\bar{E}$ as in Lemma 6.3, where $\bar{P}_a$ is the orthogonal projection of $\bar{E}$ that extends $\pi(a, \cdot)$.

Axiom (K4) is essential to Theorem C: the structure described in the proof of Proposition 10.7(2), with Z empty, satisfies every axiom of T except (K4), and its measure takes only the values 0 and 1.

 **Remark 11.3 (Scalars).** The vocabulary fixes the complex numbers as scalars. For real Hilbert spaces, replace $\mathbb{Q}(i)$ by $\mathbb{Q}$ and omit the predicate $\text{Im}\langle x \mid y \rangle$; for quaternionic spaces, use the quaternions with rational coordinates, acting on the left, and four predicates for the real components of the form. In the reading of (At4), “complex number of modulus one” then becomes “real number of modulus one” or “quaternion of modulus one”. Each choice yields the analogue of Theorem 8.3 for the corresponding case of Theorem 3.6, and the proof of Theorem C applies in all three cases, since the vectors of Table 1 have real coordinates.

 **Remark 11.4 (The logical form of the axioms).** The axioms (H1), (OL), (P1)–(P3), (Sub0), (Sub4), (At1), (At2), (K1), (K2), (K4), (Sp1), (M1) and (M2) are universal closed conditions built from finitary formulas. Axioms (H2), (Sub2), (Sub3), (K3), (Sp2), (M3) and the formula od use countable suprema or infima of uniformly bounded families. Axioms (P4), (Sub1), (At4), (At5), (Sp2), (Sp3) and (M3) use the formula $\lceil \varphi \rceil$ of (2), a countable join of a family with common bound 1 that records whether φ is positive. The functions at and span take values in the Solèr sort, on which the two-valued predicate e defines the discrete topology; neither function is continuous from the norm topology of its vector sort to that topology, a discontinuity that the logic of Section 5 permits, since its function symbols need not be continuous. Axiom (At3) uses the formula $\text{nonatom}(a)$, which recognizes atoms and contains the connective sup_b , a supremum over the Solèr sort.

 **Remark 11.5 (Redundancy).** In standard models, several axioms follow from others: (Sub1) from (Sub2) and (P4); (Sub3) from (Sub2), (Sub4), (OL) and (P4); (At3) from (At1), (At2) and Proposition 7.3; and (At4) from (At1), (P4) and Proposition 7.3. These axioms are kept nonetheless, so that each property has an axiom of its own.

 **Historical note.** Löwenheim proved in 1915 that a first-order sentence with a model has a countable model [Lö15]. Skolem extended the result to countable sets of sentences in

1920, and in 1922 he observed that axiomatic set theory, in which uncountable sets exist, therefore has countable models [Sko70]. The countable models of Remark 11.2 are of the same kind: they satisfy every axiom of a theory whose standard models are uncountable. That an incomplete inner product space can be recognized by the absence of states goes back to Hamhalter and Pták, who proved in 1987 that a separable inner product space is complete exactly when the lattice of its orthogonally closed subspaces admits a σ -additive state [HP87]. The theorem of Dvurečenskij used in Remark 11.2 belongs to the same line of work [Dvu95].

APPENDIX A. THE SPACE OF HENKIN TYPES AND THE MODEL EXISTENCE GAME

The model existence theorem of Section 10 and the omitting types theorem of Aubone, Dueñez and Iovino [ADI26] are two treatments of the Henkin construction: in the first, the description of a structure whose elements are named by witnesses is built move by move in the plays of a game; in the second, such descriptions are points of the space of Henkin types recalled in Section 5, and structures are obtained from the Baire category theorem. In this appendix we recall the facts about that space which bear on the game, referring to their paper for the proofs, and we distinguish the correspondences between the two treatments that rest on proof from those that are only analogies.

