RESEARCH REPORT • QUANTUM FOUNDATIONS × MATHEMATICAL LOGIC

How mathematical logic refutes a hidden world

What our paper contributes, for a general mathematical audience

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Research papers are written for readers who already speak their language. Ours, *Finite refutations in countably additive quantum probability*, speaks three at once: it brings new results to the foundations of physics, to mathematical logic and to mathematics at large, and it presents them in the idiom of specialists. This note retells those results for every mathematician, keeping the story and leaving the proofs to the paper, whose current version, like this one, lives at utsa-quantum-math.github.io.

QUANTUM FOUNDATIONS

From a finite check to a finite search

Egli è scritto in lingua matematica.
It is written in the language of mathematics.
Galileo, *Il Saggiatore* (1623), of the book of nature

Imagine preparing a particle the same way many times and asking it the same yes-or-no question. Quantum mechanics assigns a probability to each answer. A deterministic *hidden-variable* theory proposes that each particle carries further information that fixes its answer in advance, so that the probabilities merely record our ignorance of that information. The history of the subject is the story of finding out which assumption about such theories quantum mechanics rules out. The question is whether answers fixed in advance can agree with the rules of quantum probability, and the story turns on how much it takes to expose a conflict.

In 1932 von Neumann gave an argument that no hidden-variable theory could exist. In 1935 Einstein, Podolsky and Rosen argued instead that quantum mechanics is incomplete. By their criterion of reality, if the outcome of a measurement on one particle can be predicted with certainty by measuring its distant partner, without disturbing the first particle, then that outcome corresponds to an element of reality fixed in advance. Their argument applies to incompatible quantities such as position and momentum, which no quantum state fixes together, so the quantum description cannot be complete. In 1952 Bohm, developing de Broglie's pilot wave of 1927, wrote down a deterministic theory reproducing the predictions of nonrelativistic quantum mechanics, so von Neumann's argument had assumed too much.

Bell identified the excess: quantum averages do add, but von Neumann required the values assigned to individual systems to add in the same way, even for measurements that cannot be performed together. Bell then located the real conflict in two places. For a pair of distant particles,

This version: 23 November 2026. The most recent version is available at https://utsa-quantum-math.github.io/resources/gleason-kochen-specker-game/.

it is locality: no local hidden-variable theory reproduces the quantum correlations (Bell, 1964), so the predetermined local values that Einstein, Podolsky and Rosen asked for cannot exist. For a single system, it concerns a projection that appears in two different measurements. A projective measurement has one outcome for each member of an orthogonal decomposition of the identity, two such measurements can share a member, and a deterministic assignment selects exactly one outcome in each measurement. The requirement that a shared projection keep its answer in both is *noncontextuality*, and with it the answers form a single assignment on projections, with values summing to one on every measurement. Comparing a measurement with one in which several of its outcomes have been grouped into one gives additivity on orthogonal events, and the same comparison for measurements with countably many outcomes gives countable additivity: the assignment is a probability measure that takes only the values 0 and 1. Gleason's theorem of 1957 says that every countably additive probability measure on the subspaces of a Hilbert space of dimension at least three comes from a density operator, and the measure $P \mapsto \mathrm{Tr}(\rho P)$ of a density operator ρ never takes only the values 0 and 1; so, as Bell observed, such assignments are impossible.

Compactness makes the refutation finite

Gleason's argument uses a continuum of directions, and no experiment can ask a continuum of questions. In 1967 Kochen and Specker found a finite set of directions in three-dimensional space on which noncontextual predetermined answers are already impossible, and the conflict became a finite check. Kochen, a model theorist, and Specker, a logician, reached it by turning the question into one about representation. Within each family of commuting projections the events obey the familiar Boolean rules, and different families overlap; hidden variables would represent all of these families in a single Boolean algebra, preserving their agreement on shared events, and for propositions this means a homomorphism onto the two truth values. Here mathematical logic, the part of mathematics that makes refutation precise, supplies its tools: Kochen and Specker treated the relations among the events as axioms whose models are the Boolean algebras into which the events map homomorphically, and used the completeness theorem of first-order logic, in effect a compactness argument, to reduce the failure of such a representation to finitely many of these relations; the result is a formula of propositional logic that is classically valid but refutable in quantum mechanics. The logician J. L. Bell (not John S. Bell) and Pitowsky later made the compactness explicit: by compactness, a finite configuration follows from Gleason's theorem itself.

