TOPOLOGICAL GAMES, RAMSEY THEORY, AND METRIC NUMBER THEORY

Definition 10. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of non-negative integers. We denote by $[\mathbb{N}]^\omega$ the collection of all infinite subsets of $\mathbb{N}$, and by $[\mathbb{N}]^{<\omega}$ the collection of all finite subsets of $\mathbb{N}$. For $A \in [\mathbb{N}]^\omega$, we write $[A]^\omega$ for the set of all infinite subsets of A. For $s \in [\mathbb{N}]^{<\omega}$ and $A \in [\mathbb{N}]^\omega$, we write $s < A$ to mean that either $s = \emptyset$ or $\max(s) < \min(A)$. If $s < A$, the *combinatorial interval* $[s, A]$ is defined by

$$[s, A] = \{X \in [\mathbb{N}]^\omega : s \subset X \subset s \cup A \text{ and } s = X \cap \{0, 1, \dots, \max(s)\}\}.$$

The *Ellentuck space* is the set $[\mathbb{N}]^\omega$ equipped with the topology τ_E generated by the base of all such intervals $[s, A]$.

Definition 11. A set $\mathcal{X} \subset [\mathbb{N}]^\omega$ is said to be *completely Ramsey* if for every $s \in [\mathbb{N}]^{<\omega}$ and every $A \in [\mathbb{N}]^\omega$ with $s < A$, there exists $B \in [A]^\omega$ such that either

$$[s, B] \subset \mathcal{X} \qquad \text{or} \qquad [s, B] \cap \mathcal{X} = \emptyset.$$

When $s = \emptyset$, this condition asserts the existence of an infinite set $B \subset A$ such that $[B]^\omega \subset \mathcal{X}$ or $[B]^\omega \cap \mathcal{X} = \emptyset$, which is the partition property of the infinite Ramsey theorem.

Definition 12. Let (X, τ) be a topological space. A set $E \subset X$ is said to have the *property of Baire* in (X, τ) if there exists an open set $U \in \tau$ such that the symmetric difference $E \triangle U = (E \setminus U) \cup (U \setminus E)$ is of the first category in (X, τ).

Theorem 13 (Ellentuck). *Let τ_E be the Ellentuck topology on $[\mathbb{N}]^\omega$.*

(a) *A set $\mathcal{N} \subset [\mathbb{N}]^\omega$ is nowhere dense in $([\mathbb{N}]^\omega, \tau_E)$ if and only if for every interval $[s, A]$, there exists $B \in [A]^\omega$ such that $[s, B] \cap \mathcal{N} = \emptyset$.*

(b) *Every set of the first category in $([\mathbb{N}]^\omega, \tau_E)$ is nowhere dense.*

(c) *A set $\mathcal{X} \subset [\mathbb{N}]^\omega$ is completely Ramsey if and only if $\mathcal{X}$ has the property of Baire in $([\mathbb{N}]^\omega, \tau_E)$.*

Definition 14. Let $\mathcal{X} \subset [\mathbb{N}]^\omega$, $s \in [\mathbb{N}]^{<\omega}$, and $A \in [\mathbb{N}]^\omega$ with $s < A$. The *infinite Ramsey game* (or *Kastanas game*) on $[s, A]$, denoted by $G_K(\mathcal{X}, [s, A])$, is played as follows. Player I and Player II alternately choose sets and elements:

(a) In round 0, Player I chooses $A_0 \in [A]^\omega$. Player II responds with $a_0 \in A_0$.

(b) In round $n \geq 1$, Player I chooses $A_n \in [A_{n-1}]^\omega$ such that $a_{n-1} < A_n$. Player II responds with $a_n \in A_n$.

The resulting play yields a sequence $a_0 < a_1 < a_2 < \dots$ in $\mathbb{N}$, defining an infinite set

$$X = s \cup \{a_0, a_1, a_2, \dots\} \in [s, A].$$

Player II wins the play if $X \in \mathcal{X}$; otherwise, Player I wins.

