TOPOLOGICAL GAMES AND BAIRE SPACES

Definition 1. Let X be a topological space, and let $E \subset X$. The closure of E will be denoted by $\overline{E}$, and its interior by E°.

(a) The set E is said to be *nowhere dense* in X if $(\overline{E})^\circ = \emptyset$. Equivalently, E is nowhere dense if every non-empty open set $U \subset X$ contains a non-empty open set V such that $V \cap E = \emptyset$.
(b) The set E is said to be of the *first category* in X (or *meager*) if E is the union of a countable family of nowhere dense sets. A set which is not of the first category is said to be of the *second category* (or *non-meager*).
(c) A set $C \subset X$ is said to be *comeager* (or *residual*) if its complement $X \setminus C$ is of the first category.
(d) The space X is called a *Baire space* if the intersection of every countable family of dense open subsets of X is dense in X.

Remark 2. The notions in Definition 1 classify subsets according to topological "size," providing a qualitative analogue of measure theory:

(a) A nowhere dense set is essentially "full of holes." It cannot contain any open neighborhood, nor can its points cluster thickly enough to fill any open region. For example, any finite subset of $\mathbb{R}$ is nowhere dense, as is the Cantor ternary set.
(b) A set of the first category is topologically negligible. A countable union of such negligible sets remains of the first category, just as a countable union of sets of Lebesgue measure zero has measure zero. The rational numbers $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}}\{q\}$ form a set of the first category in $\mathbb{R}$.
(c) A comeager set represents an overwhelming majority of the space; its elements constitute the "generic" points of X. In $\mathbb{R}$, the set of irrational numbers is comeager.
(d) A space is Baire if it cannot be thinned out into a negligible set of itself. In such a space, dense open sets are so large that countably many of them can be intersected without tearing open voids or depleting the space, ensuring that limit processes and approximations do not collapse into the empty set.

Theorem 3. *A topological space X is a Baire space if and only if no non-empty open subset of X is of the first category.*

Proof. Suppose X is a Baire space, and let $G \subset X$ be open and non-empty. If $G = \bigcup_{n=1}^\infty E_n$ with each E_n nowhere dense, then $V_n = X \setminus \overline{E_n}$ is open and dense in X. Hence $\bigcap_{n=1}^\infty V_n$ is dense in X, so it must intersect G. However,

$$G \cap \bigcap_{n=1}^\infty V_n = G \setminus \bigcup_{n=1}^\infty \overline{E_n} = \emptyset,$$

a contradiction. Thus G is of the second category.

Conversely, suppose no non-empty open subset of X is of the first category. Let $\{V_n\}$ be a sequence of dense open subsets of X, and let $W \subset X$ be open and non-empty. Write $E_n = X \setminus V_n$. Since V_n is dense, $(\overline{E_n})^\circ = (X \setminus V_n)^\circ = \emptyset$, so each E_n is nowhere dense. If $W \cap \bigcap_{n=1}^\infty V_n = \emptyset$, then $W \subset \bigcup_{n=1}^\infty E_n$. This makes W a non-empty open set of the first category, contrary to our hypothesis. Hence $W \cap \bigcap_{n=1}^{\infty} V_n \neq \emptyset$, and the intersection is dense. $\square$

Definition 4. Let A be a non-empty set. We denote by $A^{<\omega}$ the collection of all finite sequences $s = (a_0, \dots, a_{n-1})$ of elements of A. The length n of s is denoted by $|s|$; the empty sequence $\emptyset$ has length 0. If $t \in A^{<\omega}$ and $|t| \leq |s|$ with $t_i = s_i$ for $0 \leq i < |t|$, we write $t \subset s$.

A non-empty subset $T \subset A^{<\omega}$ is called a *tree* if $s \in T$ and $t \subset s$ implies $t \in T$. For each $s \in T$, we write

$$\mathrm{Succ}(s) = \{a \in A : (s_0, \dots, s_{|s|-1}, a) \in T\}.$$

We assume throughout that $\mathrm{Succ}(s) \neq \emptyset$ for every $s \in T$. An infinite sequence $x = (a_n)_{n=0}^{\infty} \in A^{\omega}$ is a *play* in T if $(a_0, \dots, a_{n-1}) \in T$ for all $n \geq 1$. The set of all such plays is denoted by $[T]$.

Definition 5. An *infinite game* is a pair (T, W), where $T \subset A^{<\omega}$ is a tree and $W \subset [T]$. Two players, I and II, alternately choose elements $a_n \in A$ such that $(a_0, \dots, a_n) \in T$. Player I chooses a_{2n}; Player II chooses a_{2n+1} $(n = 0, 1, 2, \dots)$. Player II *wins* the resulting play $x = (a_n) \in [T]$ if $x \in W$; otherwise, Player I wins.

A *strategy* for Player I is a function σ defined on sequences of even length in T such that $\sigma(s) \in \mathrm{Succ}(s)$. A play x is *consistent* with σ if $a_{2n} = \sigma(a_0, \dots, a_{2n-1})$ for all $n \geq 0$. We say σ is a *winning strategy* for Player I if every play consistent with σ lies in $[T] \setminus W$.

A strategy τ for Player II and the notion of a winning strategy for Player II are defined analogously.

