GLEASON, BELL, JAUCH, KOCHEN–SPECKER, AND THE PROBLEM OF HIDDEN VARIABLES IN QUANTUM MECHANICS

NOTES BY J IOVINO

For the University of Toronto–UT San Antonio Mathematics research group

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1. Preliminaries

Throughout these notes, H will denote a complex Hilbert space with $3 \leq \dim H \leq \infty$; everything below holds verbatim for real and quaternionic Hilbert spaces as well. Let $L(H)$ be the lattice of closed subspaces of H — an *orthomodular lattice* in the usual sense — which we identify with the corresponding orthogonal projections. This lattice carries a partial order $P \leq Q$ (inclusion of ranges), a meet $P \wedge Q$ (intersection of ranges), a join $P \vee Q$ (closed span of the ranges), an orthocomplement $P \mapsto P^\perp = I - P$, and extremal elements 0 and $I = H$. Two projections are *orthogonal*, $P \perp Q$, when their ranges are orthogonal, equivalently $P \leq Q^\perp$. A *basis* of $L(H)$ is the set of rank-one projections onto the vectors of some orthonormal basis of H.

2. Quantum measures and Gleason's Theorem

2.1. **Definition** (Quantum measure). A *quantum measure* (or *state*[1]) on $L(H)$ is a function $\mu : L(H) \to [0,1]$ with $\mu(I) = 1$ that is countably additive on orthogonal families: for pairwise orthogonal $\{P_i\}$,

$$\mu\Big(\bigvee_i P_i\Big) = \sum_i \mu(P_i).$$

A *two-valued quantum measure* is a quantum measure $v : L(H) \to \{0,1\}$.

This definition is purely measure-theoretic, on an abstract orthomodular lattice; nothing about it refers to physics. (When $L(H)$ is read as the projection lattice of a physical system, a two-valued quantum measure is usually called a *valuation*; see Appendix A.)

2.2. **Theorem** (Main Theorem). *If $3 \leq \dim H \leq \infty$, no two-valued quantum measure exists on $L(H)$.*

We give two independent proofs of this: a topological one, due to Bell (Section 4, the simpler of the two) and a combinatorial one, due to Kochen and Specker (Section 5).

2.3. **Theorem** (Gleason's Theorem [Gle57]). *Let $3 \leq \dim H \leq \infty$. Every quantum measure μ on $L(H)$ has the form $\mu(P) = \mathrm{Tr}(\rho P)$ for a unique density operator ρ (positive, trace-class, $\mathrm{Tr}(\rho) = 1$; in finite dimension, Hermitian positive semidefinite with trace 1).*

We take this theorem as given; its proof (Gleason's original argument, or any later streamlining) is a substantial piece of real analysis on the unit sphere, and $\dim H \geq 3$ is exactly the hypothesis under which it is proved. The bound is optimal: it cannot be weakened to $\dim H \geq 2$, and Gleason says so, in the paper that states the theorem: "In dimension two a frame function can be defined arbitrarily on a closed quadrant of the unit circle in the real case, and similarly in the complex

[1] Gleason called this object a *measure* [Gle57]; the quantum-logic tradition (Mackey [Mac63], Piron [Pir64]) relabeled it a *state*, and much of the older literature still uses that name; later authors (e.g. [Ham03]) reverted to *quantum measure*, the term used throughout these notes.

case" [Gle57, p. 886]. (A *frame function* is Gleason's term for exactly the kind of function that μ of Definition 2.1 restricts to on rays.)

2.4. *Remark.* We show, following Gleason [Gle57, p. 886], that the lower bound $\dim H \geq 3$ is optimal. On $H = \mathbb{R}^2$, parametrize rays by angle $\theta \in [0, \pi)$. Every orthonormal basis is a pair $\{\theta, \theta + \pi/2 \pmod{\pi}\}$. Define $v : [0, \pi) \to \{0, 1\}$ by

$$v(\theta) = \begin{cases} 1, & \text{if } \theta \in [0, \pi/2), \\ 0, & \text{if } \theta \in [\pi/2, \pi). \end{cases}$$

For every such pair, exactly one of $\theta, \theta + \pi/2 \pmod{\pi}$ lands in $[0, \pi/2)$, so v satisfies additivity on every basis: v is a genuine quantum measure in the sense of Definition 2.1, two-valued, on $L(\mathbb{R}^2)$. Its associated function $g(\theta) :- v(\theta)$ is visibly discontinuous (jumps at $\theta = 0, \pi/2$), so no density operator ρ satisfies $g(\theta) = \langle \theta | \rho | \theta \rangle$: v is a direct counterexample to the conclusion of Gleason's Theorem at $\dim H = 2$.

3. The "only one winner per basis" lemma

The following elementary lemma plays a pivotal role in these notes. it is the cornerstone of both Bell's proof of Theorem 2.2 (Section 4) and the Kochen–Specker logical compactness argument (Section 6).

3.1. **Lemma.** *Let v be a two-valued quantum measure on $L(H)$ and $\{e_1, \dots\}$ an orthonormal basis of H. Exactly one index i has $v(P_{e_i}) = 1$; the rest have $v(P_{e_i}) = 0$.*

Proof. For $i \neq j$, orthogonality gives $v(P_{e_i}) + v(P_{e_j}) = v(P_{e_i} \vee P_{e_j}) \in \{0, 1\}$, so at most one index has value 1. Summing over the basis, $\sum_i v(P_{e_i}) = v(I) = 1$; since each term is 0 or 1, exactly one equals 1. ☺

3.2. *Remarks.*

(1) The proof of Lemma 3.1 does not use $\dim H \geq 3$; it holds for any orthonormal basis of cardinality at least 2, in any dimension. This dimension-independence might suggest the lemma already settles Theorem 2.2 on its own; it does not.
(2) The lemma only pins down, separately, one privileged ray in *each* orthonormal basis; it says nothing about how the privileged ray of one basis relates to that of a different, overlapping basis. The argument presented in the next section supplies exactly that missing ingredient — Gleason's classification of *all* quantum measures, plus connectedness of $S(H)$ — to force the private choice of every basis to cohere into one single global function, and then breaks that global function against itself.

