TWO GAMES, TWO DICHOTOMIES: STABILITY AND INDEPENDENCE IN BANACH SPACES

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ABSTRACT. We show that two games on the space of types of a separable Banach space decide stability and independence, each with a quantitative margin. Player II wins the commitment game for every type exactly when the space is stable, and wins the revision game, which at tolerance zero is Kiss's game, for every type exactly when the space has no independent sequence of positive gap. An infinite ladder of gap δ places some type at uniform distance at least $\delta/2$ from every polynomial in finitely many norms, and gives Player I a winning strategy at every tolerance below δ; an independent sequence of gap δ gives Player I a winning strategy at every tolerance below $\delta/4$, and makes the types act as $2^{\mathfrak{c}}$ distinct functions. A uniform gap for infinite ladders in every block subspace excludes, with a margin, every infinite-dimensional subspace close to a stable space. Every space with a basis has a stable block subspace or a block subspace with such a uniform gap, with parameters confined to one finite-dimensional space.

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The most recent version of this paper is available at https://utsa-quantum-math.github.io/resources/two-games-two-dichotomies/.

1. Introduction

In this paper we show that two games decide stability and independence in Banach spaces, and that each game yields a dichotomy. Both games are played on the compact space of types of a separable space: Player I moves toward a limit type, and Player II computes the value of a fixed type at the limit while the play goes on. In the *commitment game* the announcements of Player II are binding; in the *revision game* an announcement may be withdrawn, provided that the announcements settle down. The first dichotomy concerns stability in the sense of Krivine and Maurey: either Player II wins the commitment game for every type, and then every type is a uniform limit of polynomials in finitely many norms, or Player I wins the game for some type, by a strategy read off from an infinite ladder (Theorem A). The second dichotomy concerns independence: either Player II wins the revision game for every type, and then every type is of the first Baire class and the types act as at most $\mathfrak{c}$ distinct functions, or Player I wins the game for some type, by a strategy read off from an independent sequence, and the types act as $2^{\mathfrak{c}}$ distinct functions (Theorem B). In each dichotomy both alternatives have positive content: each comes with a winning strategy, and Player I wins with a margin given by the gap of the ladder or of the independent sequence.

Krivine and Maurey called a separable Banach space X *stable* when, for all bounded sequences $(x_m)_{m\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of vectors of X and all ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$,

$$\lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \|x_m + y_n\| = \lim_{n,\mathcal{V}} \lim_{m,\mathcal{U}} \|x_m + y_n\|,$$

where $\lim_{m,\mathcal{U}}$ denotes the limit along $\mathcal{U}$, which exists because the sequences are bounded; they proved that every infinite-dimensional stable space contains some ℓ_p, $1 \le p < \infty$, almost isometrically [11]. The spaces ℓ_p and L_p, $1 \le p < \infty$, are stable; the space c_0 is not, and Tsirelson's space, which contains no ℓ_p and no c_0 [18, 5], has no infinite-dimensional stable subspace at all. The definition of Krivine and Maurey is a symmetry, the agreement of two iterated limits, and a symmetry by itself does not measure how far a space is from being stable. Grothendieck's double limit criterion [6] turns the symmetry into the continuity of types [7], but continuity, like symmetry, is a single condition, and the criterion does not reach the weaker tameness that lies below stability. That weaker tameness is the subject of Rosenthal's ℓ_1 theorem and of the theorem of Bourgain, Fremlin and Talagrand on pointwise compact sets of Baire measurable functions [14, 15, 3], which concern sequences of continuous functions on a compact metrizable space. The two games of this paper bring both levels to the norm: the alternatives of these theorems acquire obstructions written in the norm, margins that a winning strategy makes explicit, and consequences for the geometry of subspaces.

A *type* of a separable space Y is a function $\tau(y) = \lim_{i,\mathcal{U}} \|x_i + y\|$ on Y, for a bounded sequence (x_i) in Y and an ultrafilter $\mathcal{U}$, as in the work of Krivine and Maurey. The types given by sequences in the unit ball B_Y form a compact metrizable space K_Y for pointwise convergence on Y, and a type τ acts on K_Y by $\tau(\sigma) = \lim_{i,\mathcal{U}} \sigma(x_i)$. The action depends on the sequence and the ultrafilter, and in a stable space on the type alone; statements about the action of the types on K_Y concern all the functions obtained in this way.

In the games for a bounded function f on K_Y, Player I moves through K_Y by steps that shrink geometrically, toward a limit σ. In the commitment game Player II announces nested closed intervals and must trap $f(\sigma)$ in intervals whose lengths tend to 0; in the revision game Player II announces real numbers and needs only that the numbers converge to $f(\sigma)$. Both games admit a *tolerance* $\eta \geq 0$, which relaxes the final requirement on Player II by η (Definition 2.1).

The two obstructions are configurations of vectors. An *infinite ladder of gap* δ is a pair of sequences (x_i) and (y_j) in B_Y such that, for some number r,

$$\|x_i + y_j\| \leq r \quad \text{if } i \leq j, \qquad \|x_i + y_j\| \geq r + \delta \quad \text{if } i > j.$$

A sequence (x_i) in B_Y is *independent with gap* δ if there is a number r such that, for all disjoint finite sets $A, B \subseteq \mathbb{N}$, some $y \in B_Y$ satisfies

$$\|x_i + y\| \leq r \quad \text{for } i \in A, \qquad \|x_i + y\| \geq r + \delta \quad \text{for } i \in B.$$

Against the first m vectors $x_1, \ldots, x_m$, the vectors y_j of a ladder realize only m patterns of small and large values, given by nonempty initial segments, while the vectors y of an independent sequence realize all 2^m patterns.

The first dichotomy also has an analytic face. A *norm polynomial* is a function $\sigma \mapsto P(\sigma(a_1), \ldots, \sigma(a_n))$ on K_Y, for vectors $a_1, \ldots, a_n$ of Y and a real polynomial P; at a vector y of B_Y, regarded as the type $\|y + \cdot\|$, the norm polynomial computes $P(\|y + a_1\|, \ldots, \|y + a_n\|)$ from finitely many measurements, the distances from y to the points $-a_1, \ldots, -a_n$, by one algebraic formula. By the lattice form of the Stone–Weierstrass theorem, maxima and minima of affine combinations of the measurements may replace the polynomials.

Theorem A (the first dichotomy). *For a separable Banach space Y, the following are equivalent:*

(i) *the space Y is stable;*
(ii) *every type of Y acts on K_Y as a uniform limit of norm polynomials;*
(iii) *Player II has a winning strategy in the commitment game for every type of Y acting on K_Y.*

If Y carries an infinite ladder of gap δ, then some type of Y acts on K_Y at uniform distance at least $\delta/2$ from every norm polynomial, and for that type and every $\eta < \delta$, Player I has a winning strategy in the commitment game with tolerance η.