Later configurations made the finite check strikingly simple. The eighteen-ray example of Cabello, Estebaranz and García-Alcaine consists of nine measurements in a four-dimensional space, each made of four orthogonal rays, and each of its eighteen rays occurs in exactly two of them. A predetermined assignment chooses one outcome in each measurement, and noncontextuality requires a ray chosen in one measurement to be chosen in the other measurement that contains it. Count the pairs formed by a measurement and the ray chosen in it. There is one pair for each measurement, nine in all; but each chosen ray lies in exactly two measurements, so the pairs come two by two and their number is even. Nine is odd; we call this the parity argument.

Hrushovski and Pitowsky, a model theorist and a philosopher of physics, carried the model-theoretic method from answers to probabilities. Working in the first-order theories of the real and complex fields, they used compactness to find finite sets of rays on which the probabilities specifying a quantum state constrain further probabilities to any prescribed accuracy, and the decidability of those theories to make the construction effective; Gleason's theorem thereby acquires a constructive proof. The finite configurations now carry positive information about quantum probabilities, not only the impossibility of predetermined answers. Both advances run through compactness, and both take place in finite dimension, where an orthonormal basis is finite.

Where compactness fails

The state space of a particle on a line, for instance, is an infinite-dimensional Hilbert space. The paper works in a separable infinite-dimensional Hilbert space, where probabilities are countably additive: along an orthonormal basis they sum to one. Gleason's theorem still rules out predetermined noncontextual answers there, through its continuum of directions; a refutation by finite checks meets two obstacles.

The first obstacle is physical. Place the eighteen rays in a four-dimensional subspace, a *block*, of an infinite-dimensional space. The four rays of each measurement now span only the block, so each of the nine measurements needs one more outcome, the orthogonal complement of the block, and all nine share it. On these nine measurements, a consistent set of answers is to choose the complement every time: the block receives value zero, every ray inside it receives zero, and the parity argument never starts.

The second is logical. To refute the assignment just described, the theory must force the probability off the complement of every block, and for that the axioms must be able to speak about a whole orthonormal basis. We write quantum probability as a countable set of axioms T in an infinitary logic, described in the logic section; they speak about a whole basis through sequences of vectors, and besides these about vectors, events and the probability of each event. A *model* of the axioms is any structure in which they hold; in a *standard* model the vectors and the sequences of vectors are the full closed unit balls of complex Hilbert spaces. We express the hidden-variable hypothesis in this language by adjoining to T the axiom that every event has probability zero or one; call the resulting theory T^{01}. Every finite subset of T^{01} has a model (Proposition 8.3). On the terms Kochen and Specker set, the infinite-dimensional theory therefore looks beyond reach: no finite list of its axioms is contradictory, so the compactness argument used above cannot produce a finite inconsistent subset of T^{01}. In the models constructed in Proposition 8.3, the sequences have only finitely many nonzero entries and every finite-dimensional event has probability zero, which is the block of value zero once more; these models can even satisfy the continuity axiom for the probability on the sequences they contain. In such a model the continuity axiom has no basis sequence to test: what finitely many axioms fail to supply is the sequence whose prescribed entries make that axiom test the entire named basis, and the full theory supplies it.

Countable additivity opens a way

The condition that defeats compactness is the one that rescues the refutation. Since the probabilities along an orthonormal basis sum to one, some finite initial segment of the basis carries positive probability, and for a two-valued assignment that segment contains a ray of value one. A block containing this ray has value one, the nine measurements must select outcomes inside it, and the parity argument applies.