4. Bell's 1964 proof of Theorem 2.2, based on Gleason's Theorem and elementary topology

In these notes, we provide two proofs of the non-existence of hidden variables (Theorem 2.2). In this section, we present Bell's proof, which is based on Gleason's Theorem and elementary topology, and in the next two sections, we present the Kochen–Specker proof, which is based on combinatorics and a logical compactness argument. The two arguments are independent of each other and date from the mid-1960s. In the Timeline section, Section 7, we give a detailed list of historical developments.

Proof of Theorem 2.2, *via Gleason's Theorem and connectedness.* Suppose v is a two-valued quantum measure on $L(H)$. By Gleason's Theorem, $v(P) = \mathrm{Tr}(\rho P)$ for a fixed density operator ρ, so $g(\psi) := v(P_\psi) = \langle \psi | \rho | \psi \rangle$ is continuous on $S(H)$ (a matrix element of the fixed bounded operator ρ) and $\{0, 1\}$-valued. Since $S(H)$ is connected, a continuous $\{0, 1\}$-valued function on it is constant:

$g \equiv c$ for a fixed $c \in \{0, 1\}$. But by Lemma 3.1, on any orthonormal basis g equals 1 at exactly one point and 0 at the rest — it takes *both* values there, contradicting $g \equiv c$. ☺

4.1. *Remark.* The hypothesis $\dim H \geq 3$ is only used to invoke Gleason's Theorem. Nothing else in the proof depends on dimension.

Historical context of Bell's result. Credit here divides among three researchers, and it is worth being precise about the division, since the history that follows can otherwise read as though it points in two directions at once. Gleason proved Gleason's Theorem in 1957, with no stated connection to hidden variables. Jauch, in 1963, was the one who noticed that Gleason's already-published Theorem was relevant to the hidden-variables question and pointed it out to Bell — a real contribution, but not a mathematical one: nowhere in this history did Jauch construct an argument of his own. The proof given in this section — the argument that combines Gleason's Theorem with connectedness, or, equivalently, the value-clash argument of the Provenance paragraph just after this one, to derive Theorem 2.2 — was constructed by Bell. That division is exactly why this section is titled for Bell alone: Gleason's Theorem enters as a named, cited ingredient, and Jauch's role, real as it was, belongs to the history of how Bell came to write the proof, not to the authorship of the proof.

In more detail: in 1932 von Neumann gave the first no-hidden-variables argument, in *Mathematische Grundlagen der Quantenmechanik* [vN32]: it required any dispersion-free (hidden-variable) assignment to be linear even across non-commuting observables, an assumption later criticized by Bell as physically unmotivated, since there is no operational reason a hidden-variable value assigned to $A + B$ should equal the sum of the values assigned to A and B when A, B do not commute and so cannot be jointly measured (see the Timeline section, Section 7, for Bell's own, more colorful, verdict on this point). Gleason's 1957 Theorem [Gle57] was motivated by a question of G. Mackey's about measures on projection lattices and von Neumann algebras [Mac63], again with no stated bearing on the hidden-variables debate. The connection to hidden variables was drawn six years later: J. M. Jauch (1914–1974), then directing the Institute of Theoretical Physics at the University of Geneva, pointed out Gleason's Theorem to Bell in 1963. At that time, Jauch was supervising Constantin Piron's doctoral thesis at Geneva, *Axiomatique quantique* [Pir64], which recasts the propositional structure of quantum mechanics as an orthomodular lattice satisfying further axioms (the same setting used throughout these notes). Gleason's Theorem, a classification of measures on exactly such a lattice, sat squarely inside Jauch's research program; its bearing on von Neumann's argument would have been immediately apparent to him. Bell later recalled, specifically, that Jauch pointed out that Gleason's Theorem gives a strengthening of von Neumann's conclusion, with the additivity hypothesis required only for commuting (orthogonal) observables rather than for all observables [Bel87] — exactly the distinction between a *quantum measure* and a full lattice homomorphism; this too is an observation relating two existing theorems, not a new argument. Bell records the debt in a footnote of [Bel66]: "I am much indebted to Professor Jauch for drawing my attention to this work." Bell's own professional appointment at CERN was in particle and accelerator physics; quantum foundations was, by his own account and that of his colleagues, pursued alongside that work rather than as part of it [Ber14]. He wrote up the present consequence in 1964 while on a leave of absence from CERN at the Stanford Linear Accelerator Center (SLAC) — a sabbatical that freed him to work on foundational questions directly [Ber14] — and after a two-year publication delay it appeared as *On the Problem of Hidden Variables in Quantum Mechanics*, Rev. Mod. Phys. **38** (1966), 447–452 [Bel66]. (The Timeline section, Section 7, gives the fuller chronology. Kochen and Specker's work came in two stages: papers presented in 1963 and 1964 and published in 1965 [KS65b, KS65a], which introduce partial Boolean algebras, and the 1967 paper on hidden variables [KS67]. Bell's 1966 paper appeared between the two and is independent of both.)

Provenance of the argument. The preceding argument is, in its essential content, due to Bell. His own argument did not use the language of continuity or connectedness: he showed directly, using an auxiliary third orthogonal direction (available exactly because $\dim H \geq 3$), that two rays with v-value 1 and 0 cannot be arbitrarily close; since both values must occur on every basis (Lemma 3.1), two such close, oppositely valued rays are unavoidable, a contradiction. What we have included here is the now-standard topological restatement of Bell's argument.

5. Kochen and Specker's 1967 proof of Theorem 2.2, based on a finite combinatorial obstruction

The proof of Theorem 2.2 given in this section requires no measure theory and no continuity. It replaces Bell's use, in Section 4, of Gleason and connectedness with a finite, checkable substitute supplying the same missing ingredient: a way to force the separate "one winner per basis" choices of Lemma 3.1 to interact.

We work in $H = \mathbb{R}^4$. Gleason's Theorem and Theorem 2.2 hold there by the same proofs as for $\mathbb{C}^4$.