Theorem B (the second dichotomy). *For a separable Banach space Y, the following are equivalent:*

(i) *every type of Y acts on K_Y as a function of the first Baire class;*
(ii) *Player II has a winning strategy in the revision game for every type of Y acting on K_Y;*
(iii) *the space Y carries no independent sequence of positive gap.*

If Y carries an independent sequence of gap δ, then for some type of Y and every $\eta < \delta/4$, Player I has a winning strategy in the revision game with tolerance η, and the types of Y act on K_Y as $2^{\mathfrak{c}}$ distinct functions; otherwise the types act as at most $\mathfrak{c}$ distinct functions.

The key observation is that both games make sense for a single bounded function on a compact metric space, where each game, at tolerance zero, is determined and decides a classical property (Proposition 2.2). Player II wins the commitment game exactly when the function is continuous, and wins with tolerance η when the oscillation of the function stays below η. At tolerance zero the revision game is the game of Kiss, who proved that Player II wins exactly when the function is of the first Baire class and that the game is determined [10]. The two games differ in a single rule, whether Player II may withdraw an announcement, and the two dichotomies differ in exactly that rule. A ladder gives Player I a winning strategy in the commitment game: the type given by the vectors x_i is small at the limit σ of the vectors y_j and large at each y_j, and Player I moves from σ to a nearby vector y_j as soon as Player II commits to an interval shorter than the gap. An independent sequence gives Player I a winning strategy in the revision game: the sequence maps a closed set of types continuously onto the Cantor set, and on a minimal such set Player I moves back and forth between types at which a type of Y is small and types at which the same type is large, so that Player II must change the announced value infinitely often.

A stable space carries no independent sequence of positive gap, since continuous functions are of the first Baire class. The space c_0 is wild in both games: for disjoint finite sets $A, B \subseteq \mathbb{N}$, the vector $y = \frac{1}{2}\sum_{i\in B} e_i$ satisfies $\|e_i + y\| = 1$ for $i \notin B$ and $\|e_i + y\| = \frac{3}{2}$ for $i \in B$, so the unit vectors form an independent sequence of gap $\frac{1}{2}$. For single functions the two levels are distinct; whether some Banach space is unstable and yet carries no independent sequence of positive gap is Question 7.3.

Compactness of K_Y enters the first dichotomy through Grothendieck's criterion, and a third game shows how far compactness can be dispensed with. Namioka's theorem on separately continuous functions [13], in the form that Christensen and Saint-Raymond gave it, with a topological game in place of compactness [4, 16], yields a uniform statement: in a stable space, at the points of a dense G_δ subset of any set of types on which one player of that game has no winning strategy, closeness of types at finitely many vectors forces closeness of their actions uniformly on K_Y (Proposition 5.3).

The games decide the regularity of the types of one space; among block subspaces, a single positive gap for infinite ladders gives a uniform geometric obstruction. When the same gap occurs without parameters in every block subspace, the obstruction reaches every infinite-dimensional subspace and excludes sufficiently small distortions of stable spaces (Theorem C), and this consequence motivates the following dichotomy problem.

Question 1.1 (the dichotomy problem). Does every Banach space with a basis have a block subspace Y such that either Y is stable, and therefore contains some ℓ_p almost isometrically, or there is $\delta > 0$ such that every block subspace of Y carries an infinite ladder of gap δ?

By Theorem A, the dichotomy problem asks for a block subspace Y in which either Player II wins the commitment game for every type of every block subspace, or Player I wins the commitment game with one tolerance $\eta > 0$ for some type of every block subspace. The wild alternative excludes, with a quantitative margin, every infinite-dimensional subspace that is close to a stable space.

Theorem C. *Let Y be a block subspace of a space with a basis, and suppose that every block subspace of Y carries an infinite ladder of gap δ. Then no infinite-dimensional subspace of Y is $(1+\varepsilon)$-isomorphic to a stable space for $\varepsilon < \min(\delta/9, 1/4)$; in particular, Y contains no ℓ_p, $1 \le p < \infty$, almost isometrically.*

A stabilization argument yields a uniform obstruction with *parameters*, fixed vectors c added inside the norm, $\|x_i + y_j + c\|$ (Definition 6.1), confined to one finite-dimensional space. If there is no stable block subspace, there are a further block subspace, a positive gap and a finite-dimensional space of parameters, all fixed before the block subspaces are considered, such that every block subspace carries an infinite ladder of this gap with a parameter

from the fixed space; the parameter may vary with the block subspace, the gap and the finite-dimensional space do not.

Theorem D. *Let X be a Banach space with a basis. Then either X has a stable block subspace, or there are a block subspace Y, a finite-dimensional subspace F of X and $\delta > 0$ such that every block subspace of Y carries an infinite ladder of gap δ with a parameter from F.*

The parameter arises when the first vectors of a block basis are discarded: the discarded parts of the vectors of a ladder converge in a finite-dimensional space and are frozen into a single vector (Lemma 6.3). Confining these parameters to one finite-dimensional space yields the uniform stabilization theorem. The distinction between parameters inside and outside a subspace has geometric content: in a suitably normed space $Z \oplus \mathbb{R}$, with Z a Hilbert space, the stable subspace Z carries an infinite ladder of a fixed gap when a parameter from the extra dimension is allowed (Example 6.5). Ladders with parameters therefore do not certify instability, whereas Theorem C uses a hereditary gap without parameters. The dichotomy problem asks for the elimination of the parameter in Theorem D. Tsirelson's space is the natural first test, and an arbitrarily distortable space, such as the space of Schlumprecht, is the natural place to look for a counterexample. Infinite ladders, and not long finite ladders, are what matter here: the stable space $(\bigoplus_n \ell_\infty^n)_{\ell_2}$ carries finite ladders of gap 1 of every length, and long finite ladders measure instead the failure of superstability, the stability of every ultrapower (Example 6.6).

Organization of the paper. The commitment and revision games for a function on a compact metric space, and the determinacy of both games, occupy Section 2. The first dichotomy, Theorem A, is proved in Section 3, and the second, Theorem B, in Section 4. The game of Christensen and Saint-Raymond is applied to the types of a stable space in Section 5. Infinite ladders among block subspaces are studied in Section 6, where Theorems C and D are proved, and the questions that remain, together with both dichotomies in the language of model theory, close the paper in Section 7.

2. Two games decide continuity and the first Baire class

The commitment and revision games of the Introduction concern a single function on a compact metric space, and in that generality each game, at tolerance zero, is determined and decides a classical property of the function. Throughout the section K is a compact metric space with metric d, $B(\sigma, \rho)$ is the open ball of radius ρ about σ, and $f\colon K \to \mathbb{R}$ is a bounded function. The *oscillation* of f at a point σ is $\operatorname{osc}_f(\sigma) = \inf_{\rho>0} \operatorname{diam} f(B(\sigma, \rho))$, and f is continuous at σ exactly when $\operatorname{osc}_f(\sigma) = 0$.