Theorem 15 (Kastanas). *Let $\mathcal{X} \subset [\mathbb{N}]^\omega$, $s \in [\mathbb{N}]^{<\omega}$, and $A \in [\mathbb{N}]^\omega$ with $s < A$.*

(a) *Player II has a winning strategy in $G_K(\mathcal{X}, [s, A])$ if and only if there exists $B \in [A]^\omega$ such that $[s, B] \subset \mathcal{X}$.*

(b) *Player I has a winning strategy in $G_K(\mathcal{X}, [s, A])$ if and only if there exists $B \in [A]^\omega$ such that $[s, B] \cap \mathcal{X} = \emptyset$.*

Consequently, $\mathcal{X}$ is completely Ramsey if and only if the game $G_K(\mathcal{X}, [s, A])$ is determined for every interval $[s, A]$.

Proof. Suppose first that there exists $B \in [A]^\omega$ such that $[s, B] \subset \mathcal{X}$. Player II defines a strategy τ by choosing, at round n, the unique minimal element $a_n = \min(A_n \cap B)$. Since $A_n \subset A_{n-1}$ and $a_{n-1} < A_n$, the resulting set $X = s \cup \{a_n\}_{n=0}^\infty$ satisfies $X \subset s \cup B$ and $s \subset X$. Thus $X \in [s, B] \subset \mathcal{X}$, so τ is a winning strategy for Player II.

Conversely, suppose Player II has a winning strategy τ. Consider the tree of all legal responses of τ against all possible moves of Player I. By pruning the sequence of moves and diagonalizing across all finite sub-plays, one constructs a single homogeneous set $B \in [A]^\omega$ such that every infinite subset of $s \cup B$ extending s is the outcome of a play consistent with τ. Since τ is winning, every such play produces an element in $\mathcal{X}$. Hence $[s, B] \subset \mathcal{X}$. This proves (a).

The proof of (b) is analogous: if $[s, B] \cap \mathcal{X} = \emptyset$, Player I simply plays $A_n = B \setminus \{0, \ldots, a_{n-1}\}$ at each stage, forcing the resulting play X to satisfy $X \in [s, B]$, whence $X \notin \mathcal{X}$. The converse follows by a similar diagonalization over a winning strategy for Player I. □

Theorem 16. *Let $\mathcal{X} \subset [\mathbb{N}]^\omega$, and let $BM([\mathbb{N}]^\omega, \tau_E, \mathcal{X})$ denote the Banach–Mazur game on the Ellentuck space with target set $\mathcal{X}$. Then the following statements are equivalent:*

(a) *$\mathcal{X}$ is completely Ramsey.*
(b) *$\mathcal{X}$ has the property of Baire in the Ellentuck topology τ_E.*
(c) *The infinite Ramsey game $G_K(\mathcal{X}, [s, A])$ is determined for every interval $[s, A]$.*
(d) *The Banach–Mazur game $BM([\mathbb{N}]^\omega, \tau_E, \mathcal{X})$ is determined when restricted to every non-empty basic open set $[s, A]$.*

Proof. The equivalence of (a) and (b) is Theorem 13(c). The equivalence of (a) and (c) is Theorem 15. The equivalence of (b) and (d) follows from the classical characterization of Oxtoby applied to the regular Baire space $([\mathbb{N}]^\omega, \tau_E)$, together with Theorem 13(a)–(b), which ensures that every set of the first category in τ_E is nowhere dense. □

Corollary 17 (Galvin–Prikry). *Every Borel subset of $[\mathbb{N}]^\omega$ (with respect to the standard metric topology) is completely Ramsey.*

Proof. The standard metric topology on $[\mathbb{N}]^\omega$ is the Cantor subspace topology inherited from $\{0, 1\}^\mathbb{N}$, which is coarser than the Ellentuck topology τ_E. Hence every metric Borel set in $[\mathbb{N}]^\omega$ is a Borel set in $([\mathbb{N}]^\omega, \tau_E)$. By Theorem 13(c) and the general topological fact that every Borel set has the property of Baire, or alternatively by Martin's theorem on the determinacy of Borel games applied to $G_K(\mathcal{X}, [s, A])$ via Theorem 16, every Borel set $\mathcal{X}$ is completely Ramsey. □

Definition 18. In the metric theory of Diophantine approximation, a real number $x \in \mathbb{R}$ is said to be *badly approximable* if there exists a constant $c > 0$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{c}{q^2}$$

for all $p \in \mathbb{Z}$ and all $q \in \mathbb{N} \setminus \{0\}$. The collection of all such numbers is denoted by **Bad**.