5.1. **Lemma** (Cabello–Estebaranz–García-Alcaine [CEGA96])**.** *The following* 9 *orthonormal bases of* $\mathbb{R}^4$ *(each a list of* 4 *pairwise orthogonal vectors) are built from* 18 *distinct vectors in total, each of which occurs in exactly* 2 *of the* 9 *bases.*

$$\begin{aligned}
B_1 &= \{(0,0,0,1),\ (0,0,1,0),\ (1,1,0,0),\ (1,-1,0,0)\}\\
B_2 &= \{(0,0,0,1),\ (0,1,0,0),\ (1,0,1,0),\ (1,0,-1,0)\}\\
B_3 &= \{(1,-1,1,-1),\ (1,-1,-1,1),\ (1,1,0,0),\ (0,0,1,1)\}\\
B_4 &= \{(1,-1,1,-1),\ (1,1,1,1),\ (1,0,-1,0),\ (0,1,0,-1)\}\\
B_5 &= \{(0,0,1,0),\ (0,1,0,0),\ (1,0,0,1),\ (1,0,0,-1)\}\\
B_6 &= \{(1,-1,-1,1),\ (1,1,1,1),\ (1,0,0,-1),\ (0,1,-1,0)\}\\
B_7 &= \{(1,1,-1,1),\ (1,1,1,-1),\ (1,-1,0,0),\ (0,0,1,1)\}\\
B_8 &= \{(1,1,-1,1),\ (-1,1,1,1),\ (1,0,1,0),\ (0,1,0,-1)\}\\
B_9 &= \{(1,1,1,-1),\ (-1,1,1,1),\ (1,0,0,1),\ (0,1,-1,0)\}
\end{aligned}$$

Verification. Orthogonality within each B_k is a finite check of $\binom{4}{2} = 6$ dot products, e.g. for B_3: $(1,-1,1,-1)\cdot(1,-1,-1,1) = 1+1-1-1 = 0$, $(1,-1,1,-1)\cdot(1,1,0,0) = 0$, $(1,-1,1,-1)\cdot(0,0,1,1) = 0$, $(1,-1,-1,1)\cdot(1,1,0,0) = 0$, $(1,-1,-1,1)\cdot(0,0,1,1) = 0$, $(1,1,0,0)\cdot(0,0,1,1) = 0$; the other eight bases check the same way. For the incidence count, tabulate the occurrences of each vector: $(0,0,0,1) \in B_1, B_2$; $(0,0,1,0) \in B_1, B_5$; $(1,1,0,0) \in B_1, B_3$; $(1,-1,0,0) \in B_1, B_7$; $(0,1,0,0) \in B_2, B_5$; $(1,0,1,0) \in B_2, B_8$; $(1,0,-1,0) \in B_2, B_4$; $(1,-1,1,-1) \in B_3, B_4$; $(1,-1,-1,1) \in B_3, B_6$; $(0,0,1,1) \in B_3, B_7$; $(1,1,1,1) \in B_4, B_6$; $(0,1,0,-1) \in B_4, B_8$; $(1,0,0,1) \in B_5, B_9$; $(1,0,0,-1) \in B_5, B_6$; $(0,1,-1,0) \in B_6, B_9$; $(1,1,-1,1) \in B_7, B_8$; $(1,1,1,-1) \in B_7, B_9$; $(-1,1,1,1) \in B_8, B_9$. Each of the 18 vectors occurs in exactly 2 bases, and 9 bases $\times$ 4 vectors $= 36 = 18 \times 2$, confirming the count is exhaustive. ☺

Proof of Theorem 2.2, *combinatorial, after Kochen and Specker* [KS67]*, in the streamlined form of* [CEGA96]. Suppose a two-valued quantum measure v exists on $L(\mathbb{R}^4)$. Restrict attention to the 18 rays of Lemma 5.1. Since each B_k is an orthonormal basis, additivity forces

$$\sum_{u \in B_k} v(u) = v(I) = 1 \qquad \text{for each } k = 1, \dots, 9.$$

Summing over all 9 bases,

$$\sum_{k=1}^{9} \sum_{u \in B_k} v(u) = 9.$$

Reorganize the left side by vector instead of by basis: each of the 18 vectors u occurs in exactly 2 of the B_k, so it contributes $2v(u)$ to the double sum. Hence,

$$\sum_{k=1}^{9} \sum_{u \in B_k} v(u) = \sum_{u} 2v(u) = 2 \sum_{u} v(u),$$

which is an even integer. However, the same quantity was computed above to equal 9, an odd integer. We have a contradiction. ☺

⧗ **Historical context of the Kochen–Specker result.** Kochen and Specker published *The Problem of Hidden Variables in Quantum Mechanics* [KS67] in 1967, giving an independent, self-contained combinatorial proof using 117 vectors in $\mathbb{R}^3$, phrased in the language of *partial Boolean algebras*: a model-theoretic framework treating the compatible (commuting) observables at each context as an ordinary Boolean algebra, glued together across contexts, and asking whether this partial structure embeds in a single total Boolean algebra, i.e. admits a two-valued homomorphism. Both authors came to the problem as logicians rather than physicists, which accounts for the difference in idiom from Bell's route through Gleason's Theorem (Section 4). Ernst Specker (1920–2011), professor at ETH Zürich from 1955, worked across mathematical logic, set theory, and combinatorics — among other results, he had disproved the axiom of choice in Quine's New Foundations a decade earlier — and had already announced the present obstruction, without proof, in 1960 [Spe60], saying only that "an elementary geometric argument" (*ein elementargeometrisches Argument*) establishes it. Simon Kochen (b. 1934) wrote his 1958 Princeton doctoral thesis, under Alonzo Church, on ultraproducts and arithmetical extensions of models, and in these same years was completing, with James Ax, the papers on valued fields that became the Ax–Kochen theorem; he joined the Princeton mathematics faculty in 1967, the year this paper appeared. This common background as logicians — model theory specifically for Kochen, logic, set theory, and combinatorics for Specker — is what makes the partial-Boolean-algebra embedding formulation the natural way for these two authors to have posed the question, rather than Bell's measure-theoretic route through Gleason's Theorem. (The Timeline section, Section 7, gives the fuller chronology.) The proof given above is not Kochen and Specker's original 117-vector construction but Cabello, Estebaranz, and García-Alcaine's 1996 streamlining of it to 18 vectors [CEGA96]. The **1996** entry of the Timeline section (Section 7) records how that reduction came about, by way of intermediate constructions of Peres and Kernaghan.