Definition 2.1. Let $\eta \geq 0$. In each of the following games, Player I plays points $\sigma_0, \sigma_1, \ldots$ of K with $d(\sigma_n, \sigma_{n+1}) \leq 2^{-n}$ for every n, and Player II answers each move of Player I. The points of Player I form a Cauchy sequence, with limit a point σ of K such that $d(\sigma_n, \sigma) \leq 2^{1-n}$ for every n.

(1) In the *commitment game* $\mathcal{C}(f, \eta)$, Player II plays closed bounded intervals $J_0 \supseteq J_1 \supseteq \cdots$, and Player II wins if $f(\sigma) \in \bigcap_n J_n$ and the lengths $|J_n|$ satisfy $\lim_n |J_n| \leq \eta$.
(2) In the *revision game* $\mathcal{R}(f, \eta)$, Player II plays real numbers $r_0, r_1, \ldots$, and Player II wins if $\limsup_n |r_n - f(\sigma)| \leq \eta$.

The number η is the *tolerance*, and $\mathcal{C}(f)$ and $\mathcal{R}(f)$ are the games with tolerance 0.

An interval announced in the commitment game binds Player II for the rest of the play, while a number announced in the revision game can be withdrawn at the next move, provided that the announcements eventually settle near $f(\sigma)$.

Proposition 2.2. *Let $\eta \geq 0$.*

(a) *If $\operatorname{osc}_f(\sigma) > \eta$ for some $\sigma \in K$, then Player I has a winning strategy in $\mathcal{C}(f, \eta)$; if $\operatorname{osc}_f(\sigma) < \eta$ for every $\sigma \in K$, then Player II has a winning strategy in $\mathcal{C}(f, \eta)$. Player II has a winning strategy in $\mathcal{C}(f)$ if f is continuous, and Player I has one otherwise.*
(b) *If there are a nonempty closed set $F \subseteq K$ and real numbers $a < b$ with $b - a > 4\eta$ such that $F \cap \{f \leq a\}$ and $F \cap \{f \geq b\}$ are both dense in F, then Player I has a winning strategy in $\mathcal{R}(f, \eta)$. Player II has a winning strategy in $\mathcal{R}(f)$ if f is of the first Baire class, and Player I has one otherwise.*

The second assertion of (b) is the theorem of Kiss [10]: at tolerance zero the revision game is his game, and we include the proof to make the margins of the tolerant games explicit.

Proof. Let M be a bound for $|f|$.

(a) Suppose that $\operatorname{osc}_f(\sigma) > \eta' > \eta$. Player I plays σ until Player II plays an interval J_n with $|J_n| < \eta'$, and Player I plays σ forever if no such interval comes, or if $f(\sigma) \notin J_n$; in both cases Player II loses. Otherwise, since $\operatorname{diam} f(B(\sigma, 2^{-n})) > \eta' > |J_n|$, there is $\tau \in B(\sigma, 2^{-n})$ with $f(\tau) \notin J_n$, and Player I plays τ from move $n+1$ on, which is legal since $d(\sigma, \tau) < 2^{-n}$; the limit of the play is τ, and $f(\tau) \notin \bigcap_k J_k$.

Suppose next that $\operatorname{osc}_f < \eta$ everywhere. For each $\sigma' \in K$ choose $\rho_{\sigma'} > 0$ with $\operatorname{diam} f(B(\sigma', 2\rho_{\sigma'})) < \eta$, cover K by finitely many balls $B(\sigma'_k, \rho_{\sigma'_k})$, and let ρ be the least of the radii $\rho_{\sigma'_k}$; every ball $B(\sigma', \rho)$ lies in one of the balls $B(\sigma'_k, 2\rho_{\sigma'_k})$, so that $\operatorname{diam} f(B(\sigma', \rho)) < \eta$ for every σ'. Player II plays $[-M, M]$ while $2^{1-n} \geq \rho$, and afterwards plays $J_n = J_{n-1} \cap I_n$, where I_n is the closed interval spanned by f on the closed ball of radius 2^{1-n} about σ_n. The limit σ lies in that ball, so $f(\sigma) \in I_n$ for every such n, and $|J_n| \leq |I_n| < \eta$.

If f is continuous, then f is uniformly continuous, with modulus $\omega(t) = \sup\{|f(\sigma') - f(\sigma'')| : d(\sigma', \sigma'') \le t\}$, and $\omega(t) \to 0$ as $t \to 0$. Player II plays $J_0 = [-M, M]$ and $J_n = J_{n-1} \cap [f(\sigma_n) - \omega(2^{1-n}), f(\sigma_n) + \omega(2^{1-n})]$; each interval contains $f(\sigma)$, and $|J_n| \le 2\omega(2^{1-n}) \to 0$. If f is not continuous, then $\mathrm{osc}_f(\sigma) > 0$ at some point σ, and Player I wins $\mathcal{C}(f)$ by the first paragraph.

(b) Choose $\varepsilon > 0$ with $b - a > 4\eta + 3\varepsilon$, and let $\alpha = a + \eta + \varepsilon$ and $\beta = b - \eta - \varepsilon$. Player I plays a point $p \in F$ with $f(p) \le a$ until Player II plays a number $r_n \le \alpha$; then Player I moves to a point $q \in F$ with $f(q) \ge b$ and $d(p, q) \le 2^{-n}$, which exists because $F \cap \{f \ge b\}$ is dense in F, and plays q until Player II plays a number $r_m \ge \beta$; then Player I moves to a point of $F \cap \{f \le a\}$ within 2^{-m} of q, and so on. If Player I stays at one point p from some move on, then p is the limit of the play; if $f(p) \le a$, the numbers r_n exceed $\alpha \ge f(p) + \eta + \varepsilon$ from that move on, and if $f(p) \ge b$, the numbers fall below $\beta \le f(p) - \eta - \varepsilon$, so Player II loses. If Player I moves infinitely often, then the limit σ of the play lies in the closed set F, and Player II plays numbers at most α and numbers at least β infinitely often. Were $|r_n - f(\sigma)| \le \eta + \varepsilon/2$ for all large n, then $\beta - \alpha \le 2\eta + \varepsilon$, that is, $b - a \le 4\eta + 3\varepsilon$, which is false; so Player II loses.