The space of Henkin types. The vocabulary $\mathcal{L}$ and the theory T are as in the setting of Section 10; the continuous connectives come from a countable set, as in Section 5, whence the fragment $\mathcal{F}$ generated by the sentences of the axioms of T is countable (see [ADI26, Section 6]). For the set C of new constants of Section 5 we take the set $\mathcal{W}$ of witnesses of Section 10; the fragment $\mathcal{F}_C$ that $\mathcal{F}$ generates in $\mathcal{L}_{\mathcal{W}}$ then has a countable set $S_C^{\mathcal{F}}$ of sentences. This set contains every sentence that occurs in a play of the game $G(T)$ of Definition 10.2, since such a sentence is an atomic sentence of $\mathcal{L}_{\mathcal{W}}$ or is obtained from a subformula of an axiom by substituting closed terms. We write $\prod S_C^{\mathcal{F}}$ for the product of the closed intervals $I_{\varphi} = [-r_{\varphi}, r_{\varphi}]$, for $\varphi \in S_C^{\mathcal{F}}$, and $\varphi^{\mathbf{m}}$ for the coordinate at φ of a point $\mathbf{m}$ of the product. By Tychonov's theorem, the product is a compact Hausdorff space; it contains the space $\mathcal{H}_C^{\mathcal{F}}$ of Henkin types as a subspace. For a set T' of closed conditions on sentences of $S_C^{\mathcal{F}}$, we write $\langle\langle T' \rangle\rangle$ for the set of points that satisfy every condition of T' ; the set $\langle\langle \emptyset \rangle\rangle$ is thus the whole product.

The correspondence rests on three results of [ADI26]:

- (i) A structure for $\mathcal{L}_{\mathcal{W}}$ is *canonical* when every element is the value of a closed term. The family of the values of the sentences of $S_C^{\mathcal{F}}$ in a canonical structure, its $S_C^{\mathcal{F}}$ -type, belongs to $\mathcal{H}_C^{\mathcal{F}}$; conversely, every member of $\mathcal{H}_C^{\mathcal{F}}$ is the $S_C^{\mathcal{F}}$ -type of a canonical structure, the term structure of its restriction to the atomic sentences [ADI26, Lemma 2].
- (ii) For a countable vocabulary and a countable fragment, every structure has a substructure whose sorts are countable and in which every sentence of the fragment keeps its value [ADI26, Lemma 3 and its proof]. A structure for $\mathcal{L}$ whose sorts are countable has an expansion to $\mathcal{L}_{\mathcal{W}}$ that names every element by a witness, and this expansion is canonical [ADI26, proof of Proposition 4].
- (iii) The space $\mathcal{H}_C^{\mathcal{F}}$ is a G_{δ} subset of $\prod S_C^{\mathcal{F}}$ [ADI26, Proposition 5]. Every Henkin condition defines a closed set except the upper bounds on suprema, which say that the value of $\sup_x \varphi(x)$, or of a countable join, does not exceed the supremum of the values of the sentences $\varphi(t)$, or of the members of the join; each of them defines a countable intersection of open sets.

These facts are proved there, with Henkin Realizability and the Čech-completeness of the space of Henkin types, for a many-sorted logic without continuous connectives, in which the a priori bound of a join is the supremum of the bounds of its members. The proofs apply unchanged to the logic of this paper once the Henkin conditions include, for each continuous connective f , the condition that the value of $f(\varphi_1, \dots, \varphi_\ell)$ is the value of f at the values of $\varphi_1, \dots, \varphi_\ell$: this condition, an equation in finitely many coordinates, defines a closed set; each induction on formulas acquires a case for f that follows from the definitions, as the case of $\varphi \vee \psi$ does; and the bound r that accompanies a join in Section 5 serves only to fix the interval of its values.