6. The Kochen–Specker argument that reduces the infinite case to a finite one, via logical compactness

The combinatorial proof of Section 5 hands you 18 explicit vectors. Bell's proof of Section 4 proves non-existence over the whole (infinite, uncountable) sphere without ever exhibiting a finite culprit. The following observation (which is genuine mathematical logic, not physics) shows that a finite culprit was *guaranteed* to exist the moment Bell's proof (Section 4) was known, before anyone constructed one.

Fix $\dim H = d < \infty$. Let $V = S(H)/{\sim}$ be the set of rays (unit vectors modulo sign, say), and let $\mathcal{E}$ be the set of all orthonormal bases of H, each basis being a set of exactly d rays. Treat each ray $u \in V$ as a propositional symbol φ_u, and to each basis $e = \{u_1, \ldots, u_d\} \in \mathcal{E}$ associate the propositional formula

$$\Phi_e :- \big(\varphi_{u_1} \vee \cdots \vee \varphi_{u_d}\big) \;\wedge \bigwedge_{1\le i<j\le d} \neg\big(\varphi_{u_i} \wedge \varphi_{u_j}\big) \qquad (\text{“exactly one } \varphi_{u_i} \text{ is true”}),$$

a formula in the d propositional symbols of e alone. A two-valued quantum measure on $L(H)$ is exactly a truth assignment to $\{\varphi_u : u \in V\}$ satisfying every Φ_e simultaneously (Lemma 3.1 shows “exactly one per basis” is the correct, and only, translation of the quantum-measure axioms into this propositional language).

6.1. **Proposition.** *If every finite subset of bases $\mathcal{E}_0 \subseteq \mathcal{E}$ admits* some *truth assignment to the (finitely many) rays occurring in $\mathcal{E}_0$ satisfying Φ_e for every $e \in \mathcal{E}_0$, then a two-valued quantum measure exists on all of $L(H)$.*

Proof. This proposition is an instance of the Compactness Theorem for propositional logic (equivalently, compactness of the product space $\{0,1\}^V$ in the product topology, i.e. Tychonoff's theorem for the two-point discrete space; see e.g. [End01]): a set T of propositional formulas, each in finitely many of a (possibly infinite or uncountable) set of variables, is simultaneously satisfiable if and only if every finite subset of T is. Apply this to $T = \{\Phi_e : e \in \mathcal{E}\}$; each Φ_e involves only the d propositional symbols of its own basis e, so a finite $T_0 \subseteq T$ corresponds exactly to a finite $\mathcal{E}_0 \subseteq \mathcal{E}$, and satisfiability of T_0 is exactly satisfiability of $\{\Phi_e : e \in \mathcal{E}_0\}$ by some assignment to the finitely many rays involved. ☺

6.2. **Corollary.** *For* $\dim H \ge 3$*, there exists a finite set of rays and a finite set of orthonormal bases drawn from them admitting no assignment satisfying “exactly one per basis” on every one of those bases.*

Proof. Contrapositive of Proposition 6.1: Theorem 2.2 (via Section 4) says no two-valued quantum measure exists on $L(H)$, i.e. T is not satisfiable; by compactness some finite $T_0 \subseteq T$ is not satisfiable, i.e. some finite set of bases $\mathcal{E}_0$, drawn from finitely many rays, admits no valid assignment. ☺

6.3. *Remark.* Corollary 6.2 is exactly “a finite Kochen–Specker set exists,” derived purely from Section 4 without constructing one. This derivation is genuinely non-constructive: the Compactness Theorem, proved via an ultrafilter/Tychonoff argument, uses (a weak form of) the Axiom of Choice, and gives no bound on the size or location of the finite witness — contrast the combinatorial proof of Section 5, which hands you one of size 18 with no appeal to choice at all. The two finite-flavored facts — “a finite witness exists” and “here is one, with 18 vectors” — are logically different achievements: the first is a two-paragraph corollary of connectedness plus a compactness theorem from 1930s logic; the second required constructing an explicit combinatorial object, and took from 1967 (Kochen–Specker, 117 vectors) to 1996 (Section 5, 18 vectors) to do economically, with the question of the true minimum still open in dimension 3 (see the **2024** entry of the Timeline section, Section 7).

7. Timeline

➔ **1932.** Von Neumann gives the first no-hidden-variables argument, in *Mathematische Grundlagen der Quantenmechanik* [vN32]; see Section 4 for the argument itself and its later criticism by Bell. Bell put that criticism more colorfully in a later interview: “the proof of von Neumann is not merely false but foolish!” [MC88].

➔ **1957.** Gleason proves Gleason's Theorem [Gle57], motivated by a question of G. Mackey's about measures on projection lattices and von Neumann algebras [Mac63], with no stated connection to the hidden-variables debate; see Section 4.

➔ **1960.** In *Die Logik nicht gleichzeitig entscheidbarer Aussagen* [Spe60], Specker asserts that the closed subspaces of a three-dimensional Hilbert space cannot be embedded in a Boolean algebra compatibly with the operations on orthogonal elements, saying only that "an elementary geometric argument" (*ein elementargeometrisches Argument*) shows this; no proof is given.

➔ **1963.** J. M. Jauch (1914–1974), a Swiss-American theoretical physicist then directing the Institute of Theoretical Physics at the University of Geneva, points out Gleason's Theorem to Bell as relevant to the hidden-variables question, at the same time as he is supervising Piron's doctoral thesis on a lattice-theoretic reformulation of quantum mechanics [Pir64]. See Section 4 for the fuller context, including Bell's own account of what Jauch specifically pointed out [Bel87] and Bell's acknowledgment of him in a footnote of [Bel66].

➔ **1963–1965.** Kochen and Specker present *Logical Structures Arising in Quantum Theory* at the 1963 Berkeley symposium on the theory of models and *The Calculus of Partial Propositional Functions* at the 1964 Jerusalem congress; both appear in 1965 [KS65b, KS65a]. These papers introduce partial Boolean algebras; the application to hidden variables would come later, in their 1967 paper.