Suppose that f is the pointwise limit of continuous functions $g_1, g_2, \ldots$, with moduli of continuity ω_k defined as in (a). Let $k(n)$ be the largest $k \le n$ with $\omega_k(2^{1-n}) \le 2^{-k}$, when there is one. The sequence $k(n)$ is defined for all large n, is nondecreasing, and tends to infinity, since $\omega_k(t) \to 0$ as $t \to 0$ for every k. Player II plays $r_n = g_{k(n)}(\sigma_n)$, and $r_n = 0$ when $k(n)$ is undefined. Then $|r_n - g_{k(n)}(\sigma)| \le \omega_{k(n)}(2^{1-n}) \le 2^{-k(n)}$, and $g_{k(n)}(\sigma) \to f(\sigma)$, so $r_n \to f(\sigma)$.

Suppose finally that f is not of the first Baire class. By Baire's theorem [8, Theorem 24.15], there is a nonempty closed set $F_0 \subseteq K$ such that the restriction of f to F_0 has no point of continuity. At every point x of F_0 the upper and lower limits of f along F_0 differ, so there are rational numbers $a < b$ such that every neighborhood of x meets both $F_0 \cap \{f \le a\}$ and $F_0 \cap \{f \ge b\}$. The set $C_{a,b}$ of such points x is closed in F_0, and F_0 is the union of the countably many sets $C_{a,b}$; by the Baire category theorem, some $C_{a,b}$ has nonempty interior V in F_0. Every nonempty open subset of V meets both $F_0 \cap \{f \le a\}$ and $F_0 \cap \{f \ge b\}$, in points of V, so for the closure F of V both $F \cap \{f \le a\}$ and $F \cap \{f \ge b\}$ are dense in F, and Player I wins $\mathcal{R}(f)$ by the first paragraph of (b). □

In the next two sections the function is the action of a type on K_Y, and the two games become the two dichotomies.

3. The commitment game decides stability

The first dichotomy joins three descriptions of stability: the symmetry of Krivine and Maurey, uniform approximation by norm polynomials, and

a winning strategy for Player II in the commitment game. Let Y be a separable Banach space. The space K_Y of types $\sigma(y) = \lim_{j,\mathcal{V}} \|y_j + y\|$ given by sequences (y_j) in the unit ball B_Y is a closed subset of the product $\prod_{y\in Y}[0, 1 + \|y\|]$, hence compact for pointwise convergence on Y, and the vectors $y \in B_Y$, regarded as the types $\|y + \cdot\|$, are dense in K_Y. Since every type is 1-Lipschitz on Y, convergence on a countable dense subset $\{a_k : k \in \mathbb{N}\}$ of Y is convergence on Y, and $d(\sigma, \sigma') = \sum_k 2^{-k} \min\{1, |\sigma(a_k) - \sigma'(a_k)|\}$ is a metric for the topology of K_Y. For $x \in Y$ the evaluation $e_x(\sigma) = \sigma(x)$ is continuous on K_Y, and $|e_x| \le 1 + \|x\|$. For a type $\tau = \lim_{i,\mathcal{U}} \|x_i + \cdot\|$ with (x_i) bounded in Y, the function $\tau(\sigma) = \lim_{i,\mathcal{U}} \sigma(x_i)$ on K_Y is the pointwise limit along $\mathcal{U}$ of the evaluations e_{x_i}, and on the vectors of B_Y the function coincides with the type τ.

Proof of Theorem A. Suppose that Y is stable. By Grothendieck's criterion [6], a pointwise limit of a uniformly bounded family of continuous functions on a compact space is continuous when the two iterated limits of the family agree along every sequence of points and every sequence of functions; for the evaluations e_{x_i} and points $\sigma_j \in K_Y$ this is the stability of Y, extended from vectors to types by the separate continuity of the convolution of types of a stable space [7, Proposition 13.2 and Remark 13.3]. Hence every type τ is a continuous function on K_Y. The norm polynomials form an algebra of continuous functions on K_Y that contains the constants and separates points, since a type is determined by its values at the vectors of Y; by the Stone–Weierstrass theorem, the algebra is dense in the continuous functions on K_Y for the uniform norm. A uniform limit of continuous functions is continuous, so Player II wins the commitment game for every type by Proposition 2.2(a); hence (i) implies (ii), and (ii) implies (iii).

For the quantitative statement, let (x_i) and (y_j) form an infinite ladder of gap δ in Y, with number r, let $\mathcal{U}$ be a nonprincipal ultrafilter, and let $\tau = \lim_{i,\mathcal{U}} \|x_i + \cdot\|$ and $\sigma = \lim_{j,\mathcal{U}} \|y_j + \cdot\| \in K_Y$. At the vector y_j, $\tau(y_j) = \lim_{i,\mathcal{U}} \|x_i + y_j\| \ge r + \delta$, since only $i > j$ counts; at the limit σ, $\tau(\sigma) = \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}} \|x_i + y_j\| \le r$, since only $j \ge i$ counts. Let Q be a norm polynomial with $|Q - \tau| \le \eta$ on K_Y. Since Q is continuous and the vectors y_j converge to σ along $\mathcal{U}$ in K_Y, $Q(\sigma) = \lim_{j,\mathcal{U}} Q(y_j) \ge r + \delta - \eta$, while $Q(\sigma) \le \tau(\sigma) + \eta \le r + \eta$. Hence $\eta \ge \delta/2$. The same configuration gives Player I the strategy of the last statement. Every neighborhood of σ in K_Y contains vectors y_j, since the vectors converge to σ along $\mathcal{U}$, and τ takes values at least $r + \delta$ at the vectors y_j and at most r at σ; so $\operatorname{osc}_\tau(\sigma) \ge \delta$, and Player I wins $\mathcal{C}(\tau, \eta)$ for every $\eta < \delta$ by Proposition 2.2(a).

Finally, if Y is not stable, then Y carries an infinite ladder of some positive gap, by Lemma 6.2 with the parameter 0, and the last statement, with $\eta = 0$, gives a type for which Player I has a winning strategy in the commitment game, so that Player II has none; hence (iii) implies (i). □

The proof shows where the obstruction lives, and that the obstruction and the winning strategy of Player I are the same object: the type τ is large on the vectors y_j and small at their limit, a jump of size δ that no continuous function, and in particular no norm polynomial, can follow within less than $\delta/2$. Player I exploits the jump by moving from σ to a nearby vector y_j after Player II has committed to an interval shorter than δ.

4. The revision game decides independence

The second dichotomy replaces continuity by the first Baire class, and the infinite ladder by the independent sequence; the functions e_x of Section 3 are again the starting point.

Proof of Theorem B. By Proposition 2.2(b), a type acts on K_Y as a function of the first Baire class if and only if Player II wins the revision game for the type.