By (iii), the space $\mathcal{H}_C^\mathcal{F}$ is a G_δ subset of the product, and so is $\langle\langle T \rangle\rangle \cap \mathcal{H}_C^\mathcal{F}$, since each axiom $\psi \leq r$ or $\psi \geq r$ of T defines the intersection of the open sets $\{\mathbf{m} : \psi^\mathbf{m} < r + \frac{1}{n}\}$ or $\{\mathbf{m} : \psi^\mathbf{m} > r - \frac{1}{n}\}$, for $n \geq 1$; let K be either. The closure of K in $\prod S_C^\mathcal{F}$ is compact (see Engelking [Eng89, Theorem 3.1.2]), hence a compactification of K whose remainder, the trace of the complement of a G_δ set, is a countable union of closed sets; by Theorem 3.9.1 of Engelking [Eng89] and the definition that follows it, the space K is Čech-complete, a notion that Aubone, Dueñez and Iovino define by the equivalent internal characterization of Theorem 3.9.2 there. In K , the union of countably many nowhere dense subsets has dense complement, by the Baire category theorem [Eng89, Theorem 3.9.3].

Positions and plays. Let $\mathbf{p}$ be a position of $G(T)$ whose sentences belong to $S_C^\mathcal{F}$, as do those of every position reached in a play. It defines the set $\langle\langle \mathbf{p} \rangle\rangle$ of the points $\mathbf{m} \in \prod S_C^\mathcal{F}$ such that $\varphi^\mathbf{m} \in I$ for every commitment $\varphi \in I$ of $\mathbf{p}$, and $\theta(F(\bar{w}))^\mathbf{m} = \theta(w')^\mathbf{m}$ for every record $F(\bar{w}) \mapsto w'$ of $\mathbf{p}$ and every atomic formula $\theta(x)$ of $\mathcal{L}_\mathcal{W}$ whose only free variable is x ; the equations say that $F(\bar{w})$ and w' are indiscernible by atomic formulas with parameters, as equal elements are. Since the projections are continuous, the sets $\langle\langle \mathbf{p} \rangle\rangle$ and $\langle\langle T' \rangle\rangle$ are closed. A position, which commits sentences to closed intervals, thus defines a closed set and not, as finitely many strict conditions would, a basic open set of the product. In a structure, a position is *satisfied* under an interpretation of the witnesses that occur in it when its commitments hold and $F(\bar{w})$ and w' have the same value for each of its records $F(\bar{w}) \mapsto w'$.

Proposition A.1. *Let $\mathbf{p}$ be a position whose sentences belong to $S_C^\mathcal{F}$, and let T' be a subset of T . The set $\langle\langle \mathbf{p} \rangle\rangle \cap \langle\langle T' \rangle\rangle \cap \mathcal{H}_C^\mathcal{F}$ is nonempty if and only if $\mathbf{p}$ is satisfied in some model of T' under some interpretation of the witnesses. In particular, the theory T has a model if and only if $\langle\langle T \rangle\rangle \cap \mathcal{H}_C^\mathcal{F}$ is nonempty.*

Proof. Let $\mathbf{m}$ be a point of the set; by (i), it is the $S_C^\mathcal{F}$ -type of a canonical structure $\mathfrak{N}$. As at the end of the proof of Theorem 10.3, the quotient of $\mathfrak{N}$ by indiscernibility in atomic formulas with parameters is reduced, and the quotient map preserves the value of every formula; the reduct of this quotient to $\mathcal{L}$ is therefore a model of T' , which satisfies $\mathbf{p}$ when each witness is interpreted as the class of its value, the equations defining $\langle\langle \mathbf{p} \rangle\rangle$ placing the values of $F(\bar{w})$ and w' in one class. Conversely, let a model $\mathfrak{M}$ of T' satisfy $\mathbf{p}$ under an interpretation h . By (ii), applied to the expansion of $\mathfrak{M}$ by constants for the elements $h(w)$ and to the fragment generated by $\mathcal{F}$ and the sentences of $\mathbf{p}$, the structure $\mathfrak{M}$ has a substructure $\mathfrak{M}_0$ with countable sorts that contains these elements and satisfies T' and $\mathbf{p}$ under h , the records included. The infinitely many witnesses of each sort that do not occur in $\mathbf{p}$ can name all the elements of $\mathfrak{M}_0$; the resulting expansion is canonical, and by (i) its $S_C^\mathcal{F}$ -type belongs to $\langle\langle \mathbf{p} \rangle\rangle \cap \langle\langle T' \rangle\rangle \cap \mathcal{H}_C^\mathcal{F}$. ☺