➔ **1964–1966.** Bell writes *On the Problem of Hidden Variables in Quantum Mechanics* [Bel66], submitted in 1964 but delayed in publication for two years, until Rev. Mod. Phys. **38** (1966), 447. It both criticizes von Neumann's 1932 argument and gives an elementary route — close in spirit to Section 4 — from Gleason's Theorem to the non-existence of non-contextual hidden variables, deliberately avoiding the full strength of Gleason's proof. Bell's paper thus appeared after Kochen and Specker's papers on partial Boolean algebras (see the **1963–1965** entry) but before their 1967 paper on hidden variables, and was written independently of both.

➔ **1967.** Kochen and Specker publish *The Problem of Hidden Variables in Quantum Mechanics* [KS67], an independent, self-contained combinatorial proof using 117 vectors in $\mathbb{R}^3$, phrased in the language of partial Boolean algebras; see Section 5 for the logical idiom of that proof and for Kochen's and Specker's own backgrounds as logicians.

➔ **1990.** Mermin and, independently and shortly after, Peres give the "magic square" proof: a 3×3 grid of two-qubit Pauli observables whose row and column products force an algebraic contradiction for any non-contextual ± 1 assignment. It proves a related but formally different statement (state-independent contextuality for an algebra of observables, rather than the ray-coloring statement of Theorem 2.2) and is, like the proof of Section 5, a finite parity argument.

➔ **1991.** Peres reduces Kochen and Specker's construction to 33 vectors in $\mathbb{R}^3$ [Per91].

➔ **1996.** Cabello, Estebaranz, and García-Alcaine give the 18-vector construction in $\mathbb{R}^4$ of Section 5 [CEGA96]. Their construction did not start from scratch: it sharpens a further, 24-vector reduction Peres had found in $\mathbb{R}^4$ (sometimes called the "quantum tesseract" configuration) and a subsequent identification, by Kernaghan, of 96 distinct 20-vector critical subsets within it. Cabello, Estebaranz, and García-Alcaine found that certain 18-vector subsets are themselves critical, and noted, in their own words, that none of these is contained in any of the 96 known 20-vector critical sets, "which probably explains why they were not obtained previously." Adán Cabello, the lead author of the paper, received his Ph.D. from the Universidad Complutense de Madrid that same year and is now a professor of physics at the Universidad de Sevilla; he has since become one of the principal figures in the study of quantum contextuality, including the 2003 extension to POVMs of Appendix B.

➔ **2003.** Cabello extends contextuality to POVMs for a single qubit [Cab03] (Appendix B).

➔ **2004.** Barrett and Kent identify the finite-precision loophole for fuzzy, probabilistic assignments [BK04] (Appendix B).

➔ **2005.** Spekkens gives the operational definition of generalized non-contextuality closing that loophole [Spe05] (Appendix B).

➔ **2011.** Arends, Ouaknine, and Wampler establish, by computer-assisted search, that 18 is in fact the minimum possible size of a Kochen–Specker vector set in dimension 4 [AOW11]: the proof of Section 5 is not merely economical but optimal.

➔ **2024.** Ramanathan generalizes the outcome alphabet from $\{0, 1\}$ towards $[0, 1]$ [Ram24] (Appendix B). In parallel, the analogous minimality question in dimension 3 — open since Kochen and Specker's original 117-vector construction — remains unsettled: the best known explicit set has 31 vectors (an unpublished construction of Conway and Kochen, reported in [Per93]), while the best proved lower bound, obtained by SAT-solver and computer-algebra methods, stands at 24 [LBG24].

Why Kochen and Specker, and not Gleason, became the famous result. Given Bell's proof of Section 4, it is fair to ask why the timeline just given runs through 1967 at all: the non-existence of a two-valued quantum measure follows in a few lines from Gleason's 1957 Theorem, so why is the 1967 paper, and not the 1957 one, the result every student of quantum foundations learns by name? A few distinct things are responsible, and none of them is that the later authors found a technically deeper fact.

First, Gleason's own paper contains no trace of the hidden-variables question. It was written to settle a question of Mackey's about measures on projection lattices, with no stated bearing on non-contextuality, dispersion-free states, or the EPR debate. Nothing in it flags the two-valued case as distinguished. The consequence for hidden variables is visible to us because the valuation formalism connecting it to them (Appendix A) is already set up; in 1957 that connection had not been drawn by anyone, Gleason included. It took until 1963 (see the **1963** entry of the Timeline) for anyone to notice it could be used this way — six years in which the fact sat in print doing no work on the question it is now most famous for answering.

Second, once noticed, Bell's proof had bad luck getting into print (see the **1964–1966** entry of the Timeline): the two-year publication delay meant his paper appeared only in 1966, just a year before Kochen and Specker's 1967 paper on hidden variables, which became the standard reference.

Third, and most importantly, Kochen and Specker's proof is not a corollary of anything. It is self-contained: finite, combinatorial, checkable by hand, and it borrows nothing from measure theory or continuity. This self-containedness is a real asset independent of priority. A reader deciding whether non-contextual hidden variables are possible can verify the combinatorial proof of Section 5 completely without trusting an external classification theorem about the whole continuum of rays of a Hilbert space — Gleason's proof, by contrast, is substantial analysis, read by a comparatively narrow specialist audience even today. Being finite and elementary also made the result generative in a way a measure-theoretic corollary is not: it invited thirty years of shrinking the witness (from 117 to 33 vectors in $\mathbb{R}^3$, and from 24 to 20 to 18 in $\mathbb{R}^4$, the last since proved optimal [AOW11]), it generalizes to settings — POVMs, abstract hypergraphs of contexts, generalized probabilistic theories — with no Hilbert-space trace structure for Gleason's Theorem to even apply to, and it describes a finite, fixed set of compatible measurements that can be, and has been, physically implemented in a laboratory, which "all quantum measures on $L(H)$ at once" cannot.

So the historical asymmetry is not that Kochen and Specker found something Gleason missed; by the letter of the mathematics, he had it first. It is that recognizing the significance of a fact for a specific open question, and then re-deriving it in the most economical and self-contained form for the audience already concerned with that question, is itself a distinct and evidently more consequential contribution than being the first to establish the fact in some other, unrelated context. The same pattern recurs with Bell's own inequality theorem, which eclipsed any more abstract precursor for exactly the same reasons: elementary, checkable, and aimed directly at the question the field already cared about.