Where the segment ends depends on the unknown assignment, so the refutation becomes a search. We present it as a game between a builder, who defends a proposed assignment by committing to bounds on its values, and a challenger, who questions those commitments. The challenger asks the builder to name a finite initial segment of the basis whose join has probability greater than 7/8. The builder's answer is an index N, and it tells her which rays to examine. Examining them, she obtains from further commitments a ray whose recorded lower bound exceeds 15/16, and this ray selects the block in which the parity argument applies (Proposition 8.6). Every play ends after finitely many questions in a contradiction. The builder has infinitely many possible indices at this move; an answer of zero already leads to a contradiction, since the empty join has probability zero. More generally, every winning strategy has a node with infinitely many immediate continuations (Corollary 8.4), as the logic section explains. A fixed finite list of axioms gives way to an inquiry: in every run the challenger asks finitely many questions; the index fixes which finite segment she examines, and later commitments select the block.

Theorem C. The theory T^{01} has no models, standard or not.

Gleason's theorem concerns measures on a genuine Hilbert space; Theorem C refutes the hypothesis in every model of the axioms, by finite contradictions and without a continuum argument, and its proof uses neither Gleason's theorem nor standardness, which the refutation does not need; they enter in the results on standard models (Theorems B and D).

What the contradiction is made of

Once the block is found, the finite contradiction also shows what each assumption contributes. Write one zero-one mark for each of the thirty-six occurrences of a ray (nine measurements of four rays), require exactly one mark equal to one in each of the nine measurements, and require the two marks of each ray to agree: these eighteen agreements cannot be satisfied together, but dropping any one of them leaves conditions that can be. Noncontextuality is exactly the requirement that probability depend only on the projection, and each agreement is one instance of it; the zero-one requirement enters separately, as what makes each measurement select exactly one ray (Section 9.3). The contradiction is also stable. Theorem 9.2 keeps the nine bases and their shared rays fixed, and requires the two recorded values for a shared ray to be equal; it relaxes only the zero-one requirement and the normalization. If each value is only within δ of zero or one and the values in each measurement sum only to within η of one, the contradiction persists whenever $4\delta + \eta < 1$, and this threshold is sharp (Theorem 9.2 and the remark after it). These tolerances concern the hypothetical assignment, not experimental error.

What physicists gain

- Predetermined, noncontextual outcomes are refuted in the infinite-dimensional, countably additive setting by a strategy every play of which is finite.
- The refutation must adapt to the answers it receives, and countable additivity is what makes the adaptation possible.
- The zero-one hypothesis fails in every model of the axioms, standard or not, and the proof does not use Gleason's theorem.
- In the parity argument with one mark per occurrence and one selection per measurement, each of the eighteen agreements on shared rays is indispensable, and each comes from the rule that equal projections receive equal probabilities.
- The parity contradiction survives approximately zero-one values and approximate sums, up to a sharp threshold.

MATHEMATICAL LOGIC

Finite contradictions survive the loss of compactness

Calculemus.
Leibniz, on settling disputes by calculation

The physics leaves the logician with two facts about one theory, T^{01}. Every play of the challenger's strategy ends in a finite contradiction, yet every finite subset of T^{01} has a model. In finite dimension, compactness turned an impossible assignment into a finite configuration, for Kochen and Specker and for Hrushovski and Pitowsky; in first-order logic the two facts could not coexist, since

compactness would turn the inconsistency into a finite subset without a model. In the infinitary logic of our axioms they do coexist, and logic must say what a finite refutation of the whole theory can be when no finite set of its axioms is contradictory.

For two-valued $L_{\omega_1\omega}$ the classical answers are a calculus with infinitary rules (Karp, López-Escobar), the model existence theorem for consistency properties (Makkai, Keisler), and Väänänen's model existence game, whose winning strategies describe the classical completeness proofs. Our theorem carries the game answer to the logic in which our axioms live: the real-valued infinitary logic $L_{\omega_1\omega}(\mathbb{R})$, the infinitary extension of Keisler's model theory of general structures. Its formulas take real values in bounded intervals, countable suprema and infima of formulas (infinitary connectives) are allowed besides the quantifiers $\sup_x$ and $\inf_x$, and there is no equality symbol, no prescribed metric and no modulus of uniform continuity. The events form a lattice in our vocabulary, and its operations need this generality: they are not continuous for the operator-norm distance between projections, since the meet of two distinct rays is the zero subspace however close the rays are.