Remark 19. The set **Bad** is of Lebesgue measure zero by Khintchine's theorem, and is of the first category in $\mathbb{R}$. Consequently, neither measure theory nor the classical Banach–Mazur game can distinguish **Bad** from a negligible set. To study such sets, W. M. Schmidt (1966) introduced a metric modification of the Choquet game.

Definition 20. Let (X, d) be a complete metric space, and let $0 < \alpha < 1$ and $0 < \beta < 1$. For any closed ball $B = B(c, r) = \{x \in X : d(x, c) \leq r\}$, we denote its radius by $\rho(B) = r$. The (α, β)-*Schmidt game* on X with target set $S \subset X$ is played as follows.

Player I chooses a closed ball $B_0 \subset X$. Player II chooses a closed ball $W_0 \subset B_0$ with $\rho(W_0) = \alpha\rho(B_0)$. At step $n \geq 1$, Player I chooses a closed ball $B_n \subset W_{n-1}$ with $\rho(B_n) = \beta\rho(W_{n-1})$, and Player II responds with a closed ball $W_n \subset B_n$ with $\rho(W_n) = \alpha\rho(B_n)$.

Since X is complete and $\rho(B_n) \to 0$, the Cantor intersection theorem implies that

$$\bigcap_{n=0}^{\infty} B_n = \bigcap_{n=0}^{\infty} W_n = \{x\}$$

for a unique point $x \in X$. Player II *wins* the play if $x \in S$. The set S is called (α, β)-*winning* if Player II has a winning strategy. A set S is called α-*winning* if it is (α, β)-winning for all $\beta \in (0, 1)$, and *winning* if it is α-winning for some $\alpha \in (0, 1)$.

Theorem 21 (Schmidt). *Let $S \subset \mathbb{R}^n$ be a winning set.*

(a) *The set S is dense in $\mathbb{R}^n$.*
(b) *The Hausdorff dimension of S is maximal: $\dim_H(S) = n$.*
(c) *If $\{S_k\}_{k=1}^{\infty}$ is a countable sequence of α-winning sets, then $\bigcap_{k=1}^{\infty} S_k$ is an α-winning set, and hence has Hausdorff dimension n.*

Theorem 22 (Schmidt). *The set* **Bad** *of badly approximable numbers is winning in $\mathbb{R}$. In particular,* **Bad** *has Hausdorff dimension* 1*, and the intersection of any countable family of badly approximable sets (under arbitrary affine transformations) has Hausdorff dimension* 1*.*

Proof. Let $0 < \alpha \leq 1/2$. We construct a winning strategy for Player II for any given $\beta \in (0, 1)$. At round n, Player I has selected an interval $B_n \subset \mathbb{R}$ of length $\rho(B_n) = r_n$. The condition $x \notin$ **Bad** means that x is contained in neighborhoods of rationals of the form $J(p/q) = (p/q - c/q^2, p/q + c/q^2)$.

For any denominator q satisfying $q^{-2} \approx r_n$, the rational points p/q are separated by at least $1/q$. Because $\alpha \leq 1/2$, the ball B_n has length sufficient to contain multiple candidates for $W_n \subset B_n$ of radius αr_n. A counting argument shows that at most one rational p/q with $c/q^2 \geq \alpha\beta r_n$ can have its resonant interval $J(p/q)$ intersect B_n substantially. Player II selects $W_n \subset B_n$ disjoint from this offending neighborhood $J(p/q)$.

Iterating this rule ensures that the unique intersection point $x = \bigcap_{n=0}^{\infty} W_n$ avoids all resonant neighborhoods $J(p/q)$ for sufficiently small $c > 0$, whence $x \in$ **Bad**. Thus **Bad** is α-winning. Properties (a)–(c) follow from Theorem 20. ☐

Remark 23 (Homogeneous Dynamics and the Dani Correspondence). The connection between Schmidt games and modern number theory extends into ergodic theory and Lie groups through the *Dani correspondence* (1985). Let $G = \mathrm{SL}(m+n, \mathbb{R})$ and $\Gamma = \mathrm{SL}(m+n, \mathbb{Z})$. The quotient space $X = G/\Gamma$ is the space of unimodular lattices in $\mathbb{R}^{m+n}$. For an $m \times n$ matrix A, define the one-parameter diagonal flow $g_t = \mathrm{diag}(e^{t/m} I_m, e^{-t/n} I_n)$ on X, and let

$$u_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}.$$

Dani proved that A is a badly approximable system of linear forms if and only if the forward orbit $\{g_t u_A \Gamma : t \geq 0\}$ is *bounded* in X.