Suppose that Y carries no independent sequence of positive gap, and let (x_i) be a bounded sequence in Y, and choose $R \geq 1$ with $\|x_i\| \leq R$ for every i. The evaluations $e_{x_i}(\sigma) = \sigma(x_i)$ are continuous on K_Y and uniformly bounded by $R+1$. By Rosenthal's dichotomy [14], either some subsequence of (e_{x_i}) converges pointwise on K_Y, or there are a subsequence (x_{i_k}) and numbers $r < s$ such that, for all disjoint finite sets $A, B \subseteq \mathbb{N}$, some $\sigma \in K_Y$ satisfies $\sigma(x_{i_k}) < r$ for $k \in A$ and $\sigma(x_{i_k}) > s$ for $k \in B$. The second alternative is impossible: the conditions on σ are open, the vectors of B_Y are dense in K_Y, and a vector $y \in B_Y$ that satisfies the conditions shows, after division of the vectors x_{i_k} and y by R, that (x_{i_k}/R) is independent with gap $(s-r)/R$, a contradiction. Hence every sequence in $\{e_x : x \in RB_Y\}$ has a pointwise convergent subsequence, and by the theorem of Rosenthal and of Bourgain, Fremlin and Talagrand [15, 3] the pointwise closure in $\mathbb{R}^{K_Y}$ of every countable subset of $\{e_x : x \in RB_Y\}$ consists of functions of the first Baire class. A type given by a sequence (x_i) and an ultrafilter acts as a point of the closure of $\{e_{x_i} : i \in \mathbb{N}\}$, and so as a function of the first Baire class.

Suppose now that (x_i) is independent with gap δ and number r. Let S be the closed set of types $\sigma \in K_Y$ such that, for every i, either $\sigma(x_i) \leq r$ or $\sigma(x_i) \geq r + \delta$, and let $\pi(\sigma) = \{i : \sigma(x_i) \leq r\}$ for $\sigma \in S$. The map $\pi\colon S \to 2^{\mathbb{N}}$ is continuous, since on S the set $\{\sigma(x_i) \leq r\}$ coincides with the open set $\{\sigma(x_i) < r + \delta\}$. The map is onto: for $Z \subseteq \mathbb{N}$, independence gives vectors $y_n \in B_Y$ with $\|x_i + y_n\| \leq r$ for $i \in Z$, $i \leq n$, and $\|x_i + y_n\| \geq r + \delta$ for $i \notin Z$, $i \leq n$, and a cluster point σ of the vectors y_n in K_Y lies in S and satisfies $\pi(\sigma) = Z$. By Zorn's lemma and compactness, there is a closed set $F \subseteq S$ with $\pi(F) = 2^{\mathbb{N}}$ such that no proper closed subset of F is mapped onto $2^{\mathbb{N}}$. For a nonempty relatively open set $O \subseteq F$, the image $\pi(F \setminus O)$ is a proper closed subset of $2^{\mathbb{N}}$, and so the set O contains $\pi^{-1}(N) \cap F$ for a nonempty basic open set $N \subseteq 2^{\mathbb{N}}$.

Let $\mathcal{W}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and let $\tau_{\mathcal{W}}$ be the action of the type given by (x_i) and $\mathcal{W}$, so that $\tau_{\mathcal{W}}(\sigma) \le r$ if $\pi(\sigma) \in \mathcal{W}$, and $\tau_{\mathcal{W}}(\sigma) \ge r+\delta$ otherwise, for $\sigma \in S$. A basic open set N prescribes membership for finitely many integers, so N contains a cofinite set, which belongs to $\mathcal{W}$, and a finite set, which does not. Hence every nonempty relatively open subset of F meets both $\{\tau_{\mathcal{W}} \le r\}$ and $\{\tau_{\mathcal{W}} \ge r+\delta\}$, and Player I wins $\mathcal{R}(\tau_{\mathcal{W}}, \eta)$ for every $\eta < \delta/4$ by Proposition 2.2(b); in particular $\tau_{\mathcal{W}}$ is not of the first Baire class. For distinct nonprincipal ultrafilters $\mathcal{W}$ and $\mathcal{W}'$, choose $Z \in \mathcal{W} \setminus \mathcal{W}'$ and $\sigma \in S$ with $\pi(\sigma) = Z$; then $\tau_{\mathcal{W}}(\sigma) \le r < r+\delta \le \tau_{\mathcal{W}'}(\sigma)$. There are $2^{\mathfrak{c}}$ nonprincipal ultrafilters on $\mathbb{N}$, so the types act as $2^{\mathfrak{c}}$ distinct functions. When every type acts as a function of the first Baire class, the types act as at most $\mathfrak{c}$ functions, since a function of the first Baire class on a compact metric space is the pointwise limit of a sequence of continuous functions, and there are $\mathfrak{c}$ such sequences. □

In the proof, Player I moves inside the set F, between types on which $\tau_{\mathcal{W}}$ is small and types on which $\tau_{\mathcal{W}}$ is large, and forces Player II to change the announced value infinitely often.

5. A game takes the place of compactness

Grothendieck's criterion concerns limits of functions of one variable. For functions of two variables the corresponding statement is Namioka's theorem: a separately continuous function on the product of a Čech-complete space and a compact space is jointly continuous at every point of $G \times Z$, for a dense G_δ subset G of the first factor and the whole compact factor Z [13]. Christensen and Saint-Raymond replaced the hypothesis on the first factor by a topological game [4, 16], and in this section the game, not compactness, decides where the types of a stable space depend uniformly on the type.

Definition 5.1. Let T be a topological space. In the *σ-game* on T, Player β plays a nonempty open set $U_0 \subseteq T$; Player α answers with a nonempty open set $V_0 \subseteq U_0$ and a point $p_0 \in T$; Player β plays a nonempty open set $U_1 \subseteq V_0$, and so on. Player α wins if the sequence (p_n) has a cluster point in $\bigcap_n V_n$. The space T is *β-unfavorable* if Player β has no winning strategy.

In a complete metric space Player α wins by answering with open balls whose radii tend to 0, each ball with closure inside the set just played by Player β, and with their centers as points: the intersection of the balls is a single point, and the centers converge to it. Every Polish space is therefore β-unfavorable, and so is every G_δ subset of K_Y.

Theorem 5.2 (Saint-Raymond [16]). *Let T be a β-unfavorable space, let Z be a compact space, and let $\Phi\colon T \times Z \to \mathbb{R}$ be separately continuous. Then there is a dense G_δ subset G of T such that Φ is jointly continuous at every point of $G \times Z$.*

In a stable space every type $\sigma \in K_Y$ acts on K_Y by a function $\hat{\sigma}$ that depends only on σ. Indeed, if $\sigma = \lim_{j,\mathcal{V}} \|y_j + \cdot\|$ and $\tau = \lim_{i,\mathcal{U}} \|x_i + \cdot\|$ with

(x_i) and (y_j) in B_Y, then by stability

$$\lim_{j,\mathcal{V}} \tau(y_j) = \lim_{j,\mathcal{V}} \lim_{i,\mathcal{U}} \|x_i + y_j\| = \lim_{i,\mathcal{U}} \lim_{j,\mathcal{V}} \|x_i + y_j\| = \lim_{i,\mathcal{U}} \sigma(x_i);$$

the left side depends only on τ and the sequence that gives σ, the right side only on σ and the sequence that gives τ, and so the common value $\Phi(\sigma,\tau) = \hat{\sigma}(\tau) = \hat{\tau}(\sigma)$ depends only on σ and τ.