In the model existence game for a theory, the challenger maintains that the theory has no model and the builder tries to build one, answering challenges about sentence values, witnesses, function values, and countable suprema and infima. The challenger may name any axiom, and the builder must commit to it; most answers are commitments that place the value of a sentence in a closed interval with rational endpoints, and others supply witnesses or values of functions. Asking for the initial segment in the physics is a typical move: a commitment that a countable supremum of formulas exceeds a bound is challenged by asking for one formula of the family, named by its index, together with a bound for it at the tolerance specified by the challenge. Such challenges about countable suprema and infima of formulas are the only ones that leave the builder infinitely many answers; every other challenge leaves finitely many.

Theorem A. A nonempty countable theory in a countable vocabulary of $L_{\omega_1\omega}(\mathbb{R})$, whose axioms have the form $\psi \le r$ or $\psi \ge r$ for sentences ψ and rational numbers r, has no model if and only if the challenger has a winning strategy. The tree of plays following any such strategy is well-founded, and each play ends at a finite position, a set of commitments and of records of function values, that is satisfied in no structure under any interpretation of its witnesses.

The loss of compactness does not cost us finite refutations; it changes their shape. In our example every finite subset of T^{01} has a model, even one with zero-one probabilities, yet a winning strategy still ends in finitely many incompatible commitments on every play.

Each direction of the theorem names its strategy. If the theory has a model, the builder wins by answering truthfully. If it has none, the challenger wins by bookkeeping: she schedules her challenges so that every infinite play would be fair, meaning that every challenge that becomes available is eventually made, and a fair play that the builder survived would yield a model, assembled from the Henkin types of Aubone, Dueñez and Iovino, assignments of values to sentences that are realized by structures built from closed terms; so no play survives. The game is therefore determined without appeal to the Gale–Stewart theorem.

The theorem thus sets two kinds of finiteness side by side:

- a finite subset of the axioms has no model;
- every play of a winning strategy ends at a finite position whose commitments and function records cannot be jointly realized.

Compactness gives the first in first-order logic; Theorem A gives the second, and the commitments

may concern infinitary sentences. Only challenges about countable suprema and infima of formulas give a node of the tree infinitely many successors. A winning tree that branched only finitely would be finite by König's lemma and would name only finitely many axioms; a model of those axioms would let the builder survive every play, so the tree could not be winning. Physics hands logic a natural theory on which the two kinds of finiteness come apart. For T^{01} every finite subset has a model of a special kind: its sequences have only finitely many nonzero entries, and its probability vanishes on every finite-dimensional subspace. Finitely many axioms allow this, because the full axioms require a sequence that prescribes a nonzero multiple of each basis vector, and finitely many axioms prescribe only finitely many entries (Proposition 8.3). This is the block of value zero again, now on every block at once. Hence every winning strategy for T^{01} has a node with infinitely many immediate continuations (Corollary 8.4): this is the logical form of the physicist's adaptive search.

Whether T^{01} has a winning strategy whose plays have a common finite bound on their lengths remains open (Question 8.7). Infinite branching does not settle it: a root with infinitely many leaves branches infinitely and has plays of length one, while a root with one branch of each finite length has plays of unbounded length, and the rank of a well-founded tree is finite exactly when its plays have bounded length. What is known is that the lengths are bounded once the countable answers are fixed: for each finite sequence of answers to the countably branching challenges, the plays whose answers to those challenges form exactly that sequence have bounded length. Finally, the paper gives a topological reading of the challenger's bookkeeping and compares it with the Čech-complete space of Henkin types and with Keisler's model-theoretic forcing.