Appendix A. Physical interpretation and extensions

Nothing in this appendix is a proof of Theorem 2.2. The appendix explains what the theorem is *about* physically, and why a purely mathematical uniqueness-of-classification fact (Gleason) does not, by itself, close off the questions the theorem was designed to answer. Readers who are only interested in mathematics can skip this appendix.

What a two-valued quantum measure represents. When H is the state space of a quantum system, closed subspaces correspond to yes/no physical properties (a system either does or does not have its state in the range of P), and a quantum measure $\mu(P)$ is read as the probability that a measurement of the property P returns "yes." A *two-valued* quantum measure v is then the mathematical shadow of a specific physical hypothesis: that every property has a definite, pre-existing truth value, the same value however and whenever it is measured, jointly consistent with the sum rule that mutually compatible (orthogonal) properties obey. This hypothesis is precisely what is meant by a *non-contextual hidden-variable assignment.* So Theorem 2.2, read physically, says: quantum mechanics (for any system with $\dim H \geq 3$) admits no such assignment. This physical reading is the sense in which the proof of Theorem 2.2 in Section 4 — which is nothing but Gleason's Theorem plus Lemma 3.1 plus connectedness — already *is* a complete proof that Gleason's Theorem rules out non-contextual hidden variables.

Why prove it again, physically. Section 6 and the discussion following the Timeline already gave the purely mathematical reasons (logical strength, constructivity) for the further proofs. Two more reasons live squarely in physics, not mathematics, and are worth stating as such rather than dressed up as further rigor:

(1) *Experimental realizability.* The proof of Gleason's Theorem concerns *all* quantum measures on the whole (continuum-sized) $L(H)$ at once; no laboratory ever measures "all of $L(H)$." The 18 rays and 9 bases of Section 5, by contrast, are a finite, fixed list of compatible measurement settings that has actually been implemented experimentally to test contextuality. A physical no-go theorem that hands you a finite protocol is answering a different, additional question — "can this be tested?" — that a classification theorem about the whole lattice does not address.

(2) *Portability beyond Hilbert space.* The statement "no consistent $\{0,1\}$-assignment respecting the sum rule exists on this hypergraph of contexts" makes sense for any orthomodular lattice, or indeed any abstract hypergraph of measurement contexts, whether or not it arises from a Hilbert space at all. Gleason's Theorem, tied intrinsically to the operator-algebraic structure $\mathrm{Tr}(\rho\,\cdot)$, does not. This asymmetry is why the contextuality literature — including the generalizations of Appendix B, and applications treating contextuality as a resource for quantum computation — is phrased combinatorially, in the language of Theorem 2.2, and essentially never in the language of Gleason's Theorem.

Appendix B. Attempts beyond classical propositional logic

The proofs of Section 4 and Section 5 work throughout with propositional, two-valued ($\{0,1\}$-valued) logic. Two natural ways to go beyond that setting are to keep the values binary but enrich the logic, adding bound variables and quantifiers, or to keep the propositional structure but enrich the values, allowing outcomes in $[0,1]$ rather than $\{0,1\}$. This appendix records what has been done in each direction.

Richer logics: quantifiers and first-order quantum logic.

1981. Gaisi Takeuti, already known for Boolean-valued analysis, publishes *Quantum Set Theory* [Tak81]. The construction mirrors the Scott–Solovay Boolean-valued models of ZFC exactly, with the complete Boolean algebra replaced by the (σ-complete) orthomodular lattice L of projections of a von Neumann algebra: a quantum universe $V^{(L)}$ is built by transfinite recursion, and quantifiers receive a literal semantics, $[\![(\forall x)\varphi(x)]\!] = \bigwedge_{x\in V^{(L)}} [\![\varphi(x)]\!]$, an infimum taken in L. Real numbers in this universe become self-adjoint operators; this is genuine first-order (indeed set-theoretic) quantification, evaluated in a non-classical, non-Boolean truth-value algebra.

1998–2002. Isham and Butterfield, joined for one paper by Hamilton, publish the four-part series *A Topos Perspective on the Kochen–Specker Theorem* [IB98, BI99, HIB00, IB02]. Motivated by the objection that the orthomodular lattice is too poorly behaved (non-distributive) to serve as a logic with an ordinary semantics at all, they discard it and use instead the internal, intuitionistic, genuinely quantified higher-order logic of a topos of presheaves over the category of commuting subalgebras ("contexts"). Distributivity is recovered; truth values become context-dependent sieves in a Heyting algebra instead of elements of an orthomodular lattice. The first paper in the series reformulates Theorem 2.2 itself as the non-existence of a global element of the associated *spectral presheaf* — a direct, quantifier-rich redescription of the central result of these notes. Isham and Döring later extend this into a full topos-theoretic reformulation of physics, *A Topos Foundation for Theories of Physics* [DI08].

1999. Satoko Titani's *A Lattice-Valued Set Theory* [Tit99] generalizes Takeuti's construction from orthomodular lattices to arbitrary complete lattices as the truth-value algebra, placing quantum set theory inside a wider family alongside Heyting-valued (intuitionistic) set theories, with general completeness results.

2007–2020s. Masanao Ozawa turns quantum set theory from a suggestive construction into a working technical apparatus. *Transfer Principle in Quantum Set Theory* [Oza07] proves a rigorous dictionary carrying ZFC theorems about the reals over to theorems about self-adjoint operators inside $V^{(L)}$, in the spirit of the transfer principle of nonstandard analysis; *Orthomodular-Valued Models for Quantum Set Theory* [Oza09] tightens the model theory; later work reforms the semantics to recover De Morgan's laws and extends the framework to quantum conditionals. Ozawa's own motivation throughout is to state and prove a rigorous, quantifier-based version of Heisenberg's uncertainty principle — exactly the kind of first-order reasoning a purely propositional logic cannot support.

What about garden-variety (classical) first-order logic? Here the honest answer has a sting in it: it was tried, directly, in the most literal sense — treat the orthogonality relation on the closed subspaces of a Hilbert space as a classical first-order structure and ask the ordinary model-theoretic questions — and the theorems of the field itself say that it provably cannot do the job.