In these terms, Josh's truthful answers in a model of T , in the proof that (1) implies (2) in Theorem 10.3, lead to positions that the same model satisfies, and therefore keep the set $\langle\langle \mathbf{p} \rangle\rangle \cap \langle\langle T \rangle\rangle \cap \mathcal{H}_C^\mathcal{F}$ nonempty. At a leaf of the tree of a winning strategy of Olivia, on the

other hand, the last position $\mathbf{p}$ is satisfied in no structure (see Corollary 10.5(2)); by the proposition, applied with T' empty, the closed set $\langle\langle\mathbf{p}\rangle\rangle$ does not meet $\mathcal{H}_C^\mathcal{F}$.

A fair play that Josh survives determines a nonempty closed subset of $\langle\langle T \rangle\rangle \cap \mathcal{H}_C^\mathcal{F}$. Let $\mathbf{p}_0 \subseteq \mathbf{p}_1 \subseteq \dots$ be its positions, and let $\mathbf{p}_\infty$ be their union. The set $\bigcap_i \langle\langle \mathbf{p}_i \rangle\rangle \cap \mathcal{H}_C^\mathcal{F}$, which is closed in $\mathcal{H}_C^\mathcal{F}$, is contained in $\langle\langle T \rangle\rangle$, since every axiom of T is named in a challenge (Ax). It is nonempty, because the structure $\mathfrak{M}$ built in the proof that (2) implies (1) in Theorem 10.3 is canonical, its elements being the witnesses, each named by itself, and satisfies every commitment and every record of $\mathbf{p}_\infty$; by (i), its $S_C^\mathcal{F}$ -type lies in the set. The point so obtained is in general not the only one: if a predicate symbol P other than $\mathbf{1}$ occurs in no axiom of T , for instance, then no sentence that occurs in the play contains P ; the construction leaves the value of $P^\mathfrak{M}$ at each tuple of witnesses free in $[-r_P, r_P]$, and different choices yield canonical structures whose $S_C^\mathcal{F}$ -types differ at the sentences $P(\bar{w})$.

Closed conditions, strict conditions and compactness. The contrast between closed and strict conditions drawn in Remark 10.6 has a topological counterpart: the Henkin types that satisfy a closed condition form a closed subset of $\mathcal{H}_C^\mathcal{F}$, and those that satisfy a strict condition an open subset. In the compact space $\prod S_C^\mathcal{F}$, a family of closed sets with empty intersection has a finite subfamily with empty intersection, whereas a family of open sets need not. For the theory $\{P > 0\} \cup \{P < \frac{1}{n} : n \geq 1\}$ of Remark 10.6, each finite part has a model, whence the open set it defines meets the space of Henkin types by (i) and (ii), while the whole theory defines the empty set; the tree of Olivia's strategy there branches infinitely at its root. For a theory T of closed conditions without models, the compactness of the product would yield, if $\mathcal{H}_C^\mathcal{F}$ were closed, a finite subset T_0 of T with $\langle\langle T_0 \rangle\rangle \cap \mathcal{H}_C^\mathcal{F} = \emptyset$, that is, by Proposition A.1, a finite set of axioms without models; but $\mathcal{H}_C^\mathcal{F}$ is only a G_δ subset of the product, to which the compactness of the product need not pass. By (iii), the Henkin conditions that do not define closed sets are the upper bounds on suprema, which in the game Josh certifies by naming an instance: a witness under $(\sup \geq)$ and $(\inf \leq)$, where the freshness of witnesses leaves him finitely many choices, and an index under $(\bigvee \geq)$ and $(\bigwedge \leq)$, where he has infinitely many, as parts (3) and (4) of Corollary 10.5 record. This correspondence concerns the rules of the game only: we do not derive Corollary 10.5 from the topology of $\mathcal{H}_C^\mathcal{F}$.