What the axioms can say about probabilities

Like Hrushovski and Pitowsky, the paper also asks what the logic can say about the probabilities themselves. A yes-or-no test on events is a formula that takes only two values, +1 and −1. A formula with one free event variable defines a set $D \subseteq [0,1]$ in a standard model if, at every event, it takes the value +1 when the event's probability lies in D and −1 otherwise; D is *uniformly definable* if a single formula defines it in every standard model. Coordinate projection enters through the probabilities themselves: for a formula with values ±1 that depends on both of its events only through their probabilities, the quantifier sup over the second event gives a test of the first probability alone, returning +1 exactly when some second probability passes the test, and every value in $[0,1]$ occurs as a probability in every standard model. Theorem D identifies the uniformly definable sets exactly: the uniformly definable sets are the σ-projective subsets of $[0,1]$, the class generated from the Borel subsets of the cubes $[0,1]^n$ by complements, countable unions and coordinate projections $[0,1]^{n+1} \to [0,1]^n$, which is strictly larger than the class of projective sets; the classification is proved in ZFC. Much freedom returns if a model may be chosen separately for each set. A standard model is not determined by its Hilbert space and its density operator alone: it also chooses a representative unit vector in each ray, and these choices can carry information. Choosing a standard model for each set, one fixed formula defines every subset of the open unit interval, the choice between a vector and its negative carrying the set (Proposition 7.9). For a nonnegative integer k, a set uniformly definable at quantifier depth at most k lies in $\mathbf{\Delta}^1_{k+1}$ (Proposition 7.8), so no formula of finite quantifier depth defines a nonprojective set uniformly. The countable suprema and infima of formulas and the quantifiers produce this expressive power together, the countable suprema being what yield exact tests of inequalities; whether formulas without countable suprema and infima uniformly define any set other than $\emptyset$ and $[0,1]$ remains open. The classification also reaches the limits of ZFC: if suitable large cardinals are consistent, ZFC neither proves nor refutes that every uniformly definable set of probabilities is Lebesgue measurable.

What logicians gain

- A model existence theorem, in game form, for countable theories of closed conditions in a countable vocabulary of $L_{\omega_1\omega}(\mathbb{R})$: the absence of a model is witnessed by a well-founded tree of finite contradictions, and the game is determined without Gale–Stewart.
- A natural theory, coming from physics, that has such a witness but no finite subset without a model, and so forces infinite branching.
- An exact classification of the uniformly definable sets of probabilities, a bound by quantifier depth, and a measurability question that ZFC does not settle if suitable large cardinals are consistent.

HILBERT SPACE AND LATTICES

The axioms capture quantum probability exactly

To treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.
Hilbert, the sixth of his problems (1900)

The refutation takes place in all models of the axioms, and the definability results in the standard ones, whose vectors and sequences are full closed unit balls of Hilbert spaces. The mathematical question is what else the axioms force in a standard model. In the usual description of quantum probability, events are the closed subspaces of a Hilbert space, a state is a density operator ρ, and the event whose orthogonal projection is P has probability $\mathrm{Tr}(\rho P)$. In 1900 Hilbert asked that the physical sciences, the theory of probabilities first among them, be treated by axioms as geometry had been. In that spirit, Theorem B says that our axioms force exactly this description: among standard models, the axioms recover precisely the Hilbert-space events and the trace probabilities, and every Hilbert space and density operator of the specified kind is represented.

Theorem B. In every standard model of the axioms T, the Hilbert space whose unit ball forms the vectors is separable, infinite-dimensional and complex, and the quotient event lattice, obtained by identifying events that determine the same subspace, is isomorphic as an ortholattice to its full lattice of closed subspaces. A unique density operator gives the probabilities; conversely, every separable infinite-dimensional complex Hilbert space, with any density operator on it, arises from a standard model.

Three ingredients do the work. Von Neumann's alternating projection theorem recovers intersections of subspaces from projections; a sort of square-summable sequences of vectors supplies closed spans; and Gleason's theorem, which identifies the countably additive probability measures on the closed subspaces of a Hilbert space of dimension at least three, supplies the trace representation.

The geometry and the probabilities can also be separated. A weaker theory keeps the vectors and the events but has neither sequences nor a probability predicate, and its event lattices may lack joins. The finite-dimensional subspaces and their orthogonal complements form the event lattice of one of its standard models (standard here meaning that the vectors form the full unit ball of a Hilbert space), and there the rays of the even-numbered vectors of an orthonormal basis have no join: their closed span has infinite dimension and an infinite-dimensional orthogonal complement, so it is not in the lattice, and every upper bound in the lattice has finite codimension, so it contains a nonzero vector perpendicular to the span; intersecting it with that vector's orthogonal complement gives a strictly smaller upper bound in the lattice. Even so, on the event lattice of every standard model

of the weaker theory, each probability measure that adds probabilities over countable orthogonal families whose join exists comes from a unique density operator (Theorem 6.6). The probabilities, in other words, do not wait for the geometry to be complete.