1963–1964. Before any of the above, Mackey's axiomatic reconstruction of quantum mechanics [Mac63] and Piron's representation theorem [Pir64] inaugurate a different and much more classical tradition: state the physical postulates (a set of "questions," or an atomistic, orthomodular, irreducible lattice satisfying the covering law) as ordinary axioms in ordinary set-theoretic mathematics, with no non-classical logic anywhere, and *prove*, as a classical representation theorem, that any structure satisfying them is isomorphic to the subspace lattice of a generalized Hilbert space. The "quantum" content lives entirely in which structures satisfy the axioms, not in the logic used to reason about them.

1972 onward. Foulis and Randall's *manuals* and *test spaces* [FR72] continue this classical-metatheory tradition at a more operational level: outcomes and tests are ordinary sets, and non-Boolean structure emerges only downstream, as a theorem about which propositions can be jointly tested, not as a feature of the logic used to derive it.

1984. Robert Goldblatt's *Orthomodularity is not elementary* [Gol84] asks the model-theoretic question directly: is the class of Hilbert lattices axiomatizable by classical first-order sentences in the language of the orthogonality relation? He proves that it is not: an infinite-dimensional pre-Hilbert space and its metric completion are elementarily equivalent as first-order structures, yet only the complete one is orthomodular. Metric completeness — exactly the ingredient Theorem 2.2 and Gleason's Theorem both quietly rely on (closedness of subspaces, convergence on $S(H)$) — is provably invisible to classical first-order logic here.

2001, 2007. Hardy's *Quantum Theory from Five Reasonable Axioms* [Har01] and Barrett's *Information Processing in Generalized Probabilistic Theories* [Bar07] revive the Mackey/Piron operational program in a modern, information-theoretic idiom — again ordinary classical mathematics, with quantum theory singled out as one solution, among a classified family, of first-order-statable convexity and composition axioms.

2016. Tobias Fritz's *Quantum Logic is Undecidable* [Fri16], building on Slofstra's work on embeddability problems for groups, shows that the classical first-order theory of complex Hilbert lattices is undecidable already at the level of quasi-identities: there is no algorithm deciding whether an implication between equations and orthogonality relations holds in every Hilbert space.

B.1. *Remark.* So garden-variety first-order logic was not overlooked; it was tried on the most natural target (the orthogonality relation on subspaces) and shown, by the theorems of the field itself, to be structurally too weak to pin down the class of Hilbert lattices at all (Goldblatt) and undecidable where it does apply (Fritz). This double failure — too weak to pin down the class, undecidable where it does apply — is a large part of why the two serious efforts at a genuinely quantified quantum logic (Takeuti's line and the topos line) both abandoned classical semantics rather than push harder on it: Takeuti moved the non-classicality into the truth-value algebra, Isham–Butterfield moved it into an intuitionistic internal logic, and both, not coincidentally, gained back exactly the property (extra structure that a bare classical first-order language cannot see) that Goldblatt's theorem says is missing. The Mackey/Piron/Foulis–Randall/Hardy/Barrett line, meanwhile, never tried to axiomatize Hilbert lattices in first-order logic at all: it uses classical first-order logic only as ordinary mathematics is always written, aimed at representation theorems about a differently chosen structure (a convex state space, a test space) rather than at characterizing subspace lattices up to elementary equivalence.

Richer values: $[0,1]$-valued and POVM-based assignments. Theorem 2.2 rules out *sharp*, two-valued, non-contextual assignments. A natural physical question is whether an analogous no-go survives once outcomes, or the physical measurements themselves, are allowed to be *unsharp*: valued in $[0,1]$ rather than $\{0,1\}$, as with positive-operator-valued measures (POVMs) and their effects $0 \leq E \leq I$. The answer depends on which of two things is fuzzed, and pulls in opposite directions.

Fuzzing the measurements, keeping assignments sharp: the no-go strengthens, even at $\dim H = 2$. Cabello [Cab03] constructed a Kochen–Specker-type contextuality proof for a single qubit using POVMs: although $L(\mathbb{R}^2)$ itself admits a two-valued quantum measure (Remark 2.4), the richer structure of POVM effects on $\mathbb{C}^2$, together with the operational constraints relating different POVMs realizing the same effect, admits no consistent non-contextual assignment at all. The obstruction genuinely extends past sharp projections, into a regime Theorem 2.2 does not reach.

Fuzzing the assignments, keeping measurements sharp: the naive no-go can be evaded, and needs a better physical definition to survive. Replace the deterministic $v(P) \in \{0,1\}$ by a probabilistic response function $\xi_P(\lambda) \in [0,1]$: the probability of outcome 1 given hidden ontic state λ. Barrett and Kent [BK04] observed that once a real experiment resolves outcomes only to finite precision, non-contextual hidden-variable models built from such fuzzy response functions can reproduce quantum statistics to arbitrary accuracy — so naive basis-independence becomes physically toothless once assignments are allowed to be fuzzy. Spekkens [Spe05] closed this gap with a fully operational definition of non-contextuality (operationally indistinguishable procedures must receive identical ontological representations), under which fresh no-go theorems survive, including ones that already hold at $\dim H = 2$.

Extending the outcome alphabet directly. Ramanathan [Ram24] proves a Kochen–Specker-type theorem for assignments valued in a finite alphabet $\{0, p, 1-p, 1\} \subset [0,1]$ rather than only $\{0,1\}$, for finite vector configurations in $\mathbb{C}^3$; the case $p = \frac{1}{2}$ rules out "fundamentally binary" hidden-variable theories.

B.2. *Remark.* Fuzzing the *measurements* strengthens Theorem 2.2 (Cabello, reaching $\dim H = 2$); fuzzing the *assignments* under the naive definition of non-contextuality weakens it (Barrett–Kent), which is why Spekkens's operational redefinition was needed to recover a robust no-go there; extending the outcome set while keeping assignments deterministic recovers a close cousin of the original theorem outright (Ramanathan). In each case the substantive work is a fresh rigidity argument playing the role Gleason's Theorem played earlier in these notes — never a restatement of the bare fact that local assignments respect the connectives.

References

[AOW11] M. Arends, J. Ouaknine, and C. W. Wampler, *On searching for small Kochen–Specker vector systems*, Graph-Theoretic Concepts in Computer Science (WG 2011), Lecture Notes in Computer Science, vol. 6986, Springer, 2011, pp. 23–34.

[Bar07] J. Barrett, *Information processing in generalized probabilistic theories*, Phys. Rev. A **75** (2007), 032304.