Baire category. Olivia's bookkeeping can also be read as a Baire category argument. Let σ' be a winning strategy for Josh, and let $X_{\sigma'}$ be the set of infinite plays in which Josh follows σ' , with the topology whose basic open sets consist of the plays that extend a given finite play. Since the moves range over countable sets, the space $X_{\sigma'}$ is homeomorphic to a closed subspace of the Baire space $\mathbb{N}^\mathbb{N}$; it is therefore a Polish space, in which the Baire category theorem holds. For each challenge and each round, the set of plays in which the challenge, if available at that round, is made at that round or later is open and dense in $X_{\sigma'}$. The fair plays therefore form a countable intersection of dense open subsets of $X_{\sigma'}$, which by the Baire category theorem is dense, hence nonempty. The bookkeeping can be read, too, through the *Banach–Mazur game*, in which two players alternately choose nonempty open sets of a space, each contained in the one before, and the second player wins if the intersection of the chosen sets is contained in a given set (see Figure 8). In the Banach–Mazur game on $X_{\sigma'}$ in which the players choose basic open sets and the second player wins if the resulting play is fair, the second player has a winning strategy modeled on Olivia's bookkeeping (compare Oxtoby [Oxt57]): at its move of index k , for each $k \in \mathbb{N}$, it extends the current finite play by one round, in which Olivia makes the $l(k)$ -th challenge of her enumeration for round $j(k)$,

available because at least k rounds precede the new one and $j(k) \leq k$, and Josh answers by σ' . In the proof of Theorem 10.3, the category argument reduces to the bookkeeping, because only countably many challenges are available at each round.

The Baire category argument on plays is the analogue of the one by which Eagle [Eag14] proved the omitting types theorem for $[0, 1]$ -valued $L_{\omega_1, \omega}$, and by which Aubone, Dueñez and Iovino prove it for their logic [ADI26, Theorem 8]. In the terms of this appendix, their theorem gives the following: if T has a model and $(\Xi_n(\bar{x}_n))_{n \in \mathbb{N}}$ is a sequence of types, sets of closed conditions $\varphi(\bar{x}_n) \geq r$ with $\varphi \in \mathcal{F}$ and r rational, whose instances $\Xi_n(\bar{t})$ at tuples $\bar{t}$ of closed terms of $\mathcal{L}_{\mathcal{W}}$ meet $\langle\langle T \rangle\rangle \cap \mathcal{H}_{\mathcal{C}}^{\mathcal{F}}$ in sets with empty interior in that space, then T has a model whose sorts are countable and in which no tuple satisfies any $\Xi_n(\bar{x}_n)$. Indeed, the complements of these sets are countably many dense open subsets of the nonempty space $\langle\langle T \rangle\rangle \cap \mathcal{H}_{\mathcal{C}}^{\mathcal{F}}$, in which the Baire category theorem holds; a point of their intersection is, by (i), the $S_{\mathcal{C}}^{\mathcal{F}}$ -type of a canonical structure, whose tuples consist of values of closed terms, and in the reduct to $\mathcal{L}$ of its reduced quotient, which changes no value, no tuple satisfies any $\Xi_n(\bar{x}_n)$. The two applications of the Baire category theorem take place in different spaces, the Polish space of plays and the Čech-complete space of Henkin types, and the correspondence between them is an analogy: we know of no continuous map from the one space to the other that carries one argument onto the other, for a fair play determines, as observed above, a closed set of Henkin types and not a point.

✂ **Historical note.** The omitting types theorem goes back to Henkin, in 1954 [Hen54], and to Orey, in 1956 [Ore56]. Robinson introduced forcing into model theory in an article published in 1971 [Rob71], adapting the method that Cohen had created for set theory in 1963 [Coh63]; Barwise extended Robinson’s forcing to countable fragments of $L_{\omega_1, \omega}$ and used it to give a new proof of the omitting types theorem, in notes cited by Keisler [Kei73]. In 1973 Keisler derived the model existence theorem and the omitting types theorem from a single Generic Model Theorem, which he described as a reformulation, in terms of ordinary models, of the Boolean-valued models of Rasiowa and Sikorski [RS63]. In 2009 Ben Yaacov and Iovino developed model-theoretic forcing for an infinitary continuous logic [BYI09].

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