Recovering the probabilities on the events already present leaves open what happens to the geometry when the missing events are added. In Hilbert space, a subspace splits into any smaller subspace and its orthogonal remainder; in a lattice this is *orthomodularity*, the property that $a \leq b$ implies $b = a \vee (b \wedge a^{\perp})$. The question is whether adding the missing events keeps this property. For the lattices of the weaker theory the answer is clean: the MacNeille completion, which supplies the missing joins and meets, is the full lattice of closed subspaces, hence orthomodular (Theorem 6.4). For abstract event lattices it is a classical question of quantum logic, because the MacNeille completion of an orthomodular lattice may fail to be orthomodular.

The paper answers the question for pre-Solèr lattices. A pre-Solèr lattice satisfies the axioms for a Solèr lattice with completeness left optional; among them is Solèr's condition, an infinite orthogonal family of atoms with an additional compatibility condition expressed by harmonic conjugation. Such a lattice is represented by subspaces of a hermitian space, a vector space over a division ring with a nondegenerate hermitian form, called its coordinatizing space, and Solèr's theorem identifies the complete ones as the lattices of closed subspaces of infinite-dimensional real, complex or quaternionic Hilbert spaces. The completion of a pre-Solèr lattice is the lattice of orthogonally closed subspaces of its coordinatizing space (Theorem A.23).

In a Hilbert space, extend a closed subspace by one ray outside it, and subtract from a nonzero vector on the new ray its orthogonal projection onto the old subspace: the remainder is nonzero, perpendicular to the original subspace, and lies in the enlarged one. The atom test asks for the same direction in terms of the original lattice. The completion is orthomodular exactly when the original lattice passes this test on its atoms, its minimal nonzero events; the orthogonal of a set of atoms is the set of atoms orthogonal to all of them. The test is *MacNeille orthomodularity* (MOM): for every set of atoms equal to its double orthogonal and every atom outside it, the double orthogonal of the enlarged set contains an atom orthogonal to the original set (Theorem A.22).

No single two-valued sentence about the lattice expresses the test. Among pre-Solèr lattices, neither completeness nor (MOM) is expressed by a single sentence of two-valued $L_{\omega_1\omega}$ in the ortholattice language, even with constants naming the atoms that witness Solèr's condition (Proposition A.24). Our axioms take another route to the missing events: the closed spans of sequences of vectors, which in standard models supply every closed subspace.

What mathematicians gain

- A countable theory that recovers, in standard models, the full lattice of closed subspaces and the density-operator probabilities of a separable infinite-dimensional complex Hilbert space, with every such space and density operator represented.
- The density-operator representation for countably additive states on the possibly incomplete event lattices of standard models of the weaker theory.
- An exact criterion for when the MacNeille completion of a pre-Solèr lattice is orthomodular.

THE THREAD

What logic has given us

Kochen and Specker made a question about hidden variables into a question about representation, and a finite configuration of rays showed that the representation fails. Hrushovski and Pitowsky, a model theorist and a philosopher of physics, showed that finite configurations also carry positive information, constraining quantum probabilities to any prescribed accuracy, and that they can be found effectively. Both steps rest on compactness, and both live in finite dimension. Our paper takes the program to an infinite-dimensional theory with countably additive probabilities, on a separable Hilbert space, where compactness is lost and no finite subset of T^{01} is contradictory. The finite contradiction survives at the end of a search: every play of the challenger's strategy ends in one (Theorem A and Proposition 8.6), although the search must have a node with infinitely many immediate continuations (Corollary 8.4).