Proposition 5.3. *Let Y be a stable separable Banach space, and let $T \subseteq K_Y$ be β-unfavorable, for instance a G_δ subset of K_Y. Then there is a dense G_δ subset G of T such that the map $\sigma \mapsto \hat{\sigma}$, from T to the space of continuous functions on K_Y with the uniform norm, is continuous at every point of G.*

Proof. The function Φ is symmetric, and $\Phi(\sigma,\cdot) = \hat{\sigma}$ is continuous on K_Y for every σ by Theorem A; hence Φ is separately continuous on $T \times K_Y$. By Theorem 5.2, Φ is jointly continuous at every point of $G \times K_Y$ for a dense G_δ subset G of T. Let $\sigma_0 \in G$ and $\varepsilon > 0$. For each $\tau \in K_Y$ there are open sets $O_\tau \ni \sigma_0$ in T and $W_\tau \ni \tau$ in K_Y with $|\Phi(\sigma,\tau') - \Phi(\sigma_0,\tau)| < \varepsilon/2$ for $\sigma \in O_\tau$ and $\tau' \in W_\tau$. Finitely many sets $W_{\tau_1}, \dots, W_{\tau_m}$ cover K_Y; let $O = O_{\tau_1} \cap \cdots \cap O_{\tau_m}$. For $\sigma \in O$ and $\tau' \in W_{\tau_k}$,

$$|\hat{\sigma}(\tau') - \hat{\sigma}_0(\tau')| \le |\Phi(\sigma,\tau') - \Phi(\sigma_0,\tau_k)| + |\Phi(\sigma_0,\tau_k) - \Phi(\sigma_0,\tau')| < \varepsilon,$$

so $\sup_{K_Y} |\hat{\sigma} - \hat{\sigma}_0| \le \varepsilon$ on the neighborhood O of σ_0. □

The topology of K_Y is convergence at the vectors of Y, one vector at a time, and by Proposition 5.3, at a generic type of T, closeness at finitely many vectors already forces closeness of the actions at all types at once. The proof uses stability only to make Φ defined and separately continuous on $T \times K_Y$, and uses no compactness of T. The conclusion therefore holds for every β-unfavorable set T of types on which Φ is defined and separately continuous, closed or not.

6. Dichotomies among subspaces need a margin

In this section we pass from one space to the block subspaces of a space: we measure infinite ladders, show how the measure behaves when a block basis is shortened or perturbed, and prove Theorems C and D. Throughout the section X is a real Banach space with a basis of basis constant b, subspaces are closed and, unless they are said to be finite-dimensional, infinite-dimensional, and $[z_k : k \in I]$ is the closed linear span.

Definition 6.1. Let Z be a subspace of X, let $P \subseteq X$, and let $\delta > 0$. An *infinite ladder of gap δ in Z, with a parameter from P,* consists of sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$ in the unit ball B_Z, a vector $c \in P$ with $\|c\| \le 2$, and a real number r such that, for all i, j,

$$\|x_i + y_j + c\| \le r \quad \text{if } i \le j, \qquad \|x_i + y_j + c\| \ge r + \delta \quad \text{if } i > j.$$

An *infinite ladder* without further qualification has the parameter $c = 0$. The *measure of instability* of Z relative to P is $\mathrm{w}_P(Z) = \sup\{\delta > 0 : Z$

carries an infinite ladder of gap δ with a parameter from $P\}$, with $\mathrm{w}_P(Z) = 0$ if there is no such ladder; we write $\mathrm{w}_0(Z)$ for $\mathrm{w}_{\{0\}}(Z)$.

A subsequence of rungs $(x_{i_k}, y_{i_k})_{k\in\mathbb{N}}$, $i_1 < i_2 < \cdots$, of an infinite ladder is an infinite ladder with the same gap, parameter and number r. If $Z' \subseteq Z$ and $P' \subseteq P$, then $\mathrm{w}_{P'}(Z') \le \mathrm{w}_P(Z)$. The values $\|x_i + y_j + c\|$ lie in $[0, 4]$, so $\mathrm{w}_P(Z) \le 4$.

Lemma 6.2. *Let Z be a subspace of X and let $c \in X$. Then Z carries no infinite ladder with the parameter c/R for any $R \ge 1$ with $\|c\| \le 2R$ if and only if the two iterated limits of $\|x_m + y_n + c\|$ agree along all bounded sequences (x_m) and (y_n) in Z and all ultrafilters on $\mathbb{N}$. In particular, $\mathrm{w}_0(Z) = 0$ if and only if Z is stable.*

Proof. Suppose that (a_i) and (b_j) are bounded sequences in Z and that $A = \lim_{i,\mathcal{U}} \lim_{j,\mathcal{V}} \alpha_{ij}$ and $B = \lim_{j,\mathcal{V}} \lim_{i,\mathcal{U}} \alpha_{ij}$ differ, where $\alpha_{ij} = \|a_i + b_j + c\|$. If $\mathcal{U}$ or $\mathcal{V}$ were principal, the two iterated limits would coincide, so both ultrafilters are nonprincipal and all their members are infinite. Let $\varepsilon = |A - B|/5$, $\beta_i = \lim_{j,\mathcal{V}} \alpha_{ij}$ and $\gamma_j = \lim_{i,\mathcal{U}} \alpha_{ij}$. Choose indices $i_1 < i_2 < \cdots$ and $j_1 < j_2 < \cdots$ alternately: given the earlier ones, choose $i_k > i_{k-1}$ with $|\beta_{i_k} - A| < \varepsilon$ and $|\alpha_{i_k j_l} - \gamma_{j_l}| < \varepsilon$ for $l < k$, which is possible because the intersection of these finitely many sets of indices belongs to $\mathcal{U}$ and is therefore infinite; then choose $j_k > j_{k-1}$ with $|\gamma_{j_k} - B| < \varepsilon$ and $|\alpha_{i_l j_k} - \beta_{i_l}| < \varepsilon$ for $l \le k$, using $\mathcal{V}$. Then $|\alpha_{i_l j_k} - A| < 2\varepsilon$ for $l \le k$ and $|\alpha_{i_k j_l} - B| < 2\varepsilon$ for $l < k$. Let $R \ge 1$ bound the norms of the a_i and b_j, with $\|c\| \le 2R$. If $A < B$, the sequences $x_k = a_{i_k}/R$ and $y_k = b_{j_k}/R$ form an infinite ladder of gap ε/R with the parameter c/R and the number $(A + 2\varepsilon)/R$. If $A > B$, the sequences $x_k = b_{j_k}/R$ and $y_k = a_{i_{k+1}}/R$ form an infinite ladder of gap ε/R with the parameter c/R and the number $(B + 2\varepsilon)/R$: since the norm $\|x + y + c\|$ is symmetric in x and y, the entry $\|x_k + y_l + c/R\|$ is within $2\varepsilon/R$ of B/R when $k \le l$ and of A/R when $k > l$.