[Bel66] J. S. Bell, *On the problem of hidden variables in quantum mechanics*, Rev. Mod. Phys. **38** (1966), 447–452.

[Bel87] ———, *Speakable and unspeakable in quantum mechanics: Collected papers on quantum philosophy*, Cambridge University Press, Cambridge, 1987.

[Ber14] R. A. Bertlmann, *John Bell and the nature of the quantum world*, J. Phys. A: Math. Theor. **47** (2014), 424007.

[BI99] J. Butterfield and C. J. Isham, *A topos perspective on the Kochen–Specker theorem: II. Conceptual aspects and classical analogues*, Int. J. Theor. Phys. **38** (1999), 827–859.

[BK04] J. Barrett and A. Kent, *Non-contextuality, finite precision measurement and the Kochen–Specker theorem*, Stud. Hist. Phil. Mod. Phys. **35** (2004), 151–176, arXiv:quant-ph/0309017.

[Cab03] A. Cabello, *Kochen–Specker theorem for a single qubit using positive operator-valued measures*, Phys. Rev. Lett. **90** (2003), 190401, arXiv:quant-ph/0210082.

[CEGA96] A. Cabello, J. M. Estebaranz, and G. García-Alcaine, *Bell–Kochen–Specker theorem: A proof with 18 vectors*, Phys. Lett. A **212** (1996), 183–187.

[DI08] A. Döring and C. J. Isham, *A topos foundation for theories of physics: I–IV*, J. Math. Phys. **49** (2008), 053515–053518.

[End01] H. B. Enderton, *A mathematical introduction to logic*, 2nd ed., Academic Press, 2001, Compactness theorem for propositional logic.

[FR72] D. J. Foulis and C. H. Randall, *Operational statistics. I. Basic concepts*, J. Math. Phys. **13** (1972), 1667–1675, and subsequent papers on manuals and test spaces.

[Fri16] T. Fritz, *Quantum logic is undecidable*, 2016, arXiv:1607.05870.

[Gle57] A. M. Gleason, *Measures on the closed subspaces of a Hilbert space*, J. Math. Mech. **6** (1957), 885–893.

[Gol84] R. Goldblatt, *Orthomodularity is not elementary*, J. Symbolic Logic **49** (1984), 401–404.

[Ham03] J. Hamhalter, *Quantum measure theory*, Fundamental Theories of Physics, vol. 134, Kluwer Academic Publishers, Dordrecht, 2003.

[Har01] L. Hardy, *Quantum theory from five reasonable axioms*, 2001, arXiv:quant-ph/0101012.

[HIB00] J. Hamilton, C. J. Isham, and J. Butterfield, *A topos perspective on the Kochen–Specker theorem: III. Von Neumann algebras as the base category*, Int. J. Theor. Phys. **39** (2000), 1413–1436, arXiv:quant-ph/9911020.

[IB98] C. J. Isham and J. Butterfield, *A topos perspective on the Kochen–Specker theorem: I. Quantum states as generalised valuations*, Int. J. Theor. Phys. **37** (1998), 2669–2733, arXiv:quant-ph/9803055.

[IB02] ———, *A topos perspective on the Kochen–Specker theorem: IV. Interval valuations*, Int. J. Theor. Phys. **41** (2002), 613–639.

[KS65a] Simon Kochen and Ernst P. Specker, *The calculus of partial propositional functions*, Logic, Methodology and Philosophy of Science (Proceedings of the 1964 International Congress, Jerusalem) (Y. Bar-Hillel, ed.), North-Holland, Amsterdam, 1965, pp. 45–57.

[KS65b] ———, *Logical structures arising in quantum theory*, The Theory of Models (Proceedings of the 1963 International Symposium at Berkeley) (J. W. Addison, L. Henkin, and A. Tarski, eds.), North-Holland, Amsterdam, 1965, pp. 177–189.

[KS67] S. Kochen and E. P. Specker, *The problem of hidden variables in quantum mechanics*, J. Math. Mech. **17** (1967), 59–87.

[LBG24] Z. Li, C. Bright, and V. Ganesh, *A SAT solver and computer algebra attack on the minimum Kochen–Specker problem*, Proc. IJCAI 2024, 2024, arXiv:2306.13319.

[Mac63] G. W. Mackey, *Mathematical foundations of quantum mechanics*, W. A. Benjamin, New York, 1963.

[MC88] C. Mann and R. Crease, *Interview: John Bell*, Omni **10** (1988), no. 8, 84–86, 88, 90, 92, 121.

[Oza07] M. Ozawa, *Transfer principle in quantum set theory*, J. Symbolic Logic **72** (2007), 625–648, arXiv:math/0604349.

[Oza09] ———, *Orthomodular-valued models for quantum set theory*, Rev. Symbolic Logic (2009), arXiv:0908.0367.

[Per91] A. Peres, *Two simple proofs of the Kochen–Specker theorem*, J. Phys. A: Math. Gen. **24** (1991), L175–L178.

[Per93] ———, *Quantum theory: Concepts and methods*, Kluwer, Dordrecht, 1993, Reports the unpublished 31-vector Conway–Kochen construction in $\mathbb{R}^3$.

[Pir64] C. Piron, *Axiomatique quantique*, Helv. Phys. Acta **37** (1964), 439–468.

[Ram24] R. Ramanathan, *Generalised Kochen–Specker theorem for finite non-deterministic outcome assignments*, npj Quantum Information **10** (2024), 99.

[Spe60] Ernst Specker, *Die Logik nicht gleichzeitig entscheidbarer Aussagen*, Dialectica **14** (1960), 239–246.

[Spe05] R. W. Spekkens, *Contextuality for preparations, transformations, and unsharp measurements*, Phys. Rev. A **71** (2005), 052108, arXiv:quant-ph/0406166.

[Tak81] G. Takeuti, *Quantum set theory*, Current Issues in Quantum Logic (E. G. Beltrametti and B. C. van Fraassen, eds.), Ettore Majorana International Science Series, Plenum Press, New York, 1981, pp. 303–322.

[Tit99] S. Titani, *A lattice-valued set theory*, Archive for Mathematical Logic **38** (1999), 395–421.

[vN32] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin, 1932.