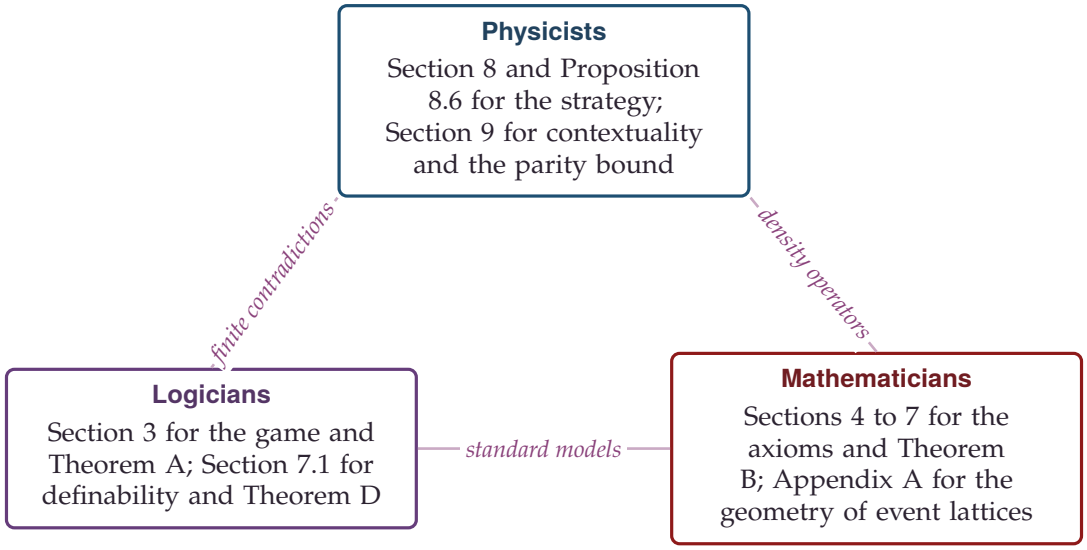
The search and the representation theorem meet in the language of the axioms. The sequences of vectors bring the countable basis within reach of the challenger, and their closed spans bring the full lattice of events within reach of the axioms in standard models (Theorem B); the logic of the search and the geometry of quantum events depend on the same addition to the language. Like Kochen and Specker, our paper refutes zero-one answers by finite contradictions; like Hrushovski and Pitowsky, it makes the probabilities themselves the object of the logic (Theorems B and D).

	Setting	**Logic**	**Certificate**
Kochen–Specker (1967)	finite dimension	first-order logic: completeness, in effect compactness	a fixed finite set of rays excluding zero-one assignments
Hrushovski–Pitowsky (2004)	finite dimension	first-order theories of the real and complex fields; compactness and decidability	finite sets of rays constraining probabilities; effective Gleason
Aubone–Iovino (2026)	separable infinite dimension, countable additivity	real-valued $L_{\omega_1\omega}(\mathbb{R})$; the model existence game, Čech-completeness and σ-projective definability	a well-founded tree whose plays end in finite contradictions; an adaptive search

Return to the particle with which we began. Suppose its answers are fixed in advance, and a projection keeps its answer in every measurement in which it occurs. The countably additive theory rules out such an assignment. In the four-dimensional example, the rays can be chosen before a single answer is given. In the countably additive theory, the builder's answers tell the challenger where to look. Once she has a block that carries probability one, each of the nine measurements must choose a ray inside it. The choices still come in pairs, and there are still nine of them. The contradiction is the one we already know. What logic has given us is a way to find it.

Where to read next

The paper, *Finite refutations in countably additive quantum probability*; theorem letters and numbers follow the paper; they refer to the current version on the project website, which the arXiv listing will match once its replacement appears.



Thoroughly examining. The game has an ancient shape. In Plato's dialogues Socrates tests a claim without claiming to know the truth himself: he asks questions until his interlocutor's own answers contradict one another, a method called the *elenchus*. The challenger tests the hidden-variable hypothesis in the same way, and Theorem A says that for theories like ours this always succeeds when the theory has no model. Socrates called himself a midwife: the knowledge was never his to give, and his questions only helped others bring forth what they already carried.

. . . the triumph of my art is in thoroughly examining whether the thought which the mind of the young man brings forth is a false idol or a noble and true birth.

Plato, *Theaetetus* 150c, translated by Benjamin Jowett

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