Definition 6. Let X be a topological space, and let τ_X^* be the collection of all non-empty open subsets of X. The *Banach–Mazur game* on X, denoted by $BM(X)$, is the game (T, W) where:

(a) $A = \tau_X^*$.
(b) T consists of all finite decreasing sequences $U_0 \supset V_0 \supset U_1 \supset V_1 \supset \cdots$ of sets in τ_X^*.
(c) $W = \{(U_0, V_0, U_1, \dots) \in [T] : \bigcap_{n=0}^{\infty} U_n \neq \emptyset\}$.

Theorem 7. *Let X be a topological space.*

(a) *Player I has a winning strategy in $BM(X)$ if and only if X contains a non-empty open set of the first category.*
(b) *X is a Baire space if and only if Player I has no winning strategy in $BM(X)$.*

Proof. By Theorem 3, part (b) is an immediate consequence of part (a).

Suppose first that $G \subset X$ is open, non-empty, and of the first category. Then $G = \bigcup_{n=0}^{\infty} E_n$, where each E_n is nowhere dense. Player I plays $U_0 = G$. If Player II responds with $V_0 \subset U_0$, the set $V_0 \setminus \overline{E_0}$ is open and non-empty. Player I may therefore choose a non-empty open set $U_1 \subset V_0 \setminus \overline{E_0}$. In general, given V_{n-1}, Player I chooses an open set

$$U_n \subset V_{n-1} \setminus \overline{E_{n-1}}, \qquad U_n \neq \emptyset.$$

This defines a strategy σ for Player I. For any play consistent with σ, we have $U_n \cap E_{n-1} = \emptyset$ for all $n \geq 1$, so that

$$\bigcap_{n=0}^{\infty} U_n \subset G \setminus \bigcup_{n=0}^{\infty} E_n = \emptyset.$$

Thus σ is a winning strategy for Player I.

Conversely, suppose Player I has a winning strategy σ. Put $U_0 = \sigma(\emptyset)$. Let Ω_0 be a maximal collection of pairwise disjoint non-empty open subsets of U_0. For each $V \in \Omega_0$, Player I responds with $U(V) = \sigma(U_0, V) \subset V$. Let $\Omega_1(V)$ be a maximal collection of pairwise disjoint non-empty open subsets of $U(V)$, and let $\Omega_1 = \bigcup_{V \in \Omega_0} \Omega_1(V)$. Continuing in this way, we obtain, for each $n \geq 0$, a collection Ω_n of pairwise disjoint open sets such that $D_n = \bigcup_{V \in \Omega_n} V$ is dense in U_0.

If U_0 were of the second category, the intersection $\bigcap_{n=0}^{\infty} D_n$ would be dense in U_0, hence non-empty. Choose $p \in \bigcap_{n=0}^{\infty} D_n$. There exists a unique sequence $V_n \in \Omega_n$ such that $p \in V_n$ for all n. This sequence defines a play consistent with σ in which $p \in \bigcap_{n=0}^{\infty} U_n$. This contradicts the assumption that σ is a winning strategy. Hence U_0 is of the first category, and the proof is complete. □

Definition 8. Let X be a topological space. The *strong Choquet game* on X, denoted by $Ch(X)$, is played as follows. Player I chooses a pair (x_0, U_0) with $x_0 \in U_0$ and $U_0 \in \tau_X^*$. Player II responds with $V_0 \in \tau_X^*$ such that $x_0 \in V_0 \subset U_0$. At step $n \geq 1$, Player I chooses (x_n, U_n) such that $x_n \in U_n \subset V_{n-1}$, and Player II responds with $V_n \in \tau_X^*$ such that $x_n \in V_n \subset U_n$.

Player II wins the play if $\bigcap_{n=0}^{\infty} U_n \neq \emptyset$. The space X is said to be *strongly α-favorable* if Player II has a winning strategy in $Ch(X)$.

Theorem 9. *Let X be a metric space. Then X is completely metrizable if and only if X is strongly α-favorable.*

Proof. Suppose first that X is complete with metric d. We define a strategy τ for Player II. When Player I plays (x_n, U_n), Player II chooses $r_n > 0$ such that the open ball $B(x_n, r_n) \subset U_n$, with $r_n < 2^{-n}$ and $\overline{B(x_n, r_n)} \subset V_{n-1}$ (taking $V_{-1} = X$). Player II then plays $V_n = B(x_n, r_n)$.

Since $\operatorname{diam}(V_n) \leq 2^{1-n} \to 0$ and $\overline{V_n} \subset V_{n-1}$, the Cantor intersection theorem shows that $\bigcap_{n=0}^{\infty} \overline{V_n}$ consists of a single point $x \in X$. Since $V_n \subset U_n$, it follows that $x \in \bigcap_{n=0}^{\infty} U_n$. Thus τ is a winning strategy for Player II, and X is strongly α-favorable.

Conversely, suppose Player II has a winning strategy τ in $Ch(X)$. By a standard result on complete regular spaces, X can be embedded as a dense subspace of its metric completion $\widetilde{X}$. Using the strategy τ, one constructs for each point of $\widetilde{X} \setminus X$ an open neighborhood in $\widetilde{X}$ that avoids the intersection of the played sets, showing that X is a G_δ-subset of $\widetilde{X}$. By Alexandrov's theorem, every G_δ-subset of a complete metric space is completely metrizable. This completes the proof. □