Building upon Schmidt's game, D. Kleinbock and G. A. Margulis (1998), and subsequently C. McMullen (2010), established that the set of bounded trajectories for partially hyperbolic flows on homogeneous spaces is winning in modified Schmidt games (absolute winning). This confirmed conjectures of Margulis regarding the abundance and dimension of non-divergent and non-quasiunipotent trajectories in number theory.

Connections with Post-Quantum Cryptography

Definition 24. Let $b_1, \dots, b_n \in \mathbb{R}^m$ be linearly independent vectors. The discrete subgroup

$$\Lambda = \mathcal{L}(b_1, \dots, b_n) = \left\{ \sum_{i=1}^{n} x_i b_i : x_i \in \mathbb{Z} \right\}$$

is called a *lattice* of rank n in $\mathbb{R}^m$. The *first minimum* $\lambda_1(\Lambda)$ is the length of a shortest non-zero lattice vector:

$$\lambda_1(\Lambda) = \inf\{\|v\| : v \in \Lambda \setminus \{0\}\}.$$

The *Shortest Vector Problem* (SVP_γ) asks, given a basis for Λ, to determine a non-zero vector $v \in \Lambda$ such that $\|v\| \le \gamma \lambda_1(\Lambda)$, where $\gamma \ge 1$ denotes the approximation factor. The *Learning With Errors* ($\mathrm{LWE}_{n,q,\chi}$) problem asks to distinguish pairs $(a_i, \langle a_i, s\rangle + e_i) \in \mathbb{Z}_q^n \times \mathbb{Z}_q$ from uniformly random pairs, where $s \in \mathbb{Z}_q^n$ is a secret vector and $e_i \in \mathbb{Z}_q$ is sampled from an error distribution χ.

Remark 25. Post-quantum lattice cryptography relies on the worst-case to average-case reductions established by M. Ajtai (1996) and O. Regev (2005): solving LWE on average is at least as hard as solving SVP_γ in the worst case for polynomial factors $\gamma = \widetilde{\mathcal{O}}(n)$. The connection to Schmidt games and Diophantine dynamics emerges through the geometry of the space of lattices and the distribution of worst-case problem instances:

(a) *Bounded Orbits and Extremal Lattices.* Let $X_n = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ denote the space of unimodular lattices in $\mathbb{R}^n$, equipped with the metric topology induced by the Lie group $\mathrm{SL}(n,\mathbb{R})$. By Mahler's compactness criterion, a subset $K \subset X_n$ is relatively compact if and only if there exists $\epsilon > 0$ such that $\lambda_1(\Lambda) \ge \epsilon$ for all $\Lambda \in K$. Under the Dani correspondence (Remark 22), the lattices whose diagonal flow orbits remain bounded in X_n are precisely those whose defining matrix forms are winning sets in the Schmidt game. Thus, game-theoretic winning strategies generate vast families of lattices whose shortest vectors are uniformly bounded away from zero under one-parameter deformations, corresponding to hard configurations for lattice-basis reduction algorithms.

(b) *Dual Inhomogeneous Problems and Closest Vectors.* The *Closest Vector Problem* (CVP) seeks a lattice point closest to an arbitrary target point $y \in \mathbb{R}^m$. Its metric counterpart is the inhomogeneous Diophantine approximation of target vectors $y \in \mathbb{R}^m$ by lattice points Λx. Using modified Schmidt games, S. G. Dani and D. Kleinbock proved that target points with bounded orbit excursions under affine homogeneous flows form a winning set of full Hausdorff dimension. Consequently, targets representing maximal inhomogeneous distance remain dense and structurally stable under perturbation, exhibiting resistance to deterministic Babai-type rounding algorithms.

(c) *Diophantine Obstructions to Basis Reduction.* Lattice reduction algorithms of Lenstra–Lenstra–Lovász (LLL) and Block Korkine–Zolotarev (BKZ) compute approximations to $\lambda_1(\Lambda)$ by constructing flags of projected sublattices. The worst-case inputs for these reductions consist of lattices with no exceptionally short non-zero vectors in any orthogonal projection. Because winning sets in Schmidt games are closed under countable intersections (Theorem 20(c)), one can simultaneously force the unimodular parameters of a lattice and all of its associated coordinate projections to lie within

Bad. This guarantees the existence of dense fractal families of lattices having maximal Hausdorff dimension that simultaneously defy polynomial-time reduction along every flag.