Conversely, let (x_i), (y_j), c/R and r form an infinite ladder of gap δ in Z. For any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the inner limit over j in $\lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}} \|x_i + y_j + c/R\|$ sees only $j \ge i$, so this iterated limit is at most r, while the inner limit over i in the other iterated limit sees only $i > j$, so that limit is at least $r + \delta$; multiplying by R gives sequences (Rx_i) and (Ry_j) along which the iterated limits of $\|x + y + c\|$ differ. The last statement is the case $c = 0$. □

Two operations on block bases change the measure in a controlled way: discarding finitely many vectors, which creates a parameter, and perturbing the vectors slightly, which does not.

Lemma 6.3 (discarding the head). *Let $(z_k)_{k\ge1}$ be a basic sequence in X with basis constant b, let $Y = [z_k : k \ge 1]$, and for $n \ge 1$ let $F_n = \operatorname{span}\{z_k : k < n\}$ and $Y_n = [z_k : k \ge n]$. Then $\mathrm{w}_0(Y) \le (1 + b)\, \mathrm{w}_{F_n}(Y_n)$ for every n.*

Proof. Let $P_n \colon Y \to F_n$ be the basis projection along Y_n, so that $\|P_n\| \le b$ and $\|I - P_n\| \le 1 + b$. Let (x_i), (y_j) and r form an infinite ladder of gap

δ in Y, and let $0 < \eta < \delta/4$. The pairs $(P_n x_i, P_n y_i)$ lie in the compact set $bB_{F_n} \times bB_{F_n}$, so some subsequence of the rungs satisfies $P_n x_i \to f$ and $P_n y_i \to g$ with $f, g \in bB_{F_n}$; discarding finitely many rungs, we may assume $\|P_n x_i - f\| \le \eta$ and $\|P_n y_j - g\| \le \eta$ for all i, j. Let $t_i = x_i - P_n x_i$ and $s_j = y_j - P_n y_j$, which lie in Y_n and have norm at most $1 + b$, and let $c = f + g \in F_n$. Then $\big|\|t_i + s_j + c\| - \|x_i + y_j\|\big| \le 2\eta$, so the vectors t_i, s_j satisfy the inequalities of an infinite ladder of gap $\delta - 4\eta$ with parameter c and number $r + 2\eta$. Dividing by $1 + b$ gives an infinite ladder of gap $(\delta - 4\eta)/(1+b)$ in B_{Y_n} whose parameter $c/(1+b)$ lies in F_n and has norm at most $2b/(1+b) \le 2$. Letting $\eta \to 0$ and $\delta \to \mathrm{w}_0(Y)$ gives the inequality. □

Lemma 6.4 (small perturbations). *Let (y_k) be a normalized basic sequence in X with basis constant at most b, and let (y'_k) be a sequence in X with $\theta = 2b\sum_k \|y_k - y'_k\| < 1$. Let $Z = [y_k : k \ge 1]$, $Z' = [y'_k : k \ge 1]$ and $\theta' = \theta/(1-\theta)$. Then $\mathrm{w}_0(Z') \ge (\mathrm{w}_0(Z) - 4\theta)/(1+\theta)$ and $\mathrm{w}_0(Z) \ge (\mathrm{w}_0(Z') - 4\theta')/(1+\theta')$.*

Proof. The coefficients of a vector $x = \sum_k a_k y_k$ satisfy $|a_k| \le 2b\|x\|$, so the operator T defined by $T y_k = y'_k$ satisfies $\|Tx - x\| \le \theta\|x\|$ on Z; hence T is an isomorphism of Z onto Z' with $\|T^{-1}x' - x'\| \le \theta'\|x'\|$ on Z'. If $\mathrm{w}_0(Z) \le 4\theta$, the first inequality is immediate from $\mathrm{w}_0(Z') \ge 0$. Let (x_i), (y_j) and r form an infinite ladder of gap $\delta > 4\theta$ in Z. For u, v in the unit ball of Z, $\big|\|Tu + Tv\| - \|u + v\|\big| \le 2\theta$, so the images Tx_i, Ty_j, of norm at most $1+\theta$, satisfy the inequalities of an infinite ladder of gap $\delta - 4\theta$ with number $r + 2\theta$; dividing by $1 + \theta$ gives an infinite ladder of gap $(\delta - 4\theta)/(1+\theta)$ in the unit ball of Z'. The second inequality is the first one for T^{-1}. □

Proof of Theorem C. Since an isometric copy is $(1+\varepsilon')$-isomorphic for every $\varepsilon' > 0$, we may assume $\varepsilon > 0$. Let V be a subspace of Y and $T\colon W \to V$ an isomorphism from a stable space W with $\|T\|\,\|T^{-1}\| \le 1+\varepsilon$; rescaling T, we may assume $\|T^{-1}\| = 1$ and $\|T\| \le 1+\varepsilon$, so that $\|v\|/(1+\varepsilon) \le \|T^{-1}v\| \le \|v\|$ for $v \in V$. By the selection principle of Bessaga and Pełczyński [12], V contains a normalized sequence (v_k) for which there is a normalized block sequence (y_k) of Y with $\theta = 2b\sum_k \|v_k - y_k\| < \varepsilon$. The block subspace $[y_k : k \ge 1]$ of Y carries an infinite ladder of gap δ, so by Lemma 6.4 the space $V' = [v_k : k \ge 1]$ carries infinite ladders of every gap below $(\delta - 4\theta)/(1+\theta)$. Let (x_i), (y_j) and r form one of them, of gap γ. The vectors $T^{-1}x_i$ and $T^{-1}y_j$ lie in the unit ball of the stable space $T^{-1}V' \subseteq W$, and for v of norm at most 2, $\big|\|T^{-1}v\| - \|v\|\big| \le 2\varepsilon$; so they satisfy the inequalities of an infinite ladder of gap $\gamma - 4\varepsilon$, which by Lemma 6.2 is impossible unless $\gamma \le 4\varepsilon$. Hence $(\delta - 4\theta)/(1+\theta) \le 4\varepsilon$, and with $\theta < \varepsilon < 1/4$ this gives $\delta < 4\varepsilon(1+\varepsilon) + 4\varepsilon < 9\varepsilon$. So no such V exists when $\varepsilon < \min(\delta/9, 1/4)$. The spaces ℓ_p, $1 \le p < \infty$, are stable [11]. □