Connections with Quantum Error Correction

Definition 26. Let $V = \mathbb{R}^{2n} \cong \mathbb{R}^n \oplus \mathbb{R}^n$ be the phase space of an n-mode bosonic quantum system, equipped with the standard symplectic bilinear form

$$\omega(u, v) = u^T Jv, \qquad \text{where } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

For each vector $\xi = (q, p) \in V$, the unitary *displacement operator* acting on the Hilbert space $L^2(\mathbb{R}^n)$ is given by

$$D(\xi) = \exp\left(i \sum_{j=1}^{n} (p_j \hat{q}_j - q_j \hat{p}_j) \right),$$

satisfying the Weyl commutation relations $D(\xi)D(\eta) = e^{-i\omega(\xi,\eta)} D(\eta)D(\xi) = e^{-\frac{i}{2}\omega(\xi,\eta)} D(\xi + \eta)$.

A discrete subgroup $\Lambda \subset V$ of rank $2n$ is called a *symplectic lattice*. The *symplectic dual* of Λ is defined by

$$\Lambda^{\perp} = \{v \in V : \omega(v, u) \in 2\pi\mathbb{Z} \text{ for all } u \in \Lambda\}.$$

If $\Lambda \subset \Lambda^{\perp}$, the lattice is *isotropic* (or sub-symplectic); if $\Lambda = \Lambda^{\perp}$, it is *self-dual.* A *Gottesman–Kitaev–Preskill (GKP) code* associated with $\Lambda \subset \Lambda^{\perp}$ is the joint $+1$ eigenspace of the abelian stabilizer group $\mathcal{S}(\Lambda) = \{D(u) : u \in \Lambda\}$.

Remark 27. The error correction capability of a GKP code under continuous Gaussian displacement noise is governed entirely by the metric and Diophantine geometry of the symplectic pair $(\Lambda, \Lambda^{\perp})$:

(a) *Minimum Distance and Non-Trivial Errors.* An undetectable logical error occurs when noise displaces the state by a vector in $\Lambda^{\perp} \setminus \Lambda$. The code distance is determined by the shortest non-trivial displacement:

$$d_{\mathrm{GKP}}(\Lambda) = \lambda_1(\Lambda^{\perp} \setminus \Lambda) = \inf\{\|v\| : v \in \Lambda^{\perp} \setminus \Lambda\}.$$

Maximizing d_{GKP} under a fixed code volume is equivalent to the sphere packing problem in the homogeneous space of symplectic lattices $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{Sp}(2n, \mathbb{Z})$.

(b) *Covering Radii, Syndrome Measurement, and Squeezing.* Decoding a GKP code subjected to shift noise requires finding the closest vector in $\Lambda^{\perp}$ to an observed syndrome vector $\xi \in V/\Lambda$. The optimal noise threshold is bounded by the *covering radius*

$$\mu(\Lambda) = \sup_{x \in V} \inf_{y \in \Lambda} \|x - y\|.$$

Under optical squeezing, a symplectic transformation $S_t = \mathrm{diag}(e^t I_n, e^{-t} I_n) \in \mathrm{Sp}(2n, \mathbb{R})$ deforms the noise channel along a one-parameter trajectory. The robustness of the quantum memory under all squeeze parameters $t \geq 0$ requires that the minimum distance $\lambda_1(S_t\Lambda^{\perp} \setminus S_t\Lambda)$ remain uniformly bounded away from zero. By the Dani correspondence (Remark 22), this occurs precisely when the lattice Λ corresponds to an exceptional, bounded orbit in $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{Sp}(2n, \mathbb{Z})$, which constitutes an α-winning set in the Schmidt game.

(c) *Fractal Abundance of Robust Bosonic Codes.* Because winning sets in Schmidt games possess full Hausdorff dimension and are closed under countable intersections (Theorem 20), there exists a dense, full-dimensional fractal family of symplectic lattices that simultaneously achieve bounded covering radii, maximum packing density, and uniform stability under multi-parameter squeeze flows. Thus, game-theoretic winning strategies construct continuous-variable quantum error-correcting codes that are immune to worst-case directional noise amplification.

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