Proof of Theorem D. Suppose that the second alternative fails: for every block subspace Y, every finite-dimensional subspace F of X and every $\delta >$

0, some block subspace Z of Y carries no infinite ladder of gap δ with a parameter from F, so that $\mathrm{w}_F(Z) \le \delta$. Build block subspaces $X = Z_0 \supseteq Z_1 \supseteq \cdots$ and finitely supported vectors $z_1, z_2, \ldots$ as follows. At stage $n \ge 1$ the vectors $z_1, \ldots, z_{n-1}$ are known; let $F_n = \operatorname{span}\{z_k : k < n\}$, choose a block subspace Z_n of Z_{n-1} with $\mathrm{w}_{F_n}(Z_n) \le 1/n$, and choose a nonzero finitely supported vector $z_n \in Z_n$ whose support lies after that of z_{n-1}. The space $Y = [z_n : n \ge 1]$ is a block subspace of X, and (z_n) has basis constant at most b. For each n, the tail $Y_n = [z_k : k \ge n]$ lies in Z_n, because $z_k \in Z_k \subseteq Z_n$ for $k \ge n$, so Lemma 6.3 gives $\mathrm{w}_0(Y) \le (1+b)\,\mathrm{w}_{F_n}(Y_n) \le (1+b)/n$. Hence $\mathrm{w}_0(Y) = 0$, and Y is stable by Lemma 6.2. □

The parameter in Theorem D comes from the head F_n of the block basis, which is chosen before the subspace Z_n; that is what confines the parameters to one finite-dimensional space. The following example shows that an ambient parameter can produce an infinite ladder on a subspace whose own norm is Hilbertian, so that ladders with parameters do not certify instability.

Example 6.5 (one extra dimension destroys the margin). On $X = (\ell_2 \oplus \ell_2) \oplus \mathbb{R}$ define

$$\|(u, v, t)\| = \max\Big\{ \big(\|u\|_2^2 + \|v\|_2^2\big)^{1/2},\ |t| + 2^{-1/2} \sup_{i>j} |u_i + v_j| \Big\}.$$

The norm is equivalent to a Hilbert norm, and on the subspace $Z = \{t = 0\}$ the norm is the Hilbert norm, since $2^{-1/2}|u_i + v_j| \le (u_i^2 + v_j^2)^{1/2}$; so Z is stable. The vectors $x_i = (e_i, 0, 0)$ and $y_j = (0, e_j, 0)$ lie in the unit ball, and for $c = (0, 0, 1)$,

$$\|x_i + y_j + c\| = 1 + 2^{-1/2} \quad \text{if } i \le j, \qquad \|x_i + y_j + c\| = 1 + 2^{1/2} \quad \text{if } i > j,$$

so Z carries an infinite ladder of gap $2^{-1/2}$ with a parameter from the one-dimensional space $\mathbb{R}c$, and so does every subspace of Z spanned by subsequences of the two unit vector bases.

Example 6.6 (stability, not superstability). The space $E = (\bigoplus_{n\ge1} \ell_\infty^n)_{\ell_2}$ is stable, since finite-dimensional spaces are stable and ℓ_2-sums of stable spaces are stable [11]. In the n-th summand, $x_i = e_i$ and $y_j = \sum_{j<k\le n} e_k$ satisfy $\|x_i + y_j\| = 1$ for $i \le j$ and $\|x_i + y_j\| = 2$ for $i > j$, so E carries finite ladders of gap 1 of every length, and an ultrapower of E is unstable, although E carries no infinite ladder. Long finite ladders measure the failure of superstability; infinite ladders measure the failure of stability, which is what Question 1.1 concerns.

7. Questions, and the view from model theory

The central question is Question 1.1, which asks for the elimination of the parameter in Theorem D. By Example 6.5, a positive answer cannot come from a version of Lemma 6.3 with parameters: the head term must be absorbed inside the subspace, by a vector of the subspace that behaves like the head against the vectors of the ladder. The parameter arises from

discarding the head of a block basis, not from a failure of compactness, so neither the games of Section 2 nor the game of Section 5 removes the parameter by itself.

Question 7.1. Does Tsirelson's space have a block subspace Y and a number $\delta > 0$ such that every block subspace of Y carries an infinite ladder of gap δ?

Tsirelson's space has no infinite-dimensional stable subspace, since it contains no ℓ_p and no c_0, and its norm is built from admissible families of blocks far out; a positive answer would confirm Question 1.1 on the space where the question matters most.

Question 7.2. Is there a Banach space with a basis, without infinite-dimensional stable subspaces, in which every block subspace Y has block subspaces of arbitrarily small positive measure of instability w_0?

Such a space would answer Question 1.1 negatively; arbitrarily distortable spaces, such as the space of Schlumprecht, are the natural candidates.

Question 7.3. Is there a separable Banach space that is not stable but carries no independent sequence of positive gap? Does every Banach space with a basis have a block subspace Y such that either Y carries no independent sequence of positive gap, or there is $\delta > 0$ such that every block subspace of Y carries an independent sequence of gap δ?

The second part of Question 7.3 is the analogue of the dichotomy problem for the second dichotomy, Theorem B; the stabilization argument of Section 6 applies to independent sequences as to ladders once the analogues of Lemmas 6.3 and 6.4 are proved, and with the same parameter.

Question 7.4. Argyros, Negrepontis and Zachariades call a space *weakly stable* when the iterated limits in the definition of stability agree for weakly convergent sequences [2]. For a weakly stable space Y, is the set of types of weakly convergent sequences in B_Y a β-unfavorable subset of K_Y on which Φ is defined and separately continuous?

A positive answer would give Proposition 5.3 for weakly stable spaces, such as c_0, where the set of types in question need not be closed and compactness is not available.

The view from model theory. An infinite ladder is an instance of the order property of model theory for the formula $\|x + y\|$, and stability in the sense of Krivine and Maurey is the stability of that formula in the space itself; the stability of every ultrapower is the stability of the formula in the theory of the space, which corresponds to the absence of long finite ladders. An independent sequence is an instance of the independence property, and condition (iii) of Theorem B is the dependence of the formula in the space itself, the NIP in a model of Khanaki and Pillay [9]; the equivalence of (i) and (iii) is the Banach space form of Simon's description of dependent formulas by Rosenthal compacta [17]. Stability is a symmetry, the commutation of two iterated limits, and abstract model theory has learned that symmetry need not be assumed: Kim's analysis of forking, in the form

that Adler gave it [1], shows that in the presence of the other axioms of independence, symmetry follows from local character. The same holds for stability of Banach spaces: a local-character condition on Morley sequences of semidefinable types, which mentions only one order of iterated limits, characterizes stability, and the commutation of the iterated limits follows from it. This characterization, whose proof is Kim's argument, will appear separately, together with the framework it requires.

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